

ID Codes in Cartesian Products of Cliques

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ABSTRACT. An identifying code in a graph G is a set D of vertices such that the closed neighborhood of each vertex of the graph has a nonempty, distinct intersection with D . The minimum cardinality of an identifying code is denoted $\gamma^{ID}(G)$. Building upon recent results of Gravier, Moncel, and Semri, we show for $n \leq m$ that $\gamma^{ID}(K_n \square K_m) = \max\{2m - n, m + \lfloor n/2 \rfloor\}$. Furthermore, we improve upon the bounds for $\gamma^{ID}(G \square K_m)$ and explore the specific case when G is the Cartesian product of multiple cliques.

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1 Introduction

A subset D of the vertices of a graph is an *identifying code* if the intersection of D with the closed neighborhood of x is nonempty and distinct for all vertices x . This concept was introduced by Karpovsky, Chakrabarty, and Levitin [7] as a question about fault detection in multiprocessor systems.

There are now numerous papers on this topic. Some of these focus on product graphs. Apart from the original paper, several papers have considered the hypercube (Hamming space), including [1, 2, 6]. There has also been work on more general Cartesian products [8, 9] and other products such as the lexicographic product [3] and the direct product [10]. Recently, Gravier, Moncel, and Semri [5] determined the minimum size of an identifying code in the Cartesian product of two cliques (complete graphs) of the same size.

In this paper we determine the minimum size of an identifying code in the Cartesian product of two general cliques. We then provide upper and lower bounds in regards to the Cartesian product of three or more cliques. For example, we show that the minimum size of an identifying code of the product of three cliques of size m is approximately m^2 , and that the minimum size of an identifying code when one clique is much larger than the others is related to the total domination number of the product of the smaller cliques.

1.1 Notation

Let G be a simple, undirected graph. Given vertex $x \in V(G)$, we let $N(x)$ denote the set of vertices adjacent to x , and let $N[x] = \{x\} \cup N(x)$ denote the *closed neighborhood* of x . An *identifying code* (ID code for short) of G is a subset D of vertices such that the intersection $N[x] \cap D$ is nonempty and distinct for every vertex $x \in V(G)$. We say that D *separates* every pair of vertices in G . A necessary consequence of this definition is that D is a dominating set of G (that is, every vertex not in D is adjacent to a vertex in D). The minimum cardinality of an ID code of the graph G is denoted by $\gamma^{ID}(G)$.

Given two graphs G and H , the *Cartesian product* of G and H , denoted $G \square H$, is the graph whose vertex set is the Cartesian product, $V(G) \times V(H)$, and whose edge set is the set of all $(g_1, h_1)(g_2, h_2)$ such that $g_1 = g_2$ and $h_1 h_2 \in E(H)$ or $h_1 = h_2$ and $g_1 g_2 \in E(G)$. We will use $\{1, \dots, m\}$ for the vertex set of the clique/complete graph K_m .

2 Two Cliques

The result for equal cliques was proved by Gravier et al.:

Theorem 2.1 ([5]). *For $m \geq 1$, $\gamma^{ID}(K_m \square K_m) = \lfloor 3m/2 \rfloor$.*

This result was also reproved in Foucard et al. [4], since as we will exploit, the graph $K_m \square K_m$ is the line graph of $K_{m,m}$. We present here the exact result for the Cartesian product of two general cliques.

Theorem 2.2. *For $2 \leq n \leq m$, we have*

$$\gamma^{ID}(K_n \square K_m) = \begin{cases} m + \lfloor n/2 \rfloor & \text{if } m \leq 3n/2, \\ 2m - n & \text{if } m \geq 3n/2. \end{cases}$$

We prove this theorem in the next two subsections.

2.1 Lower bound

We will need the edge analogue of ID codes. An *edge-ID code* of G is a set D of edges such that for each edge $e \in E(G)$, the subset of edges of D incident with e is nonempty and distinct. A set of edges D is *edge-dominating* if every edge in G is either in D or incident with an element in D .

We will use the fact that $K_n \square K_m$ is the line graph of $K_{n,m}$ by considering edge-ID codes for $K_{n,m}$. Let the partite sets of $K_{n,m}$ be $Y = \{y_1, \dots, y_n\}$ and $X = \{x_1, \dots, x_m\}$.

Lemma 2.3. *If an edge-dominating set D of $K_{n,m}$ is an edge-ID code, then*

(A) *in the spanning subgraph H with edge-set D , $|N_H(S)| \geq 2$ for any set S of two vertices in the same partite set;*

(B) *any set T of two vertices of each partite set is incident with at least three edges of D .*

Proof. (A) Suppose $S = \{x_1, x_2\}$ has $|N_H(S)| \leq 1$. Say $N_H(S) \subseteq \{y_1\}$. Then D does not separate edges x_1y_1 and x_2y_1 , a contradiction.

(B) Suppose $T = \{x_1, x_2, y_1, y_2\}$ is incident with only two edges of D . Then by property A, these two edges form a matching in T . However, then the two edges of T not in D are not separated. \square

Lemma 2.4. *For integers $2 \leq n \leq m$, an edge-ID code D of $K_{n,m}$ satisfies*

$$|D| \geq \max \left\{ 2m - n, m + \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Proof. The result is trivial if $m = 2$; so assume $m > 2$. Let D be a minimum edge-ID code of $K_{n,m}$.

Suppose some vertex of X is disjoint from D , say x_1 . By property A, each x_j for $2 \leq j \leq m$ is incident to at least two edges in D . It follows that $|D| \geq 2m - 2 \geq \max\{2m - n, m + \lfloor n/2 \rfloor\}$, as required. So we may assume every vertex of X is incident with an edge of D . For each $i \in \{1, \dots, m\}$, choose an edge $e_i \in D$ incident to $x_i \in X$, and let $T = \{e_1, \dots, e_m\}$.

Consider any $y_i \in Y$. By property A applied to each pair of vertices in $N_T(y_i)$, it follows that at most one vertex in $N_T(y_i)$ is not incident with an edge of $D - T$. Thus $|D - T| \geq \sum_{i=1}^n (d_T(y_i) - 1) = m - n$, and so $|D| \geq 2m - n$.

It remains to show that $|D| \geq m + \lfloor n/2 \rfloor$. Define

$$Y_1 = \{y \in Y \mid y \text{ is incident to exactly 1 edge in } T\}, \quad \text{and} \\ Y_2 = \{y \in Y \mid y \text{ is incident to at least 2 edges in } T\}.$$

Let $|Y_1| = n_1$ and $|Y_2| = n_2$. There are two cases:

- Suppose some vertex of Y is disjoint from D , say y_1 . By property A applied to the set $\{y_i, y_1\}$ for each $y_i \in Y - Y_2 - y_1$, we can conclude D contains at least $2(n - n_1 - n_2 - 1) + n_1 = 2n - n_1 - 2n_2 - 2$ edges disjoint from T . By property A applied to one pair of vertices

in $N_T(y_i)$ for each $y_i \in Y_2$, it is also true that D contains at least n_2 edges disjoint from T . It follows that $|D|$ is at least

$$|T| + \left\lceil \frac{2n - n_1 - 2n_2 - 2 + n_2}{2} \right\rceil = m + \left\lceil \frac{2n - n_1 - n_2 - 2}{2} \right\rceil \geq m + \left\lfloor \frac{n}{2} \right\rfloor.$$

- Suppose every vertex of Y is incident with D . By Property B applied to $\{y_i, y_j\} \cup N_T(\{y_i, y_j\})$ for each $y_i, y_j \in Y_1$, it follows that at most one vertex of Y_1 is not incident with an edge in $D - T$. At the same time, every vertex in $Y - Y_1 - Y_2$ is incident with an edge of $D - T$. Therefore, D contains at least $(n_1 - 1) + (n - n_1 - n_2) = n - n_2 - 1$ edges disjoint from T . As above, D contains at least n_2 edges disjoint from T . It follows that

$$|D| \geq |T| + \left\lceil \frac{(n - n_2 - 1) + n_2}{2} \right\rceil = m + \left\lfloor \frac{n}{2} \right\rfloor.$$

In any case, an edge-ID code D of $K_{n,m}$ satisfies $|D| \geq \max\{2m - n, m + \lfloor n/2 \rfloor\}$. \square

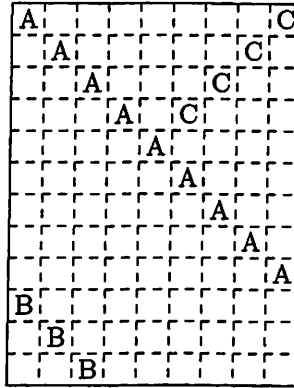
2.2 Construction

We now construct ID codes to show the lower bound is also the upper bound. Let $G = K_n \square K_m$ with $m \geq n$. Gravier et al. [5] constructed ID codes for the case $n = m$. Based on their construction, we define the following sets

$$\begin{aligned} A &= \{(i, i) \mid 1 \leq i \leq n\}, \\ B &= \{(i, n + i) \mid 1 \leq i \leq m - n\}, \\ C &= \{(n - i + 1, i) \mid 1 \leq i \leq \lfloor n/2 \rfloor\}, \text{ and} \\ X &= \{(1, i), (2, i) \mid \lceil (3n + 1)/2 \rceil \leq i \leq m\}. \end{aligned}$$

(Note that $X = \emptyset$ if $m \leq 3n/2$.) Further, define $D = A \cup B \cup C \cup X$. Note that $|D| = n + m - n + \lfloor n/2 \rfloor + m - \lceil (3n + 1)/2 \rceil + 1 = 2m - n$ if $m > 3n/2$, and that $|D| = m + \lfloor n/2 \rfloor$ otherwise.

For example, here is the picture for $n = 9$ and $m = 12$.



Notice that each column and each row of G intersects D , and therefore D dominates G . So let $u = (i, j)$ and $v = (x, y)$ be distinct vertices of G . We need to show that D separates these two vertices.

Start by considering the set A and assume that A does not separate u and v (that is, $N[u] \cap A = N[v] \cap A$), since otherwise we are done. There are two cases.

If $N[u]$ and $N[v]$ contain two vertices of A , then it must be that $i = y$ and $x = j$. But then it is easily seen that u and v are separated by C .

So assume $N[u]$ and $N[v]$ contain exactly one vertex of A . Then it must be that $i = x$, and that $j, y \in \{i\} \cup \{n+1, \dots, m\}$. If $j = i$, then C separates u and v ; so we may assume $y, j > n$.

If $j, y \leq 3n/2$, then u and v are separated by B . On the other hand, if one or both j, y is greater than $3n/2$, then the two vertices are separated by X . Thus D is an ID code, as required.

This concludes the proof of Theorem 2.2.

3 Equal Cliques

We now consider ID codes for the product of multiple copies of equal-sized cliques.

The hypercube is the Cartesian product of K_2 's and there has been considerable research on ID codes in hypercubes. Between the original paper by Karpovsky et al. [7], Exoo et al. [2] and Blass et al. [1], the minimum size of an ID code for the d -dimensional hypercube Q_d is now known for small value of d . These values are given in the table below:

d	3	4	5	6	7
$\gamma^{ID}(Q_d)$	4	7	10	19	32

Between them, it is also shown that γ^{ID} of the d -dimensional hypercube is asymptotically $(2+o(1)) \times 2^d/(d+2)$. (Note that Kim and Kim [8] consider what graph theory calls the Cartesian product of cycles, not hypercubes.)

We start with a simple construction.

Theorem 3.1. *Let $d \geq 3$ and let G be the Cartesian product of d copies of K_m . Then*

$$\gamma^{ID}(G) \leq m^{d-1}.$$

Proof. Define D as the set of vertices in G whose coordinates sum to 0 modulo m . That is, D is the set of all (x_1, x_2, \dots, x_d) with $1 \leq x_i \leq m$ for each $i \in \{1, \dots, m\}$ and $\sum_{i=1}^m x_i$ is a multiple of m . Note that $|D| = m^{d-1}$, since we can freely specify the first $d-1$ coordinates. (In coding theory D is called a *parity code*.)

We claim that D is an independent set. Consider any two vertices v_1 and v_2 in D . If they were adjacent, then v_1 and v_2 would have $d-1$ coordinates in common. However, any $d-1$ coordinates determine the last one. Therefore, D is an independent set.

We claim that D is an ID code. If vertex v is in D , then $N[v] \cap D = \{v\}$. On the other hand, if $u \in V(G) - D$, then u is adjacent to exactly d vertices in D , each of which agrees with u in exactly $d-1$ coordinates. It follows that D certainly separates any two vertices in D , as well as any pair of vertices $u \in V(G) - D$ and $v \in D$.

Thus, we need only to consider pairs of vertices in $V(G) - D$. We claim that $N[u] \cap D$ uniquely determines u for $u \in V(G) - D$. Indeed, we can determine u by simply taking the majority vote in each coordinate. \square

In general, the best lower bound we know is the bound from the original paper by Karpovsky et al. [7]. Namely, they showed that $\gamma^{ID}(G) \geq 2n/(r+2)$ for an r -regular graph G of order n .

Theorem 3.2. [7] *Let G be the Cartesian product of d copies of K_m . Then*

$$\gamma^{ID}(G) \geq \frac{2m^d}{dm - d + 2}.$$

Together with Theorem 3.1, this shows that for the Cartesian product of d copies of K_m , γ^{ID} is $\Theta(m^{d-1})$. We know almost nothing about the actual values. We can, however, improve the lower bound for three cliques, which we consider next.

4 Lower Bound For Three Equal Cliques

We will need some concepts from domination. We say that a set D *open-dominates* vertex x if $N(x) \cap D \neq \emptyset$. We say that set D is a *total dominating set* if it open-dominates every vertex of G . The minimum cardinality of a total dominating set is denoted $\gamma_t(G)$.

Consider the graph $G \square K_m$ and let D be a subset of the vertices. We will use the notation G_i to denote the i th copy of G (that is, all vertices of the form $(?, i)$), and D_i to be the subset of D in G_i . We let X_i denote all vertices in G_i that do not have a neighbor in D_i . (That is, X_i is the vertices of G_i that are not open-dominated by D_i .) Finally, let \hat{X}_i be the subset of $V(G)$ corresponding to X_i (the projection of X_i).

Lemma 4.1. *Let D be an ID code of $G \square K_m$ and let X_i be defined as above. Then the \hat{X}_i are disjoint.*

Proof. Suppose that the \hat{X}_i are not disjoint. Say $v \in \hat{X}_i \cap \hat{X}_j$. That is, $(v, i) \in X_i$ and $(v, j) \in X_j$. Then these two vertices are not separated by D , since the intersection of each of their closed neighborhoods with D is precisely the set of vertices of D in the v^{th} copy of K_m . The result follows. \square

From this lemma we obtain a lower bound for the product of three cliques:

Theorem 4.2. $\gamma^{ID}(K_m \square K_m \square K_m) \geq m^2 - o(m^2)$.

Proof. Let $G = K_m \square K_m$ and let D be an ID code of $G \square K_m$. Define D_i and X_i as in Lemma 4.1. From that lemma we have that $\sum_{i=1}^m |X_i| \leq m^2$.

On the other hand, if $|D_i| < m$ then there are at least $m - |D_i|$ rows and columns in G_i not containing a vertex of D_i , and so $|X_i| \geq (m - |D_i|)^2$. It follows that $|D_i| \geq m - \sqrt{|X_i|}$. Hence

$$|D| = \sum_{i=1}^m |D_i| \geq m^2 - \sum_{i=1}^m \sqrt{|X_i|}.$$

To bound $|D|$ from below, we need to maximize $z = \sum_{i=1}^m \sqrt{|X_i|}$, subject to the constraint $\sum_{i=1}^m |X_i| \leq m^2$. Using the fact that $\sum_{i=1}^m \sqrt{|X_i|}$ is concave, the method of Lagrange for nonlinear programs shows that the optimal value is $z^* = m\sqrt{m}$. Thus, $|D| \geq m^2 - m\sqrt{m}$. \square

For $m = 2$ we know from [7] that $\gamma^{ID}(K_2 \square K_2 \square K_2) = \gamma^{ID}(Q_3) = 4$. When $m = 3$, an exhaustive search by computer shows that $\gamma^{ID}(K_3 \square K_3 \square K_3) = 9$. We conjecture that this pattern continues:

Conjecture 4.3. For all $m \geq 1$, $\gamma^{ID}(K_m \square K_m \square K_m) = m^2$.

5 Unequal Cliques

We will need another concept from domination. A set D *doubly dominates* vertex x if $|N[x] \cap D| \geq 2$. We start with a partial converse to Lemma 4.1.

Lemma 5.1. Let D be a subset of the vertices of $G \square K_m$, and X_i defined as in Lemma 4.1. If every vertex is doubly dominated from within its copy of K_m , and the projections \hat{X}_i are disjoint, then D is an ID code.

Proof. The idea is that its neighbors in the copy of K_m determine which copy of K_m a vertex is in, and its neighbors (or lack thereof) in G_i determine which G_i it is in.

To be precise, let (u, i) and (v, j) be two vertices in $G \square K_m$. If $u \neq v$, then by the hypothesis (u, i) is dominated by at least two vertices of D within its copy of K_m , say (u, k) and (u, ℓ) (where possibly $k, \ell \in \{i, j\}$). Since (v, j) is adjacent to at most one of these vertices, D separates (u, i) and (v, j) .

So assume $u = v$. By the hypothesis, it cannot be that both $(u, i) \in X_i$ and $(u, j) \in X_j$. So say $(u, i) \notin X_i$; that is, (u, i) is dominated by some (w, i) in D ; this vertex separates (u, i) from (u, j) . Thus D is an ID code. \square

We can use Lemmas 4.1 and 5.1 to give approximate results for the minimum size of an ID code for the product of cliques where one clique is very large.

Theorem 5.2. For any isolate-free graph G of order n , and integer $m \geq 3$,

$$m \gamma_t(G) - n \leq \gamma^{ID}(G \square K_m) \leq m \gamma_t(G) + 2n - 3\gamma_t(G).$$

Proof. To prove the upper bound, construct set D as follows. Take all the vertices in two copies of G , and take a minimum total dominating set in each of the remaining copies of G except one. Then in the terminology of Lemma 5.1, each vertex is doubly dominated in its copy of K_m , and each X_i is empty except one, so D is an ID code.

To prove the lower bound, let D be an ID code and let D_i be the intersection of D with the i^{th} copy G_i of G . Since we can form a total

dominating set of G_i by adding to D_i one neighbor of each vertex in X_i , it follows that $|D_i| \geq \gamma_t(G) - |X_i|$.

On the other hand, it follows from Lemma 4.1 that $\sum_{i=1}^m |X_i| \leq n$. Thus,

$$|D| = \sum_{i=1}^m |D_i| \geq m \gamma_t(G) - \sum_{i=1}^m |X_i| \geq m \gamma_t(G) - n,$$

as required. □

Corollary 5.3. *For any fixed isolate-free graph G of order n , $\gamma^{ID}(G \square K_m) = m \gamma_t(G) \pm O(n)$.*

For us, one obvious example is the case where G is the product of equal cliques. But the values of $\gamma_t(G)$ do not appear to be known in general for three cliques or more. If one of the cliques is sufficiently large, then it is easy to determine the total domination number of the product of cliques. Indeed, we conclude with the following exact result.

Theorem 5.4. *Let H be an isolate-free connected graph of order n . If $m > 2\ell$ and $\ell > 2n$, then*

$$\gamma^{ID}(H \square K_\ell \square K_m) = n(m - 1).$$

Proof. Let $G = H \square K_\ell$ and $F = G \square K_m$. Define $i \bmod \ell$ to be the unique integer s in the range 1 to ℓ such that $i - s$ is a multiple of ℓ . Let D be the set of vertices of F given by

$$D = \{(v, i \bmod \ell, i) \mid v \in V(H), 1 \leq i \leq m - 1\}.$$

Note that $|D| = n(m - 1)$.

We claim that D satisfies the hypotheses of Lemma 5.1. To see this, consider any vertex (v, j, i) in F . Since $m > 2\ell$, there are at least two values $1 \leq i_1, i_2 \leq m - 1$ such that $i_1 \bmod \ell = i_2 \bmod \ell = j$. Thus (v, j, i) is doubly dominated within its copy of K_m . Furthermore, assume $i \neq m$. Then the vertex $(v, i \bmod \ell, i)$ is in D . So if $j \neq i \bmod \ell$, the vertex (v, j, i) has a neighbor in D in its G_i . On the other hand, if $j = i \bmod \ell$, then the vertex (v, j, i) has some neighbor (w, j, i) in D , since H is isolate-free. It follows that all the X_i except X_m are empty, and so the projections \hat{X}_i are disjoint. Thus by Lemma 5.1, the set D is an ID code.

To prove the lower bound, we use Lemma 4.1. Let D be a minimum set of vertices of F such that the \hat{X}_i are disjoint. We can assume that we take no more than n vertices from each copy G_i , since taking n vertices is enough to ensure that X_i is empty.

Consider any G_i and let $A_i(v)$ be the copy of K_ℓ whose vertices are of the form $(v, ?, i)$. Assume that $A_i(v)$ does not contain a vertex of D . Then since G_i contains at most n vertices of D , it follows that at least $\ell - n$ vertices of $A_i(v)$ are in X_i . Since $\ell > 2n$, this means that more than half of $A_i(v)$ is in X_i .

Since the \hat{X}_i are disjoint, it follows that for every v , at most one of the $A_i(v)$ does not contain a vertex of D . That is, at least $mn - n$ of the $A_i(v)$ do contain a vertex of D , and so $|D| \geq mn - n$. \square

We suspect that the conclusion of the above theorem is true for a much wider range of clique-sizes.

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