# Hamiltonian Properties of Independent Domination Critical Graphs

S. Ao, G. MacGillivray and J. Simmons

Department of Mathematics and Statistics
University of Victoria
P. O. Box 3060 STN CSC
Victoria, B.C., Canada, V8W 3R4

**Abstract**

A graph $G$ is $k$-edge-$i$-critical if it has independent domination number $i(G) = k$, and $i(G + xy) < i(G)$ whenever $xy \notin E(G)$. The following results are obtained for 3-edge-$i$-critical graphs $G$: (a) If $\delta \geq 3$, then $G$ is hamiltonian (b) If $\delta = 2$, then there is exactly one family of non-hamiltonian graphs; and (c) If $|V(G)| > 6$, then $G$ has a Hamilton path. The proofs of these results rely on a closure operation, a characterisation of the 2-connected, 3-edge-$i$-critical graphs with $\delta = 2$, and a characterisation of the 3-edge-$i$-critical graphs with a cut vertex.

*Dedicated to our friend and colleague Kieka Mynhardt. Our department is better because she is in it.*

## 1 Introduction

We consider only finite simple graphs. The vertex set and edge set of a simple graph $G$ will be denoted $V(G)$ and $E(G)$, respectively, or simply $V$ and $E$ when $G$ is understood from context. For basic graph theoretic definitions, the reader is referred to West [9].

For subsets $X, Y \subseteq V(G)$, we say that $X$ dominates $Y$ if every vertex in $Y$ is either in $X$ or adjacent to a vertex in $X$. For adjacent vertices

$x \in X$ and $y \in Y$, we also say that $x$ dominates $y$. If $Y = V(G)$, then $X$ is called a *dominating set* of $G$. The minimum cardinality of a dominating set of $G$ is the *domination number*, $\gamma(G)$. An *independent set* is a set of pairwise nonadjacent vertices, and the *independence number*, $\beta(G)$, is the maximum cardinality of an independent set. A dominating set which is also an independent set is an *independent dominating set*. The minimum size of such a set is the *independent domination number*, $i(G)$. For ease of notation, $\gamma$, $i$, and $\beta$ will be used in place of $\gamma(G)$, $i(G)$, and $\beta(G)$ when the graph $G$ under discussion is clear from context.

In general, if $e \notin E(G)$, then $\gamma(G + e) \leq \gamma(G)$. A graph $G$ is *$k$-edge-$\gamma$-critical* if $\gamma(G) = k$ and, for every edge $e \notin E(G)$, $\gamma(G + e) < k$. We say $G$ is *edge-$\gamma$-critical* if there exists $k$ such that $G$ is $k$-edge-$\gamma$-critical.

On the other hand, if $e \notin E(G)$, then it is possible for $i(G + e)$ to be greater than, less than, or equal to $i(G)$. A graph $G$ is *$k$-edge-$i$-critical* if $i(G) = k$ and, for every edge $e \notin E(G)$, $i(G+e) < k$, and $G$ is *edge-$i$-critical* if there exists $k$ such that $G$ is $k$-edge-$i$-critical.

Various notions of criticality of dominating sets are compared and contrasted in [1, 4, 7]. There seems to be no general relationship between changes in $\gamma(G)$ and changes in $i(G)$ resulting from joining a pair of non-adjacent vertices of $G$.

Sumner [5, 6] conjectured that every connected 3-edge-$\gamma$-critical graph on more than 6 vertices has a Hamilton path, and this was proven by Wojcicka in [10]. Sumner and Wojcicka (mentioned in [10]) also conjectured that every 2-connected 3-edge-$\gamma$-critical graph with minimum degree $\delta \geq 2$ has a Hamilton cycle. Hanson [3] developed an approach based on a domination closure and verified the conjecture in some circumstances. Later, in [2], it was proven that every 2-connected 3-edge-$\gamma$-critical graph with $\delta \geq 2$ satisfies $\beta \leq \delta + 2$, and is hamiltionian when $\beta \leq \delta + 1$. The conjecture was verified when, in [8], it was shown that the 2-connected 3-edge-$\gamma$-critical graph with $\delta \geq 2$ and $\beta = \delta + 2$ are also hamiltonian.

Using a closure similar to the one developed by Hanson, we show that every 2-connected 3-edge-$i$-critical graph with $\delta \geq 3$ has a Hamilton cycle. We also characterise the 2-connected, 3-edge-$i$-critical graphs with $\delta = 2$, and determine which of these are hamiltonian. By combining these results with a complete characterisation of the 3-edge-$i$-critical graphs with a cut-vertex, we establish that any connected 3-edge-$i$-critical graph with more than six vertices has a Hamilton path.

## 2 Minimum degree at least three

In this section we prove that every 2-connected, 3-edge-$i$-critical graph with $\delta \geq 3$ is hamiltonian. The main tool is a domination closure similar to [3].

Let $G$ be a 3-edge-$i$-critical graph. If $uv \notin E(G)$, then $i(G+uv) = 2$ implies that there exists a vertex $x \notin N(u) \cup N(v)$ such that either $\{u,x\}$ dominates $V(G) - v$ or $\{v,x\}$ dominates $V(G) - u$. In the former case we write $[u,x] \to v$, and in the latter case we write $[v,x] \to u$.

**Theorem 2.1** *Suppose that $G$ is 2-connected. If $[u,v] \to w$ for some vertices $u, v$ and $w$ with $d(w) \geq 3$, then $G$ is hamiltonian if and only if $G+uv$ is hamiltonian.*

**Proof:** First note that if $G$ is hamiltonian then $G+uv$ is obviously also hamiltonian.

Now consider vertices $u, v$, and $w$ such that $[u,v] \to w$ and $d(w) \geq 3$. Suppose $G+uv$ is hamiltonian while $G$ is not. Then, $G$ contains a Hamilton path $v_1v_2\ldots v_n$ from $u = v_1$ to $v = v_n$ where $n = |V|$, and $N[v_1] \cup N[v_n] = V - \{v_p\}$ with $v_p = w$. Define $M = max\{i : v_1v_i \in E\}$ and $m = min\{j : v_jv_n \in E\}$. If $v_1v_i \in E$, then $v_nv_{i-1} \notin E$, otherwise $v_1v_2\ldots v_{i-1}v_nv_{n-1}\ldots v_iv_1$ is a Hamilton cycle in $G$. Therefore neither the case $p < m < M$ nor the case $m < M < p$ is possible. The remaining cases are $M \leq m < p$ (or $p < M \leq m$), $M < p < m$, and $m < p < M$.

**Case 1:** $\underline{M \leq m < p.}$

Since $[v_1, v_n] \to v_p$, $v_1$ dominates $\{v_1, v_2, \ldots, v_M\}$, $v_n$ dominates $\{v_m, v_{m+1}, \ldots, v_n\} - \{v_p\}$, and therefore $M \leq m \leq M+1$.

We will first prove that $v_iv_j \notin E$ for all $i$ and $j$, where $1 \leq i < M$ and $m < j \leq n$. Consider $i$ and $j$ such that $1 \leq i < M$ and $m < j \leq n$. If $v_iv_j \in E$, certainly $i \neq 1$ and $j \neq n$. If $j - 1 \neq p$, then

$$v_1v_2\ldots v_iv_jv_{j+1}\ldots v_nv_{j-1}v_{j-2}\ldots v_{i+1}v_1$$

is a Hamilton cycle in $G$. Hence, assume $j - 1 = p$, that is, $j = p+1$. We will obtain a contradiction by showing that $v_p$ can have no neighbours other than $v_{p-1} = v_j$ and $v_{p+1}$. Suppose there exists $k \notin \{p-1, p+1\}$ such that $v_kv_p \in E$. If $1 \leq k \leq M-1$, then

$$v_1v_2\ldots v_kv_pv_{p+1}\ldots v_nv_{p-1}v_{p-2}\ldots v_{k+1}v_1$$

is a Hamilton cycle in $G$. If $M \leq k \leq p-2$, then

$$v_1v_2\ldots v_iv_{p+1}v_{p+2}\ldots v_nv_{k+1}v_{k+2}\ldots v_pv_kv_{k-1}\ldots v_{i+1}v_1$$

is a Hamilton cycle in $G$. Finally, if $p+2 \le k \le n$, then

$$v_1 v_2 \dots v_i v_{p+1} v_{p+2} \dots v_{k-1} v_n v_{n-1} \dots v_k v_p v_{p-1} \dots v_{i+1} v_1$$

is a Hamilton cycle in $G$. Hence $d(v_p) = 2$, a contradiction. Therefore $v_i v_j \notin E$ for all $i$ and $j$ where $1 \le i < M$ and $m < j \le n$.

Now, since $G$ is 2-connected, $m \ne M$. Thus, $m = M+1$. Since $v_M$ and $v_m$ are not cut-vertices, there must exist $i$ and $j$ with $1 \le i < M < m < j \le n$, such that $v_i v_m \in E$ and $v_M v_j \in E$. If $j-1 \ne p$, then $G$ contains the Hamilton cycle

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_{j-1} v_n v_{n-1} \dots v_j v_M v_{M-1} \dots v_{i+1} v_1.$$

Hence, assume $j-1 = p$. We will now obtain a contradiction by showing that $v_p$ can have no neighbours other than $v_{p-1}$ and $v_{p+1} = v_j$. Suppose $v_k v_p \in E$ for some $k \notin \{p-1, p+1\}$. If $1 \le k \le M-1$, then

$$v_1 v_2 \dots v_k v_p v_{p+1} \dots v_n v_{p-1} v_{p-2} \dots v_{k+1} v_1$$

is a Hamilton cycle in $G$. If $k = M$, then $G$ contains the Hamilton cycle

$$v_1 v_{i+1} v_{i+1} \dots v_M v_p v_{p+1} \dots v_n v_{p-1} v_{p-2} \dots v_m v_i v_{i-1} \dots v_1.$$

If $m \le k \le p-2$, then

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_k v_p v_{p-1} \dots v_{k+1} v_n v_{n-1} \dots v_{p+1} v_M v_{M-1} \dots v_{i+1} v_1$$

is a Hamilton cycle in $G$. Finally, if $p+2 \le k \le n$, then

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_p v_k v_{k+1} \dots v_n v_{k-1} v_{k-2} \dots v_{p+1} v_M v_{M-1} \dots v_{i+1} v_1$$

is a Hamilton cycle in $G$. It follows that $d(v_p) = 2$, a contradiction.

Since the case where $p < M \le m$ is symmetrical to the above, the proof of Case 1 is complete.

**Case 2:** $\underline{M < p < m.}$

In this case we must have $m = M+2$. Also, $v_i v_j \notin E$ for all $i$ and $j$ such that $1 \le i < M < m < j \le n$, otherwise

$$v_1 v_2 \dots v_i v_j v_{j+1} \dots v_n v_{j-1} v_{j-2} \dots v_{i+1} v_1$$

is a Hamilton cycle in $G$. Since $d(v_p) \ge 3$, by symmetry we can assume that $v_k v_p \in E$ for some $k$ with $1 < k < M$.

Suppose $v_p v_q \notin E$ for all $q$ such that $m < q < n$. Then since $v_m$ is not a cut-vertex, $v_M v_j \in E$ for some $j$ with $m < j < n$. However, this implies

$$v_1 v_2 \dots v_k v_p v_m v_{m+1} \dots v_{j-1} v_n v_{n-1} \dots v_j v_M v_{M-1} \dots v_{k+1} v_1$$

is a Hamilton cycle in $G$, a contradiction. Therefore, $v_p v_q \in E$ for some $q$ with $m < q < n$.

Since $v_p$ is not a cut-vertex, $v_i v_m \in E$ for some $i$ with $1 < i \leq M$. But now, if $i < M$ then

$$v_1 v_2 \dots v_i v_m v_{m+1} \dots v_{q-1} v_n v_{n-1} \dots v_q v_p v_{p-1} \dots v_{i+1} v_1$$

is a Hamilton cycle of $G$, and if $i = M$ then

$$v_1 v_2 \dots v_k v_p v_q v_{q+1} \dots v_n v_{q-1} v_{q-2} \dots v_m v_M v_{M-1} \dots v_{k+1} v_1$$

is a Hamilton cycle of $G$, a contradiction.

**Case 3:** $\underline{m < p < M.}$

In this case we must have $N(v_1) \supseteq \{v_2, v_3, \dots, v_{m-1}, v_{p+1}, v_{p+2}, \dots, v_M\}$, and $N(v_n) \supseteq \{v_m, v_{m+1}, \dots, v_{p-1}, v_{M+1}, v_{M+2}, \dots, v_{n-1}\}$.

As in the previous cases, we obtain a contradiction by showing that $v_p$ can have no neighbours other than $v_{p-1}$ and $v_{p+1}$. Suppose there exists $k \notin \{v_{p-1}, v_{p+1}\}$ such that $v_k v_p \in E$. If $1 \leq k \leq m-2$, then

$$v_1 v_2 \dots v_k v_p v_{p+1} \dots v_n v_{p-1} v_{p-2} \dots v_{k+1} v_1$$

is a Hamilton cycle in $G$. If $m \leq k \leq p-2$, then

$$v_1 v_2 \dots v_k v_p v_{p-1} \dots v_{k+1} v_n v_{n-1} \dots v_{p+1} v_1$$

is a Hamilton cycle in $G$. If $k = m-1$, then

$$v_1 v_2 \dots v_{m-1} v_p v_{p-1} \dots v_m v_n v_{n-1} \dots v_{p+1} v_1$$

is a Hamilton cycle in $G$. By symmetry, if $p+2 \leq k \leq n$, then $G$ contains a Hamilton cycle. Hence $d(v_p) = 2$, a contradiction.

Since all cases lead to a contradiction, we conclude that if $G + uv$ is hamiltonian, then $G$ is hamiltonian. This completes the proof. ∎

For a 2-connected graph $G$, we define the *domination closure* of $G$, denoted $D^*(G)$, to be the graph with vertex set $V(G)$ and edge set $E(G) \cup \{uv : \exists\, w \in V(G)$ where $d(w) \geq 3$ and $[u,v] \to w\}$.

**Corollary 2.2** *If $G$ is 2-connected, then $D^*(G)$ is hamiltonian if and only if $G$ is hamiltonian.*

**Proof:** If $G$ is hamiltonian, then certainly $D^*(G)$ is hamiltonian.

Suppose the converse is false and choose a minimal subset $\{e_1, e_2, \ldots, e_k\}$ of $E(D^*(G)) - E(G)$ such that $G + \{e_1, e_2, \ldots, e_k\}$ has a Hamilton cycle but $G' = G + \{e_1, e_2, \ldots, e_{k-1}\}$ does not. By Theorem 2.1, $k \geq 2$. Let $e_k = xy$. Then $G'$ has a Hamilton path $P = v_1 v_2 \ldots v_n$, where $x = v_1$ and $y = v_n$.

Since $e_k = xy$, $[x, y] \to w$ for some $w$ with $d(w) \geq 3$ in $G$. By the minimality of $\{e_1, e_2, \ldots, e_k\}$, all edges $e_1, e_2, \ldots, e_{k-1}$ must be in $P$. If neither $xw$ nor $yw$ are edges in $P$, then $[x, y] \to w$ in $G'$ (as well as in $G$). But $G'$ is 2-connected (since $G$ is a subgraph of $G'$), and $G' + xy$ is hamiltonain, so Theorem 2.1 gives that $G'$ is hamiltonian, a contradiction.

Therefore, without loss of generality, suppose $xw$ is in $P$, that is, $w = v_2$ and $[x, w] \to y$. Consider the path $w = v_2 v_3 \ldots v_n = y$ in $G'$. If there exists $k$ where $3 \leq k \leq n$ and $v_1 v_k, v_2 v_{k+1} \in E(G)$, then

$$v_2 v_{k+1} v_{k+2} \ldots v_n v_1 v_k v_{k-1} \ldots v_2$$

is a Hamilton cycle in $G + \{e_2, e_3, \ldots, e_k\}$, a contradiction. Therefore, since $[x, w] \to y$, there exists $p$ such that $N_{G'}(v_2) = \{v_3, v_4, \ldots, v_p\} \cup \{v_1\}$ and $N_{G'}(v_1) = \{v_{p+1}, v_{p+2}, \ldots, v_n\} \cup \{v_2\}$. But $[x, y] \to w$ gives $v_n v_p \in E(G)$, and hence

$$v_1 v_{p+1} v_{p+2} \ldots v_n v_p v_{p-1} \ldots v_1$$

is a Hamilton cycle in $G'$, a contradiction. The result now follows. ∎

We write $d^*(x)$ for the degree of vertex $x$ in $D^*(G)$.

**Theorem 2.3** *If $G$ is a 2-connected, 3-edge-i-critical graph with $\delta(G) \geq 3$, then $D^*(G)$ is hamiltonian.*

**Proof:** Let $w$ be any vertex of $G$, and let $\overline{N}(w) = V - N[w]$. Define the sets $A_w$ and $B_w$ by

$$A_w = \{x \in \overline{N}(w) : \exists y \in \overline{N}(w) \text{ s.t. } [x, y] \to w\}, \text{ and}$$
$$B_w = \overline{N}(w) - A_w.$$

Since $G$ is 3-edge-$i$-critical, and by the definition of $A_w$ and $B_w$, for any $x \in B_w$ there exists $y$ such that, in $G$, $[w, y] \to x$. This implies that $y$

dominates every vertex in $A_w$, so $y \in B_w$. Hence, for each $x \in B_w$ there is a $y \in B_w$ so that $[w, y] \to x$.

Now consider distinct vertices $x_1$ and $x_2$ in $B_w$. There exist $y_1$ and $y_2$ in $B_w$ so that $[w, y_1] \to x_1$ and $[w, y_2] \to x_2$. If $y_1 = y_2$, then we must have $x_1 = x_2$, a contradiction. Thus, the mapping $f$ on $B_w$ defined by $f(x) = y$ if $[w, y] \to x$ is one-to-one and onto.

¿From the above argument and the definition of $A_w$, it follows that each vertex of $\overline{N}(w)$ is incident with an edge of $D^*(G)$ which is not an edge of $G$. We call such an edge a *new edge* with respect to $w$.

Suppose $[a, c] \to b$, so that $ac$ is a new edge with respect to $b$. Then it is also a new edge with respect to each of $a$ (since $c \in \overline{N}(a)$) and $c$ (since $a \in \overline{N}(c)$). It is not a new edge with respect to any other vertex, since $\{a, c\} \subseteq \overline{N}(w)$ if and only if $w \in \{a, b, c\}$.

Each vertex $x$ is in $\overline{N}(w)$ for $|\overline{N}(x)| = |V| - |N[x]| = |V| - d(x) - 1$ vertices $w$. Each of these choices for $w$ leads to a new edge incident with $x$, and each new edge arises from exactly two choices of $w$ (if $xy$ is a new edge with respect to $z$, then the set $\{z, x, y\}$ is independent and dominating in $G$, and since $[x, y] \to w$, this is possible only if $w = y$ or $w = z$). Thus,

$$d^*(x) \geq d(x) + |\overline{N}(x)|/2 = d(x) + |V|/2 - d(x)/2 - 1/2 \geq |V|/2.$$

By Dirac's theorem [9], which states that a graph is hamiltonian if $\delta \geq |V|/2$, $D^*(G)$ is hamiltonian. ∎

**Corollary 2.4** *If $G$ is 2-connected and 3-edge-$i$-critical with $\delta \geq 3$, then $G$ is hamiltonian.*

**Proof:** By Theorem 2.3, $D^*(G)$ is hamiltonian. Thus, by Corollary 2.2, $G$ is also hamiltonian. ∎

# 3 Characterisations

In this section we give a complete description of the 3-edge-$i$-critical graphs with a cut-vertex, and a complete description of the 2-connected, 3-edge-$i$-critical graphs with $\delta = 2$.

The following three lemmata are stated without proof, as they are similar to work found in [6].

**Lemma 3.1** *Let $G$ be a 3-edge-i-critical graph and $I$ be an independent set in $G$ with size $n \geq 4$. Then, the vertices in $I$ may be ordered as $u_1, u_2, \ldots, u_n$ in such a way that there exists a path $x_1 x_2 \ldots x_{n-1}$ in $G - I$ with $[u_i, x_i] \to u_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

**Lemma 3.2** *If $G$ is 3-edge-i-critical, then no vertex of $G$ has two neighbours of degree one.*

**Lemma 3.3** *If $G$ is 3-edge-i-critical and $S \subseteq V(G)$ is a vertex cut, then $G - S$ has at most $|S| + 1$ components.*

**Lemma 3.4** *Let $G$ be a connected, 3-edge-i-critical graph. If $v$ is a cut-vertex of $G$, then $G - v$ has exactly two components $C_1$ and $C_2$, with $C_1$ complete and $i(C_2) = 2$.*

**Proof:** By Lemma 3.3, $G - v$ has exactly two components, $C_1$ and $C_2$. For $i = 1, 2$, let $x_i \in V(C_i)$ be adjacent to $v$. Since $x_1 x_2 \notin E$, there exists $x$ such that $[x_1, x] \to x_2$ or $[x, x_2] \to x_1$. In either case, the choice of $x_1$ and $x_2$ implies that $\{x, x_1, x_2\}$ is an independent dominating set not containing $v$. Without loss of generality, $x \in V(C_2)$, so that $i(C_1) = 1$ and $i(C_2) \leq 2$. Suppose there exists a vertex $z \in V(C_2)$ that dominates $C_2$. Then, since $x_1$ dominates $V(C_1) \cup \{v\}$ and $x_1 x \notin E$, the set $\{x_1, z\}$ is an independent dominating set of $G$ with size two, a contradiction. This gives $i(C_2) = 2$.

If $|V(C_1)| = 1$, certainly $C_1$ is complete. Otherwise, suppose $x, y \in V(C_1)$ and $xy \notin E$. Then there exists $z$ such that (in $G$) either $[x, z] \to y$ or $[y, z] \to x$. Since $i(C_2) = 2$, $z = v$ and $v$ dominates $C_2$. Now, for any $u \in V(C_2)$, since $xu \notin E$, there exists $w$ such that either $[x, w] \to u$ or $[u, w] \to x$. Since $i(C_2) = 2$ and $v$ dominates $C_2$, $w \in V(C_2)$. But then $y$ can not be dominated by $\{x, w, u\}$, a contradiction. Therefore $C_1$ is complete. ∎

Let $G$ and $H$ be disjoint graphs. The *join* of $G$ and $H$ is the graph $G + H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{gh : g \in V(G), h \in V(H)\}$. The join of $n \geq 3$ vertex-disjoint graphs $G_1, G_2, \ldots, G_n$ is recursively defined to be the graph $G_1 + G_2 + \cdots + G_n = (G_1 + G_2 + \cdots + G_{n-1}) + G_n$.

Let $\mathcal{Q}_{n,p}$ $(p \geq n)$ denote the set of graphs $Q_{n,p}$ on $n + p$ vertices that can be obtained from $K_p \cup \overline{K}_n$ by adding, for each vertex $v \in K_p$, an edge from $v$ to any one vertex in $\overline{K}_n$, such that $Q_{n,p}$ has no isolated vertices.

We use $T_{n,rn}$ to denote the complete $n$-partite graph in which each part contains $r$ vertices.

For distinct vertices $u$ and $v$, we use $N(u) \oplus N(v)$ to denote the symmetric difference of the neighbourhoods of $u$ and $v$.

**Theorem 3.5** *Let $G$ be a connected 3-edge-i-critical graph, and let $v$ be a cut-vertex of $G$. Then, $G - v$ has exactly two components $C_1$ and $C_2$ such that $C_1$ is complete and $i(C_2) = 2$. Furthermore, if $|V(C_1)| \geq 2$, then the following hold:*

1. *$C_2 = T_{n,2n}$ for some $n \geq 2$, and*
2. *for every $x \in V(C_1)$, $vx \in E$, and for any pair $u, u'$ of non-adjacent vertices of $C_2$, $vu \in E$ if and only if $vu' \in E$.*

**Proof:** Since $v$ is a cut vertex of $G$, by Lemma 3.4, $G - v$ has two components $C_1$ and $C_2$ with $C_1$ complete and $i(C_2) = 2$.

We first prove that $C_2 = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2 = T_{n,2n}$. Let $x \in V(C_1)$ be adjacent to $v$. For every $u \in V(C_2)$, since $xu \notin E$, there exists $u'$ such that either $[u, u'] \to x$ or $[x, u'] \to u$. Since $C_1$ is complete and $xv \in E$, $u' \in V(C_2)$, and since $|V(C_1)| \geq 2$, $[x, u'] \to u$. So, $N[u'] \supseteq V(C_2) - \{u\}$. Now, since $xu' \notin E$, there exists $w$ such that $[x, w] \to u'$ or $[u', w] \to x$. As above, $[x, w] \to u'$ and $w \in V(C_2)$, so $w = u$ and $N[u] \supseteq V(C_2) - \{u'\}$. Therefore, for every $u \in V(C_2)$, there exists $u' \in V(C_2)$ such that $u$ and $u'$ are both adjacent to every vertex of $V(C_2) - \{u, u'\}$. Hence $C_2 = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2 = T_{n,2n}$ for some $n \geq 1$.

To prove statement 2, consider $u \in V(C_2)$ with $vu \notin E$. Then there exists $y$ such that either $[u, y] \to v$ or $[v, y] \to u$. If $[u, y] \to v$, then $y \in V(C_1)$, and hence $u$ dominates $C_2$, a contradiction. Thus $[v, y] \to u$. If $y \in V(C_1)$, then $yv \notin E$ and there exists $w$ such that $[y, w] \to v$ or $[v, w] \to y$. In the first case, $w \in V(C_2)$ and $w$ dominates $C_2$, a contradiction. In the second case, since $C_1$ is complete and $[v, y] \to u$, $w \in V(C_2)$, and $w = u$. Now, for the unique non-neighbour of $u$ in $C_2$, $u'$, since $yu' \notin E$, there exists $z$ such that either $[z, y] \to u'$ or $[z, u'] \to y$. In both cases, $z = u$ (as $y$ dominates $C_1$, $u'$ dominates $C_2 - u$, and $u'v \in E$). But, since $yu, vy \notin E$, it is not possible that $[u, y] \to u'$. Therefore, $[u, u'] \to y$ and $V(C_1) = \{y\}$, contrary to $|V(C_1)| \geq 2$. Hence $y \in V(C_2)$. Since $u'$ is the only vertex of $C_2$ not adjacent to $u$, $y = u'$ and $u'v \notin E$. Thus $v$ is adjacent to either both $u$ and $u'$, or neither $u$ nor $u'$.

Suppose $v$ dominates $V(C_2)$. Then either $\{v\}$ is an independent dominating set of size 1 in $G$, or there exists $x \in V(C_1)$ such that $\{x, v\}$ is an independent dominating set of size 2 in $G$. Both cases contradict $i(G) = 3$, and therefore there exists a pair of vertices $u$ and $u'$ in $V(C_2)$ such that

$[u,v] \to u'$. It follows that $vx \in E$ for every $x \in V(C_1)$, completing the proof of statement 2. Furthermore, since $v$ does not dominate $C_2$, and $G$ is connected, it follows that $C_2 = T_{n,2n}$ for some $n \geq 2$, completing the proof of statement 1. ∎

**Corollary 3.6** *Let $G$ be a connected 3-edge-i-critical graph, and let $v$ be a cut-vertex of $G$. Let $C_1$ and $C_2$ denote the two components of $G - v$, where $C_1$ is complete and $i(C_2) = 2$. Then $|V(C_1)| \geq 2$ if and only if $\delta(G) \geq 2$.*

**Proof:** If $\delta(G) \geq 2$, certainly $|V(C_1)| \geq 2$. Conversely, suppose $|V(C_1)| \geq 2$. Then, by Theorem 3.5, $C_2 = T_{n,2n}$ for $n \geq 2$, and hence $d(x) \geq 2$ for all $x \in V(C_2)$. Also by Theorem 3.5, $vx \in E$ for every $x \in V(C_1)$, and hence $d(x) \geq 2$. Furthermore, $d(v) \geq 2$. Therefore $\delta(G) \geq 2$. ∎

**Theorem 3.7** *Let $G$ be a connected 3-edge-i-critical graph with $\delta = 1$, and let $v$ be a cut-vertex of $G$. Then, $G - v$ has exactly two components $C_1$ and $C_2$ such that $C_1$ is complete and $i(C_2) = 2$. Furthermore,*

1. *$C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_2$ or a graph $Q_{2,p} \in \mathcal{Q}_{2,p}$, $1 \leq j \leq m$, and*
2. *there exist non-adjacent vertices $u, u' \in V(C_2) - N[v]$ such that*
   (a) *$N[u] \cup N[u'] = V(C_2)$, and*
   (b) *for all $z \in N(u) \oplus N(u')$, $vz \in E$ and $N[z] \supseteq V(C_2) - \{u, u'\}$.*

**Proof:** Since $v$ is a cut-vertex of $G$, by Lemma 3.4, $G - v$ has exactly two components $C_1$ and $C_2$, with $C_1$ complete and $i(C_2) = 2$. Furthermore, by Corollary 3.6, $|V(C_1)| = 1$.

Let $V(C_1) = \{x\}$. Then, since $G$ is connected, $vx \in E$. Since $i(G) = 3$, there exists $u \in V(C_2)$ with $vu \notin E$, and $u'$ such that $[v,u'] \to u$ or $[u,u'] \to v$. But $x \notin N[u] \cup N[u']$, so it must be that $[v,u'] \to u$, where $u' \in V(C_2)$. Furthermore, the fact that $vu' \notin E$ implies there exists $y$ such that $[v,y] \to u'$ or $[u',y] \to v$. By the same reasoning, $[v,y] \to u'$, where $y \in V(C_2)$. Now, since $[v,u'] \to u$, it must be that $y = u$. Hence $[v,u] \to u'$ and $[v,u'] \to u$. We now prove statements (a) and (b) hold for these vertices.

Suppose there exists $w \in V(C_2) - N[u] \cup N[u']$. Then, from $wu \notin E$, there exists $y$ such that $[w,y] \to u$ or $[u,y] \to w$. Since $x$ must be dominated, $y$ must be $v$ or $x$. But $u' \notin N[w] \cup N[v] \cup N[u]$, a contradiction. Hence, (a) holds.

To prove (b), we first define $S_1$. Let $A_1 = N(u) \oplus N(u')$. If $A_1 = \emptyset$, then $S_1 = G[\{u, u'\}] = \overline{K}_2$. Otherwise, there exists $z \in A_1$. Consider $z \in N(u)$ such that $z \notin N(u')$. From the fact that $[v, u'] \to u$, $vz \in E$. Now, from $zu' \notin E$, we have $t$ such that $[z, t] \to u'$ or $[u', t] \to z$. Since $vz \in E$ and $u'u \notin E$, we must have $t = x$. Thus $[z, x] \to u'$ and $N[z] = (V(C_2) - \{u'\}) \cup \{v\}$. Similarly, if $z \in N(u')$ and $z \notin N(u)$, then $N[z] = (V(C_2) - \{u\}) \cup \{v\}$. Therefore $G[A_1]$ is complete. In this case ($A_1 \neq \emptyset$), set $S_1 = G[A_1 \cup \{u, u'\}]$.

We now show that $S_1 \in \mathcal{Q}_{2,p}$. Suppose $A_1 \subseteq N(u)$ and $z \in A_1$. Then, $xu \notin E$ implies there exists $y$ such that either $[x, y] \to u$ or $[u, y] \to x$. If $[x, y] \to u$, then $y \neq v$ as $xv \in E$, and $y \notin N(u)$ gives $y = u'$. But $xz$, $u'z \notin E$, a contradiction. Thus $[u, y] \to x$. Since $xv \in E$, $y \neq v$. Furthermore, since $uv \notin E$, $y \in N(v)$. But $vu' \notin E$ gives $y \neq u'$, and hence $y \notin N(u') - N(u)$, a contradiction. Therefore, $A_1 \not\subseteq N(u)$, and $S_1 \in \mathcal{Q}_{2,p}$.

Let $B_1 = N(u) \cap N(u')$. If $B_1 = \emptyset$, then $C_2 = S_1$. Otherwise, $B_1 \neq \emptyset$. If $u_1 v \in E$ for all $u_1 \in B_1$, then $G[B_1] = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2$, as in the proof of Theorem 3.5. Otherwise, there exists $u_1 \in B_1$ such that $u_1 v \notin E$. Then, again as in Theorem 3.5, it can be shown that there exists $u_1' \in B_1$, such that $u_1 u_1' \notin E$ and such that (a) and (b) hold with $u_1$ and $u_1'$ in place of $u$ and $u'$.

In general, if $B_k = N(u_{k-1}) \cap N(u'_{k-1}) = \emptyset$, then $C_2 = S_1 + S_2 + \cdots + S_k$. Suppose $B_k \neq \emptyset$. If $u_k v \notin E$ for all $u_k \in B_k$, then $G[B_k] = \overline{K}_2 + \overline{K}_2 + \cdots + \overline{K}_2$. Otherwise, there exists $u_k \in B_k$ such that $u_k v \in E(B_k)$ and $u_k v \notin E$. Thus, there exists $u_k' \in B_k$ such that $u_k u_k' \notin E$, and such that (a) and (b) hold with $u_k$ and $u_k'$ in place of $u$ and $u'$.

Since $G$ is finite, we have $B_m = \emptyset$ for some $m$, and thus $C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_2$ or $\mathcal{Q}_{2,p}$ for all $1 \leq j \leq m$. The result follows. ∎

Let $L_{p,q}$ denote the bipartite graph constructed from $K_{1,p-1} \cup K_{1,q-1}$ by adding an edge between the vertex of degree $p-1$ in $K_{1,p-1}$ and the vertex of degree $q-1$ in $K_{1,q-1}$. We will refer to the vertices incident with this new edge as the *centre vertices* of $K_{1,p-1}$ and $K_{1,q-1}$. Note that when $p = 1$ (or $q = 1$) $L_{p,q} = K_{1,q}$ (or $K_{1,p}$, respectively), and $L_{1,1} = K_{1,1} = K_2$.

Now, consider a 2-connected 3-edge-$i$-critical graph $G$ with $\delta = 2$. Let $x$ be a vertex with $d(x) = 2$. Then $N(x)$ is a vertex cut of size two. By Lemma 3.3, the graph $G - N(x)$ has at most three components. The following two theorems give a description of such graphs.

**Theorem 3.8** *Let $G$ be a 2-connected, 3-edge-$i$-critical graph with $\delta = 2$. If there exists a vertex $x$ such that $N(x) = \{v, v'\} = S$, where $vv' \notin E$, then* *either 1 or 2 holds.*

1. *The graph $G - S$ has exactly two components $C_1$ and $C_2$ such that $V(C_1) = \{x\}$ and $C_2 = S_1 + S_2 + \cdots + S_m$, where $m \geq 1$ and $S_j = \overline{K}_{1,p}$ or $\overline{L}_{p,q}$ for all $1 \leq j \leq m$. Furthermore,*
   (a) *$C_2 - (N(v) \cup N(v'))$ is a complete graph on $r \geq 1$ vertices, and*
   (b) *for $j = 1, 2, \ldots, m$, one of the following holds:*
      i. *$S_j = \overline{L}_{p,q}$ with centre vertices $\{w, w'\}$, where $w, w' \in N(v) \oplus N(v')$, and $V(S_j) - \{w, w'\} \subseteq N(v) \cap N(v')$;*
      ii. *$S_j = \overline{K}_{1,p}$ with centre vertex $u$, where $u \in N(v) \cap N(v')$ or $V(S_j) - \{u\} \subseteq N(v) \cap N(v')$.*

2. *$G - S$ has exactly three components $C_1, C_2$ and $C_3$ such that $V(C_1) = \{x\}$, $C_2 = K_1$, and $C_3 = K_q$ with $q \geq 3$. Furthermore, $N(v) = N(v')$ and there exist vertices $z_1, z_2, z_3 \in V(C_3)$ that satisfy*

$$V(C_1) \cup V(C_2) \cup \{z_1, z_2\} \subseteq N(v) \subseteq V(C_1) \cup V(C_2) \cup (V(C_3) - \{z_3\}).$$

**Proof:** By Lemma 3.3, $G - S$ has either two or three components. We consider these two cases separately.

**Case 1:** <u>$G - S$ has exactly two components $C_1$ and $C_2$.</u>

Without loss of generality, $V(C_1) = \{x\}$, and $i(C_2) \geq 2$.

We first show that for any $w \in N(v) \oplus N(v')$, there is a unique vertex $w' \in N(v) \oplus N(v')$ such that $ww' \notin E$, and furthermore, $w$ dominates $V(C_2) - (N(v) \cup N(v'))$. For any $w \in N(v) - N(v')$, $wv' \notin E$ implies that there exists $w'$ such that $[w, w'] \to v'$ or $[v', w'] \to w$. Since $\{w, w', v'\}$ must be an independent set, $w' \notin N[x]$, and hence $[v', w'] \to w$. Furthermore, $w' \in N(v) - N(v')$ and $N[w'] \supseteq V(C_2) - (N(v') \cup \{w\})$. Now, from $w'v' \notin E$ we must have $[v', w] \to w'$ and thus $N[w] \supseteq V(C_2) - (N(v') \cup \{w'\})$. Similarly, for any $u \in N(v') - N(v)$, we have $u' \in N(v') - N(v)$ such that $N[u] \supseteq V(C_2) - (N(v) \cup \{u'\})$ and $N[u'] \supseteq V(C_2) - (N(v) \cup \{u\})$. Now suppose there exists $w \in N(v) - N(v')$ and $u \in N(v') - N(v)$, such that $wu \notin E$. Then there exists $t$ such that $[w, t] \to u$ or $[u, t] \to w$. In either case, since $t \notin \{v, v'\}$ and $t$ must be dominated, $t = x$. However, neither $[w, x] \to u$ nor $[u, x] \to w$ is possible, a contradiction. Therefore $wu \in E$ for all $w \in N(v) - N(v')$ and $u \in N(v') - N(v)$.

Next, we prove that $C_2 - (N(v) \cup N(v'))$ is a complete graph with $r \geq 1$ vertices. Let $y \in V(C_2) - (N(v) \cup N(v'))$. Note that $y$ exists, else $\{v, v'\}$ is an independent dominating set. Since $vy \notin E$, there exists $t$ such that

$[v,t] \to y$ or $[y,t] \to v$. In either case, $\{v,t,y\}$ must be an independent set, and $N(y) \supseteq N(v) \oplus N(v')$ by the first paragraph. Furthermore, neither $v$ nor $y$ dominate $v'$. It follows that if $[v,t] \to y$, then $t = v'$, and if $[y,t] \to v$, then $t = v'$. Now, if $[v,v'] \to y$, then $C_2 - (N(v) \cup N(v')) = \{y\}$, a complete graph on one vertex. Otherwise, $[y,v'] \to v$ (and by the same argument, $[y,v] \to v'$), and $N[y] \supseteq V(C_2) - (N(v) \cap N(v'))$. Since $y$ is arbitrary, it follows that $C_2 - (N(v) \cup N(v'))$ is a complete graph with $r \geq 1$ vertices.

Finally, we prove that $C_2 = S_1 + S_2 + \cdots + S_m$, with $m \geq 1$ and $S_j = \overline{K}_{1,p}$ or $\overline{L}_{p,q}$ for all $1 \leq j \leq m$. Recall that $i(C_2) \geq 2$. First, consider any pair of nonadjacent vertices $w, w' \in N(v) \oplus N(v')$. If there exists a vertex $z \neq w'$ such that $wz \notin E$, then $z \in N(v) \cap N(v')$ and either $[w,x] \to z$ or $[z,x] \to w$. Since $\{w,x\}$ does not dominate $w'$, we must have $[z,x] \to w$ and $N[z] \supseteq V(C_2) - \{w\}$. Therefore $V(C_2) - N(w) - \{w'\}$ induces a subgraph $\overline{K}_{1,p}$. Similarly, $V(C_2) - N(w') - \{w\}$ induces a subraph $\overline{K}_{1,q}$. Together, these two subgraphs induce a subgraph $\overline{L}_{p,q}$ with centre vertices $w$ and $w'$. Thus the vertices in $N(v) \oplus N(v')$ together with the vertices of $N(v) \cap N(v')$ that do not dominate $N(v) \oplus N(v')$ induce $S_1 + S_2 + \cdots + S_i$, where $S_j = \overline{L}_{p,q}$ for all $1 \leq j \leq i = |N(v) \oplus N(v')|/2$.

It remains to be shown that $S_j = \overline{K}_{1,p}$ for $i+1 \leq j \leq m$. The subgraphs $S_{i+1}, S_{i+2}, \ldots, S_m$ can be found recursively as follows. To find $S_j$, consider any pair of nonadjacent vertices $y$ and $z$ in $V(C_2) - V(S_1 + S_2 + \cdots + S_{j-1})$. Such vertices exist, else $\{x,y\}$ would be a dominating set for any $y \in V - V(S_1 + S_2 + \cdots + S_{j-1})$. By fact $(a)$, either $y$ or $z$ (or both) is in $N(v) \cap N(v')$. Hence either $[y,x] \to z$ or $[z,x] \to y$, that is, either $N[y] \supseteq V(C_2) - \{z\}$ or $N[z] \supseteq V(C_2) - \{y\}$. Without loss of generality, suppose $[y,x] \to z$. Now consider all nonneighbours of $z$ in $V(C_2)$. Specifically, if $wz \notin E$, then $[w,x] \to z$. Therefore, the graph induced by $V - N(z)$ is a graph $\overline{K}_{1,p}$. The result now follows.

**Case 2:** <u>$G - S$ has exactly three components $C_1, C_2$ and $C_3$.</u>

Without loss of generality, $V(C_1) = \{x\}$.

Since $G$ is 2-connected, for $i = 2,3$ there exist vertices $y_i \in V(C_i)$ such that $vy_2 \in E$ and $v'y_3 \in E$. Now, $y_2y_3 \notin E$ implies that there exists $t$ such that $[y_2,t] \to y_3$ or $[y_3,t] \to y_2$. Since $x \notin N[y_2] \cup N[y_3]$ and $t \notin \{v,v'\}$, we must have $t = x$. Without loss of generality, $[y_3,x] \to y_2$. Hence, $V(C_2) = \{y_2\}$, and hence $C_2$ is a complete graph on one vertex. For any $z \in V(C_3)$, $y_2z \notin E$ gives $[z,x] \to y_2$ and hence $C_3$ is complete.

Furthermore, since $\delta = 2$, $v'y_2 \in E$. Also, for any $z \in V(C_3)$, if $v'z \in E$ then $vz \in E$ (otherwise $\{v,z\}$ is an independent dominating set) and if $vz \in E$ then $v'z \in E$ (otherwise $\{v',z\}$ is an independent dominating set). Therefore, $N(v) = N(v')$. Since $i = 3$, $\{v, v'\}$ is not an independent dominating set, and hence there exists $z \in V(C_3)$ such that $z \notin N(v) \cup N(v')$. Also, since $y_3$ is not a cut-vertex, $V(C_3) \cap N(v) \neq \{y_3\}$. It follows that $C_3 = K_q$ for some $q \geq 3$. Specifically, there exist distinct vertices $z_1, z_2, z_3 \in V(C_3)$ such that $z_1, z_2 \in N(v)$, $z_3 \notin N(v)$, and

$$V(C_1) \cup V(C_2) \cup \{z_1, z_2\} \subseteq N(v) = N(v') \subseteq V(C_1) \cup V(C_2) \cup (V(C_3) - \{z_3\}).$$

This completes the proof. ∎

Let $\mathcal{R}_{3,p}$ be the set of graphs $R_{3,p}$ on $3+p$ vertices with the form: $R_{3,p}$ can be obtained from $K_2 \cup K_1 \cup K_p$ by adding, for each $v \in K_p$, two edges from $v$ to vertices not in $K_p$, such that the resulting graph is 2-connected.

**Theorem 3.9** *Let $G$ be a 2-connected 3-edge-i-critical graph with $\delta = 2$, and a vertex $x$ with $N(x) = \{v, v'\} = S$, where $vv' \in E$. Then, $G - S$ has exactly two components $C_1$ and $C_2$ such that $C_1 = \{x\}$ and $C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j = \overline{K}_{1,p}$, a graph $Q_{2,p} \in \mathcal{Q}_{2,p}$, or a graph $R_{3,p} \in \mathcal{R}_{3,p}$, for all $1 \leq j \leq m$. Furthermore, there exist nonadjacent vertices $u, u' \in V(C_2) - N(v)$ such that*

1. *$N[u] \cup N[u'] = V(C_2)$, and*
2. *either*
   - *(a) for all $z \in V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$, $vz \in E$, $v'z \in E$, and $N[z] \supseteq V(C_2) - \{u, u'\}$, or*
   - *(b) there exists $u'' \in N(v) - N(v')$ such that for all $z \in V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$, $vz \in E$, $v'z \in E$, and $N[z] \supseteq V(C_2) - \{u, u', u''\}$.*

**Proof:** By Lemma 3.3, $G - S$ has either two or three components.

Suppose $G - S$ has three components $C_1, C_2, C_3$ where $V(C_1) = \{x\}$. Since $G$ is 2-connected, there exist vertices $y_2 \in V(C_2)$ and $y_3 \in V(C_3)$ such that $vy_2 \in E$ and $v'y_3 \in E$. Since $y_2y_3 \notin E$, without loss of generality, there exists a vertex $t$ such that $[y_2, t] \rightarrow y_3$. Since $x$ must be dominated by $t$ and $vy_2, v'y_3 \in E$, $t = x$. Hence $[y_2, x] \rightarrow y_3$, and $C_3 = \{y_3\}$. Since $G$ is 2-connected, $vy_3 \in E$, and $i = 3$ implies there exists $z \in V(C_2)$ such that $vz \notin E$. Again, since $zy_3 \notin E$, there exists $t$ such that either $[z, t] \rightarrow y_3$ or $[y_3, t] \rightarrow z$. In either case, $x$ must be dominated by $t$ as $xz, y_3z \notin E$, and $y_3v, y_3v' \in E$. Thus, $t = x$. Since $\{y_3, x\}$ does not dominate $y_2$, it is not possible that $[y_3, x] \rightarrow z$. Therefore, $[z, x] \rightarrow y_3$ and $\{z, v\}$ is an independent dominating set, a contradiction. Therefore $G - S$ has exactly two components $C_1$ and $C_2$, where $V(C_1) = \{x\}$.

Since $i = 3$, there exists $u \in V(C_2) - N(v)$. Now, $uv \notin E$ implies there exists $u'$ such that either $[u, u'] \to v$ or $[v, u'] \to u$. Suppose $[u, u'] \to v$. Then, $u' \notin \{x, v, v'\}$, so $x$ is not dominated, a contradiction. Therefore $[v, u'] \to u$. Since $u'v \notin E$, we have $u' \in V(C_2) - N(v)$ and $N[u'] \supseteq V(C_2) - N(v) - \{u\}$. Also, $u'v \notin E$ implies there exists $w$ such that either $[u', w] \to v$ or $[v, w] \to u'$. As above, $[v, w] \to u'$ and $w \in V(C_2) - N(v)$. Now $N[u'] \supseteq V(C_2) - N(v) - \{u\}$ implies $w = u$. Hence $[v, u] \to u'$ and $N[u] \supseteq V(C_2) - N(v) - \{u'\}$. This argument shows that every $u \in V(C_2) - N(v)$ can be paired with a unique vertex $u' \in V(C_2) - N(v)$ such that $[v, u] \to u'$ and $[v, u'] \to u$. Similarly, every $u \in V(C_2) - N(v')$ can be paired with a unique vertex $u'' \in V(C_2) - N(v')$ such that $[v', u] \to u''$ and $[v', u''] \to u$.

Consider $u \in V(C_2) - (N(v) \cup N(v'))$. We now consider the possibilities for $u$ and $u'$ defined above. Suppose $u' \in V(C_2) - (N(v) \cup N(v'))$. Since $[v', u] \to u''$ and $uu', v'u' \notin E$, $u' = u''$. Furthermore, $[v, u] \to u'$, $[v', u] \to u'$, $[v, u'] \to u$, and $[v', u'] \to u$ gives $N[u] \supseteq V(C_2) - (N(v) \cap N(v')) - \{u'\}$ and $N[u'] \supseteq V(C_2) - (N(v) \cap N(v')) - \{u\}$. Therefore, if $u' \in N(v')$, then $u'' \in N(v)$.

We have now shown that $G - S$ has exactly two components $C_1$ and $C_2$ with $C_1 = \{x\}$, and that the vertices in $V(C_2) - (N(v) \cap N(v'))$ can be uniquely partitioned into pairs of the form $\{u, u'\}$ and triples of the form $\{u, u', u''\}$. If $y$ is in a pair in the partition, we will refer to $y$ as *Type I*, and if $y$ is in a triple in the partition, we will refer to $y$ as *Type II*.

Consider $u \in V(C_2) - (N(v) \cap N(v'))$ such that $u \notin N(v)$ (the same argument applies for $u \notin N(v')$). Suppose there exists $w \in V(C_2) - N[u] - N[u']$. Since $wu \notin E$, there exists $t$ such that either $[w, t] \to u$ or $[u, t] \to w$. In either case, $x$ must be dominated by $t$, and hence $t \in \{x, v, v'\}$. Also, $u'$ must be dominated, and $u' \notin N[w] \cup N[u] \cup N[x] \cup N[v]$, so $t = v'$. Therefore $u \notin N(v')$ and $u' \in N(v')$. Similarly, $wu' \notin E$ gives $u' \notin N(v'), u \in N(v')$, a contradiction. Therefore, $N[u] \cup N[u'] = V(C_2)$.

We now show that if $y \in N(v) - N(v')$ and $u \in N(v') - N(v)$, then $uy \in E$. Suppose $uy \notin E$. Then there exists $w$ such that $[u, w] \to y$ or $[y, w] \to u$. In either case, $x$ must be dominated by $w$, and $uv', yv \in E$, so $w = x$. If $[u, x] \to y$, then $uu' \in E$, a contradiction. So $[y, x] \to u$ and $yy' \in E$, a contradiction. Hence $uy \in E$ for all $y \in N(v) - N(v')$, $u \in N(v') - N(v)$.

We have shown that statement (1) of the theorem holds. We now prove statement (2). There are two cases. After both have been considered, we subsequently show that $C_2$ has the structure claimed.

**Case 1.** $u$ is Type I.

Let $A_1 = V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$, and consider $z \in A_1$. If $u, u' \in N(v')$ (the case where $u, u' \in N(v)$ is similar), then $z \notin N(v) - N(v')$ since both $u$ and $u'$ dominate $N(v) - N(v')$, and $z \notin V(C_2) - N(v)$, so $z \in N(v) \cap N(v')$. If $u, u' \notin N(v) \cup N(v')$, then both $u$ and $u'$ dominate $V(C_2) - (N(v) \cap N(v')) - \{u, u'\}$ and hence $N(u) \cap N(u') \supseteq V(C_2) - (N(v) \cap N(v')) - \{u, u'\}$. Therefore $z \in N(v) \cap N(v')$. In either case, without loss of generality, let $uz \in E$ and $u'z \notin E$. Now $zu' \notin E$ implies there exists $t$ such that $[z, t] \to u'$ or $[u', t] \to z$. Since $xz, xu' \notin E$ and $z \in N(v) \cap N(v')$, $t = x$. Hence $[u', x] \to z$ is not possible as neither $u'$ nor $x$ dominate $u$. Therefore, $[z, x] \to u'$ and $N[z] \supseteq V(C_2) - \{u'\}$. Similarly, if $u'z \in E$ and $uz \notin E$, then $N[z] \supseteq V(C_2) - \{u\}$. Therefore $N[z] \supseteq V(C_2) - \{u, u'\}$.

**Case 2.** $u$ is Type II.

Let $A_1 = V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$, and consider $z \in A_1$. From previous results, each of $u, u', u''$ dominates every vertex in

$$V(C_2) - (N(v) \cap N(v')) - \{u, u', u''\}.$$

Therefore, $z \in N(v) \cap N(v')$. Suppose $zu \notin E$. Then, there exists $t$ such that $[z, t] \to u$ or $[u, t] \to z$. In order to dominate $x$, $t = x$. Since $u'$ must be dominated, it is not possible that $[u, x] \to z$. Therefore, $[z, x] \to u$ and hence $N[z] \supseteq V(C_2) - \{u\}$. The same argument for $zu' \notin E$ or $zu'' \notin E$ gives $N[z] \supseteq V(C_2) - \{u, u', u''\}$.

This completes the proof of (2). It remains to show that $C_2$ has the claimed structure. We begin by defining $S_1$, depending on whether $u$ is Type I or Type II.

Suppose $u$ is Type I. Let $A_1 = V(C_2) - (N(u) \cap N(u')) - \{u, u'\}$. If $A_1 = \emptyset$, then let $S_1 = G[\{u, u'\}] = \overline{K}_{1,1}$. If $A_1 \neq \emptyset$, let $S_1 = G[A_1 \cup \{u, u'\}]$. Since $N[z] \supseteq V(C_2) - \{u, u'\}$ for all $z \in A_1$, $G[A_1] = K_p$. Now consider any $z \in A_1$. Without loss of generality, $uz \in E$, and $u'z \notin E$. Also, $xu \notin E$ implies there exists $y$ such that $[x, y] \to u$ or $[u, y] \to x$. In either case, $u'$ must be dominated, so $y \in N[u']$. But $y \neq u'$ as neither $[x, u'] \to u$ nor $[u, u'] \to x$. Also, $y \in V(C_2)$ as $xy \notin E$. Therefore, $y \in N(u')$ and $y \notin N(u)$, so $y \in A_1$ and $A_1 \not\subseteq N(u)$. Therefore $S_1 \in \mathcal{Q}_{2,p}$.

Suppose $u$ is Type II. Let $A_1 = V(C_2) - (N(u) \cap N(u') \cap N(u'')) - \{u, u', u''\}$. If $A_1 = \emptyset$, let $S_1 = G[\{u, u', u''\}] = S_1 = \overline{K}_{1,2}$. If $A_1 \neq \emptyset$, let $S_1 = G[A_1 \cup \{u, u', u''\}]$. Since $N[z] \supseteq V(C_2) - \{u, u', u''\}$ for all

$z \in A_1$, again we have $G[A_1] = K_p$. By defininition of $A_1$, for any $z \in A_1$ there exists $w \in \{u, u', u''\}$ such that $wz \notin E$. Therefore there exists $t$ such that either $[w, t] \to z$ or $[z, t] \to w$. Since $zx, wx \notin E$, $x$ must be dominated by $t$, and $z$ is adjacent to both $v$ and $v'$, so $t = x$. Now $[w, x] \not\to z$, as $w$ does not dominate $\{u, u', u''\}$. Therefore $[z, x] \to w$ and $z$ dominates $\{u, u', u''\} - \{w\}$. This shows that every $z \in A_1$ is adjacent to exactly two vertices in $\{u, u', u''\}$. Suppose one of $u, u'$, or $u''$ is adjacent to every vertex in $A_1$. Specifically, suppose $u'z \in E$ for all $z \in A_1$. Then $N[u'] = V - \{x, v, u\}$. Since $u'x \notin E$, there exists $y$ such that either $[u', y] \to x$ or $[x, y] \to u'$. In either case, $\{u', y, x\}$ must be an independent set, and hence $y = u$. But neither $[u', u] \to x$ nor $[x, u] \to u'$ is true, a contradiction which implies $u'$ does not dominate $A_1$. A similar argument can be used to show that neither $u$ nor $u''$ dominate $A_1$. Therefore, every vertex in $\{u, u', u''\}$ has a neighbour in $A_1$, and specifically, $u$ has at least two neighbours in $A_1$. This proves $S_1 \in \mathcal{R}_{3,p}$.

Proceed inductively as follows. If $u$ is Type I, let $B_1 = N(u) \cap N(u')$, and if $u$ is Type II, let $B_1 = N(u) \cap N(u') \cap N(u'')$. If $B_1 \neq \emptyset$ and there exists $y \in B_1$ such that $y \in (N(v) \cup N(v')) - (N(v) \cap N(v'))$, repeat the above procedure to find $A_i, S_i, B_i$ for $i = 2, 3, \ldots, j$ (as in Theorem 3.5 Case 2), where $j$ is the least $i$ such that $B_i = \emptyset$ or there is no $y \in B_i$ such that $y \in (N(v) \cup N(v')) - (N(v) \cap N(v'))$. In the first case, $C_2 = S_1 + S_2 + \cdots + S_j$. In the second case, $y \in N(v) \cap N(v')$ for all $y \in B_j$. In this case, for any $y \in B_j$ there exists $y_1 \in B_j$ such that $yy_1 \notin E$, otherwise $\{y, x\}$ is an independent dominating set of $G$. Now $yy_1 \notin E$ implies there exists $t$ such that, without loss of generality, $[y_1, t] \to y$. Either $y_1$ or $t$ must dominate $x$, so $t = x$ and $[y_1, x] \to y$. This implies $N[y_1] \supseteq B_j - \{y\}$. For each vertex $w \in B_j - \{y, y_1\}$, either $yw \in E$ or $[w, x] \to y$ (as $[y, x] \to w$ would imply $yy_1 \in E$). So let $\{y_1, y_2, \ldots y_p\}$ be the set of vertices in $B_j$ such that $[y_i, x] \to y$, $i = 1, 2, \ldots p$. Now let $S_{j+1} = G[\{y, y_1, \ldots y_p\}] = \overline{K}_{1,p}$, $p \geq 1$. Note that every vertex in $\{y, y_1, \ldots, y_p\}$ is adjacent to every vertex in $V(C_2) - \{y, y_1, \ldots y_p\}$. If $B_j - \{y, y_1, \ldots, y_p\} = \emptyset$ then $C_2 = S_1 + S_2 + \cdots + S_{j+1}$. Otherwise, set $B_{j+1} = B_j - \{y, y_1, \ldots, y_p\}$, and repeat this argument to find $S_{j+2}, S_{j+3}, \ldots, S_m$. It is important to note that at each step, $B_j, B_{j+1}, \ldots, B_m$ each contain at least two vertices, as if $B_m = \{y\}$ then $\{y, x\}$ is an independent dominating set of $G$. Therefore $C_2 = S_1 + S_2 + \cdots + S_m$, where $S_j$ has one of the claimed structures, for $j = 1, 2, \ldots m$. ∎

# 4 Hamilton Paths and Cycles

Using the results of the previous sections, we give a characterisation of the 2-connected, 3-edge-$i$-critical graphs that are hamiltonian. We then prove that every connected, 3-edge-$i$-critical graph with more than six vertices has a Hamilton path.

Note that each graph in $\mathcal{Q}_{2,p}$ has a Hamilton path. By the definition of join we have the following lemmata. A couple of the easy proofs are omitted.

**Lemma 4.1** *If $G$ has a Hamilton cycle, then so does $G+\overline{K}_{1,q}$, $G+\overline{L}_{p,q}$, $G+Q_{2,p}$ for any $Q_{2,p}\in\mathcal{Q}_{2,p}$, and $G+R_{3,p}$ for any $R_{3,p}\in\mathcal{R}_{3,p}$.*

**Proof:** Let $v_1v_2\ldots v_n$ denote any Hamilton cycle in $G$.

Consider $G+\overline{K}_{1,q}$. Label the vertices from $\overline{K}_{1,q}$ by $x, x_1, x_2, \ldots, x_q$, where $x$ is the isolated vertex. Then

$$v_1v_2\ldots v_{n-1}xv_nx_1x_2\ldots x_qv_1$$

is a Hamilton cycle in $G+\overline{K}_{1,q}$.

Consider $G+\overline{L}_{p,q}$. Label the vertices from $\overline{L}_{p,q}$ by $x, x_1, x_2, \ldots, x_{q-1}, y$, $y_1, y_2, y_{p-1}$, where $x$ and $y$ are the vertices of degree $q-1$ and $p-1$, with neighbours $x_1, x_2, \ldots, x_{q-1}$ and $y_1, y_2, \ldots, y_{p-1}$, respectively. In $G+\overline{L}_{p,q}$,

$$v_1v_2\ldots v_{n-1}xx_1x_2\ldots x_{q-1}v_nyy_1y_2\ldots y_{p-1}v_1$$

is a Hamilton cycle. Note that this is valid even if $p=1$ or $q=1$ (or both).

Consider $G+Q_{2,p}$. Label the vertices from $\overline{K}_2$ by $x$ and $y$, and the vertices from $K_p$ by $u_1, u_2, \ldots, u_p$ such that $xu_1$ and $yu_p$ are edges. Such a labeling is possible, since $Q_{2,p}$ has no isolated vertices. Now

$$v_1v_2\ldots v_nxu_1u_2\ldots u_pyv_1$$

is a Hamilton cycle in $G+Q_{2,p}$.

Lastly, consider $G+R_{3,p}$. Label the vertex from $K_1$ by $x$, and the vertices from $K_2$ by $y_1, y_2$. Label the vertices from $K_p$ by $u_1, u_2, \ldots, u_p$ such that $u_1$ and $u_2$ are adjacent to $x$, and $u_n$ is adjacent to $y_2$. Such a labeling is possible since $d(x)\geq 2$, and at least one of the vertices from $K_2$ must be adjacent to $u_n$. Hence

$$v_1v_2\ldots v_nu_1xu_2u_3\ldots u_ny_2y_1v_1$$

is a Hamilton cycle in $G+R_{3,p}$. ∎

**Lemma 4.2** *If $G = S_1 + S_2 + \cdots + S_k$, where $k \geq 2$ and $S_j = \overline{K}_{1,p}$ or $\overline{L}_{p,q}$, for $j = 1, 2, \ldots, k$, then $G$ has a Hamilton cycle.*

**Proof:** Consider $\overline{K}_{1,p} + \overline{K}_{1,q}$. Label the vertices in each of $\overline{K}_{1,p}$ and $\overline{K}_{1,q}$ as in the proof of Lemma 4.1. Then

$$xy_1y_2\ldots y_qx_1x_2\ldots x_pyx$$

is a Hamilton cycle in $\overline{K}_{1,p} + \overline{K}_{1,q}$.

Now consider $\overline{L}_{p,q} + \overline{L}_{r,s}$, with the vertices from $\overline{L}_{p,q}$ labelled as in the proof of Lemma 4.1, and the vertices from $\overline{L}_{r,s}$ labelled analogously by $u, u_1, u_2, \ldots u_{s-1}$ and $v, v_1, v_2, \ldots, v_{r-1}$. Then

$$xx_1x_2\ldots x_{q-1}uu_1u_2\ldots u_{s-1}yy_1y_2\ldots y_{p-1}vv_1v_2\ldots v_{r-1}x$$

is a Hamilton cycle in $\overline{L}_{p,q} + \overline{L}_{r,s}$.

Finally, consider $\overline{K}_{1,p} + \overline{L}_{r,q}$, with vertices labelled analogously to the above graphs, by $x, x_1, x_2, \ldots, x_p$ in $\overline{K}_{1,p}$ and by $y, y_1, y_2, \ldots, y_{q-1}$, and $w, w_1, w_2, \ldots, w_{r-1}$ in $\overline{L}_{r,q}$. Then

$$xyy_1y_2\ldots y_{q-1}x_1x_2\ldots x_pww_1w_2\ldots w_{r-1}x$$

is a Hamilton cycle in $\overline{K}_{1,p} + \overline{L}_{r,q}$.

¿From the three cases above, it has been shown that $G = S_1 + S_2 + \cdots + S_k$ is hamiltonian if $k = 2$. By induction on $k$, the result follows from Lemma 4.1. ∎

**Lemma 4.3** *Every $R_{3,p} \in \mathcal{R}_{3,p}$ has a Hamilton cycle.*

**Proof:** Consider $R_{3,p}$ with vertices labelled as in the proof of Lemma 4.1. Let $V_1$ denote the subset of vertices from $K_p$ that are adjacent to $y_1$ and $x$, let $V_2$ denote the subset of vertices from $K_p$ that are adjacent to $y_2$ and $x$, and let $V_{12}$ denote the subset of vertices from $K_p$ that are adjacent to $y_1$ and $y_2$. Note that $\{V_1, V_2, V_{12}\}$ is a partition of the vertices in $K_p$. Since $R_{3,p}$ is 2-connected, $\delta \geq 2$ and hence both $V_1$ and $V_2$ are nonempty. Thus

$$xP_1y_1y_2P_{12}P_2x$$

is a Hamilton cycle in $R_{3,p}$ for any paths $P_1$, $P_2$, and $P_{12}$ through all vertices in $V_1$, $V_2$, and $V_{12}$, respectively. Note that $V_{12}$ may be empty. ∎

**Lemma 4.4** *If $G = S_1 + S_2 + \cdots + S_k$ where $S_j = \overline{K}_{1,p}, Q_{2,p} \in \mathcal{Q}_{2,p}$, or $R_{3,p} \in \mathcal{R}_{3,p}$ for $j = 1, \ldots, k$, then $G$ has a Hamilton cycle, unless $G = \overline{K}_{1,p}$ or $G \in \mathcal{Q}_{2,p}$. If $G \in \mathcal{Q}_{2,p}$, then $G$ has a Hamilton path.*

**Proof:** If $S_j = R_{3,p}$ for any $j$, then by Lemma 4.1 and Lemma 4.3, $G$ has a Hamilton cycle. Otherwise, since $\overline{K}_{1,p} + \overline{K}_{1,q}$, $\overline{K}_{1,p} + Q_{2,q}$, and $Q_{2,p} + Q_{2,q}$ have Hamilton cycles, by Lemma 4.1 so does $G$, except possibly if $k = 1$. If $G = Q_{2,p}$, it is easy to see that $G$ has a Hamilton path. ∎

For any $q \geq 3$, let $\mathcal{W}_{q,4}$ denote the set of graphs on $q + 4$ vertices constructed from $K_q \cup C_4$ such that two nonadjacent vertices of $C_4$ have the same neighbourhood in $K_q$ of size greater than 1 and less than $q$, and the other two vertices of $C_4$ have no neighbours in $K_q$.

**Lemma 4.5** *Each graph in $\mathcal{W}_{q,4}$ has a Hamilton path, but not a Hamilton cycle.*

**Theorem 4.6** *If $G$ is 2-connected and 3-edge-i-critical, then $G$ is hamiltonian, unless $G \in \mathcal{W}_{q,4}$, in which case it has a Hamilton path.*

**Proof:** If $\delta \geq 3$, then by Corollary 2.4, $G$ is hamiltonian. Otherwise, suppose that there exists $x \in V$ such that $d(x) = 2$. Let $S = N(x) = \{v, v'\}$. Then by Theorem 3.8 and Theorem 3.9, $G - S$ has at most 3 components and one of (1) or (2) in the statement of Theorem 3.8 holds, or $vv' \in E$.

If (1) holds, then by Lemma 4.2, $C_2$ is hamiltonian. Notice that there exists some Hamilton path in $C_2$ that starts at a point of $N(v)$ and ends at a point of $N(v')$. Hence $G$ is also hamiltonian.

If (2) holds, then $G \in \mathcal{W}_{q,4}$, and by Lemma 4.5, $G$ has a Hamilton path, but not a Hamilton cycle.

If $vv' \in E$, by the construction of $C_2$ in Theorem 3.9,

$$S_1 \in \{Q_{2,p}, \overline{K}_{1,1}, \overline{K}_{1,2}, R_{3,p}\}.$$

Note that $C_2 \neq \overline{K}_{1,p}$, otherwise $G$ is not 2-connected. Therefore, by Lemma 4.4, $C_2$ is either hamiltonian or $C_2 = Q_{2,p}$. If $C_2$ is hamiltonian and $S_j = Q_{2,p}$ or $R_{3,p}$ for some $1 \leq j \leq m$, then there exists a Hamilton path in $C_2$ from a vertex in $N(v)$ to a vertex in $N(v')$, and hence $G$ is hamiltonian. If $C_2$ is hamiltonian and $C_2 = \overline{K}_{1,p_1} + \overline{K}_{1,p_2} + \cdots + \overline{K}_{1,p_m}$, where $p_i = 1$ or 2 for $i = 1, 2, \ldots, m$, it is not difficult to find a Hamilton path in $C_2$ from a vertex in $N(v)$ to a vertex in $N(v')$, and therefore $G$ is hamiltonian. If $C_2 = Q_{2,p}$, then each of the vertices in the defining copy of $\overline{K}_2$ in $C_2$ have degree greater than one (or else $G$ has a cut vertex) and there exists a Hamilton path in $C_2$ from a vertex in $N(v)$ to a vertex in $N(v')$. Hence there exists a Hamilton cycle in $G$. ∎

**Theorem 4.7** *If $G$ is connected and 3-edge-i-critical with $|V| > 6$, then $G$ has a Hamilton path.*

**Proof:** If $G$ is 2-connected, then by Theorem 4.6, $G$ has a Hamilton path. Thus assume $G$ has a cut vertex $v$. By Theorem 3.5, $G - v$ has exactly two components $C_1$ and $C_2$, such that $C_1$ is complete, $i(C_2) = 2$, and either (1) or (2) in the statement of Theorem 3.5 holds.

If (1) holds, then since $G[C_1 \cup \{v\}]$ is complete, it has a Hamilton path $P$ that ends at $v$. Since $T_{n,2n}$ has a Hamilton path that starts at any vertex, $G[C_2 \cup \{v\}]$ has a Hamilton path $Q$ that starts at $v$. Then the path $PQ$ is a Hamilton path of $G$.

If (2) holds, then since $G$ is connected and $i(G) = 3$, $C_2 \neq \overline{K}_2$. Thus, by Lemma 4.4, $C_2$ has a Hamilton path. Furthermore, $G[\{v\} \cup V(C_2)]$ has a Hamilton path $Q$ that starts at $v$, except when $C_2 \in \mathcal{Q}_{2,2}$ (i.e. $C_2$ is a path with 4 vertices). Since $|V(G)| > 6$, this does not happen. Let $P = xv$. Then the path $PQ$ is a Hamilton path of $G$. Therefore, if $|V| > 6$ (thus, $C_2 \neq P_4$), then $G$ has a Hamilton path. ∎

**Corollary 4.8** *If $G$ is connected and 3-edge-i-critical with $|V| > 6$ even, then $G$ has a perfect matching.*

For $k \geq 4$, the question of when a $k$-edge-$i$-critical graph $G$ contains a Hamilton cycle is still open.

On the other hand, for any $k \geq 4$, there are arbitrarily large $k$-edge-$i$-critical graphs with no Hamilton path: Let $m \geq 2k$, and $p = 1 + 1 + \cdots + 1 + (m - 2k + 1)$ be a partition of $p = m - k$. Then each graph in $\mathcal{Q}_{k,p}$ is $k$-edge-$i$-critical but has $k - 1 \geq 3$ vertices of degree one, and hence no Hamilton path.

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