Perfection in Graphs, A New Look at Irredundance

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Abstract

Given a set $S \subseteq V$ in a graph $G = (V, E)$, we say that a vertex $v \in V$ is *perfect* if $|N[v] \cap S| = 1$, that is, the closed neighborhood $N[v] = \{v\} \cup \{u|\ uv \in E\}$ of v contains exactly one vertex in S. A vertex v is *almost perfect* if it is either perfect or is adjacent to a perfect vertex. Similarly, we can say that a set $S \subset V$ is *(almost) perfect* if every vertex $v \in S$ is (almost) perfect; S is *externally (almost) perfect* if every vertex $u \in V - S$ is (almost) perfect; and S is *completely (almost) perfect* if every vertex $v \in V$ is (almost) perfect. In this paper we relate these concepts of perfection to independent sets, dominating sets, efficient and perfect dominating sets, distance-2 dominating sets, and to perfect neighborhood sets in graphs. The concept of a set being almost perfect also provides an equivalent definition of irredundance in graphs.

1 Introduction

Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and *order* $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = \{u|uv \in E\}$ of vertices u that are adjacent to v; the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *open neighborhood* of a set $S \subset V$ of vertices is the set $N(S) = \bigcup_{v \in S} N(v)$, and the *closed neighborhood* of S is

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the set $N[S] = \bigcup_{v \in S} N[v]$. For a given set S, the *subgraph induced by S* is defined as $G[S] = (S, E \cap (S \times S))$. Given the definitions of open and closed neighborhoods of vertices $v \in V$ and sets S, we can define a number of well-studied types of sets in graphs, as follows. A set $S \subset V$ of vertices is:

1. *independent* if no two vertices in S are adjacent. Let $i(G)$ and $\beta_0(G)$ denote the minimum and maximum cardinalities of a maximal independent set in G.

2. a *dominating set* if $N[S] = V$, that is, every vertex $u \in V - S$ is adjacent to at least one vertex in S. Let $\gamma(G)$ and $\Gamma(G)$ denote the minimum and maximum cardinalities of a minimal dominating set in G.

3. a *distance-2 dominating set* if every vertex $u \in V - S$ is within distance 2 of at least one vertex in S. Let $\gamma_{\leq 2}(G)$ and $\Gamma_{\leq 2}(G)$ denote the minimum and maximum cardinalities of a minimal distance-2 dominating set in G, and let $i_{\leq 2}(G)$ and $\beta_{\leq 2}(G)$ denote the minimum and maximum cardinalities of a minimal independent distance-2 dominating set in G. Note that $\beta_{\leq 2}(G) < \beta_0(G)$ is possible; for example, $\beta_{\leq 2}(P_5) = 2 < \beta_0(P_5) = 3$. See Henning [12] for a thorough discussion of distance domination in graphs.

4. a *perfect dominating set* if for every vertex $u \in V - S$, $|N(u) \cap S| = 1$, that is, every vertex $u \in V - S$ is adjacent to exactly one vertex in S. Let $\gamma_p(G)$ and $\Gamma_p(G)$ denote the minimum and maximum cardinalities of a minimal perfect dominating set in G. Trivially, the entire vertex set V is a perfect dominating set for any graph G, but for some graphs the set V is a minimal perfect dominating set. An example of such a graph is the complete tripartite graph $K_{2,2,2}$ of order $n = 6$. For this graph, $\gamma_p(K_{2,2,2}) = n = 6$.

   Perfect dominating sets were first defined and studied by Cockayne, Hartnell, Hedetniemi and Laskar in 1993 [4]. Note that for the 2×3 grid graph $G = P_2 \Box P_3$, $\gamma_p(G) = 2$ while $\Gamma_p(G) = 3$.

5. an *efficient dominating set* if every vertex $v \in V$ is adjacent to exactly one vertex in S, that is, for every vertex $v \in V$, $|N[v] \cap S| = 1$. It is well known that not every graph has an efficient dominating set (e.g. the cycle C_5), but if a graph G has an efficient dominating set, then every efficient dominating set in G has the same cardinality and it equals $\gamma(G)$. Efficient dominating sets were first defined and studied by Bange, Barkauskas, Host and Slater in 1988 [2, 1]. However, efficient dominating sets are also known as *perfect codes*, the first mention of which is due to Biggs in 1973[3].

6. an *irredundant set* if for every vertex $v \in S$, $pn[v] = N[v] - N[S - \{v\}] \neq \emptyset$. The set $pn[v]$ is called the set of ***private neighbors*** of v with respect to the set S. If $v \in pn[v]$, then v is not adjacent to any vertex in $S - \{v\}$ and v is said to be *its own private neighbor.* Every vertex $w \in V - S$ for which $w \in pn[v]$ is called an *external private neighbor* of v. Let $ir(G)$ and $IR(G)$ equal the minimum and maximum cardinalities of a maximal irredundant set in G. Irredundant sets were first defined and studied by Cockayne, Hedetniemi and Miller in 1978 [5].

A thorough discussion of these types of sets can be found in the two books written and edited by Haynes, Hedetniemi and Slater [9, 10].

To these definitions we now introduce several concepts and parameters having to do with *perfection* in graphs. Some of these concepts were introduced by Fricke, Haynes, Hedetniemi, Hedetniemi and Henning in 1999 [8]. In the notation that follows, we use a subscript (α_S) to refer to a parameter α requiring some condition on the vertices in a set S, and we use a superscript (α^{V-S}) to refer to a parameter requiring some condition on vertices in the set $V - S$. If the parameter α requires some condition on all vertices $v \in V$ we do not use a subscript or a superscript.

Definition 1 *Given a set $S \subseteq V$ in a graph $G = (V, E)$, a vertex $v \in V$ is S-perfect if $|N[v] \cap S| = 1$, that is, the closed neighborhood $N[v]$ contains exactly one vertex in S.*

Definition 2 *Given a set $S \subseteq V$ in a graph $G = (V, E)$, a vertex v is almost S-perfect if it is either S-perfect or is adjacent to an S-perfect vertex.*

When a set S has been identified and is assumed, we say more simply that a vertex is *perfect* or *almost perfect* without referring to the set S.

Definition 3 *A set $S \subset V$ is perfect if every vertex $v \in S$ is perfect, and is almost perfect if every vertex $v \in S$ is almost perfect; for brevity we say that an almost perfect set is an ap set. Let $\theta_{ap}(G)$ and $\Theta_{ap}(G)$ equal the minimum and maximum cardinalities of a maximal ap set in G.*

Definition 4 *A set S is externally perfect if every vertex $u \in V - S$ is perfect, and is externally almost perfect if every vertex $u \in V - S$ is either perfect or adjacent to a perfect vertex; for brevity we say that an externally almost perfect set is an eap set. Let $\theta^{ap}(G)$ and $\Theta^{ap}(G)$ equal the minimum and maximum cardinality of a minimal eap set in G.*

Definition 5 *A set S is completely perfect if every vertex $v \in V$ is perfect, and is completely almost perfect if every vertex $v \in V$ is either perfect or is adjacent to a perfect vertex. Completely almost perfect sets are called perfect neighborhood sets in the literature (cf. [8, 6, 7, 11]). Let $\theta(G)$ and $\Theta(G)$ equal the minimum and maximum cardinalities of a perfect neighborhood set in G, and let $\theta_p^{ap}(G)$ and $\Theta_p^{ap}(G)$ equal the minimum and maximum cardinalities of an independent perfect neighborhood set in G.*

Definition 6 *A set S is an eap irredundant, eap dominating set or eap independent set if it is a maximal irredundant, minimal dominating or maximal independent set that is also eap. Thus, every vertex $v \in V - S$ is either perfect or is adjacent to a perfect vertex. Let $ir^{ap}(G)$, $\gamma^{ap}(G)$, $i^{ap}(G)$, $\beta^{ap}(G)$, $\Gamma^{ap}(G)$ and $IR^{ap}(G)$ denote the minimum and maximum cardinalities of such sets.*

In the graph in the following figure a vertex labelled "p" is perfect, while a vertex labelled "ap" is almost perfect. The three shaded vertices form a set S that is almost perfect (every vertex in S is either perfect or is adjacent to a perfect vertex) and is externally almost perfect (every vertex in $V - S$ is either perfect or is adjacent to a perfect vertex), thus, it is totally almost perfect, or a perfect neighborhood set.

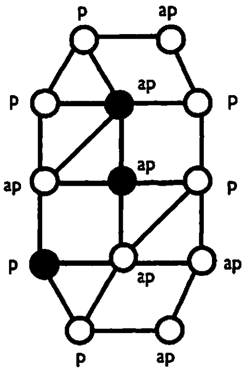

Figure 1: A perfect neighborhood set.

2 Relationships Among Perfection Parameters

Given the definitions in the previous section we can begin to relate them. In the first case we can see that the concept of a set S being perfect is equivalent to the concept of a set being independent.

Proposition 1 *A set S is perfect if and only if it is independent.*

Proof. A vertex $u \in S$ is perfect if and only if $|N[u] \cap S| = 1$, that is, it is an isolated vertex in the induced subgraph $G[S]$. Thus, if every vertex in S is perfect, then S is an independent set. Conversely, if S is an independent set, then clearly every vertex $v \in S$ satisfies the condition that $|N[v] \cap S| = 1$. Therefore S is perfect. $\Box$

Corollary 1 *For any graph G, $\theta_p^{ap}(G) \leq i(G) = i^{ap}(G)$.*

Proof. We first show that $i(G) = i^{ap}(G)$. Let S be a maximal independent set of minimum cardinality $i(G)$. By Proposition 1, S is perfect. By definition, S is also a minimal dominating set. Thus, every vertex in S is perfect, and every vertex in $V - S$ is adjacent to a perfect vertex. Thus, S is a perfect set whose complement $V - S$ is an ap set, and therefore, $i^{ap}(G) \leq |S| = i(G)$. Conversely, since every i^{ap}-set is maximal independent, it is therefore an independent dominating set, and therefore $i(G) \leq i^{ap}(G)$.

The fact that $\theta_p^{ap}(G) \leq i(G)$ follows from the observation that every maximal independent set is an independent perfect neighborhood set. Note that a subdivided star $G = S(K_{1,n})$ is an example for which $\theta_p^{ap}(G) = 1$ yet $i(G) = n$, and thus this inequality can be strict. $\Box$

Corollary 2 *For any graph G, $\beta_0(G) = \beta^{ap}(G) = \Theta_p^{ap}(G)$.*

Proof. It is well known that every β_0-set S is both a maximal independent set and a minimal dominating set. Therefore, from Proposition 1, S is a perfect set and an eap set, since every vertex in $V - S$ is adjacent to a perfect vertex in S. It is therefore an independent perfect neighborhood set, and thus, $\beta_0(G) \leq \Theta_p^{ap}(G)$. But every $\Theta_p^{ap}(G)$-set is an independent set, and therefore by definition, $\Theta_p^{ap}(G) \leq \beta_0(G)$.

Similarly, every β_0-set S is a maximal independent set that is also externally almost perfect, since every vertex in $V - S$ is adjacent to a perfect vertex in S. Thus, $\beta_0(G) \leq \beta^{ap}(G)$. But every β^{ap}-set is an independent set, and therefore by definition, $\beta^{ap}(G) \leq \beta_0(G)$. □

Proposition 2 *A set S is externally perfect if and only if S is a perfect dominating set.*

Proof. If a set S is externally perfect, then every vertex $v \in V - S$ is perfect. This means that $|N[v] \cap S| = 1$, which means that v is adjacent to exactly one vertex in S, which means that it is dominated exactly once by a vertex in S. Thus, S is a perfect dominating set. Conversely, if S is a perfect dominating set, then every vertex in $V - S$ is dominated exactly once, and is, therefore, perfect. □

Proposition 3 *A set S is completely perfect if and only if S is an efficient dominating set(or a perfect code).*

Proof. If a set S is completely perfect, then in particular every vertex $u \in S$ is perfect, which means that S is an independent set. Since S is also externally perfect, it is a perfect dominating set. The combination of being a perfect dominating set and an independent set makes S an efficient dominating set. Clearly, the converse is true, that an efficient dominating set is both independent and a perfect dominating set. □

Next, we can see that the concept of a set being almost perfect (ap) is equivalent to the concept of being irredundant.

Proposition 4 *A set S is almost perfect if and only if S is irredundant.*

Proof. If a set S is almost perfect, then every vertex $u \in S$ is either perfect or adjacent to a perfect vertex. Either u is an isolated vertex in $G[S]$, in which case it is perfect and is its own private neighbor, or u is adjacent to a perfect vertex, say w. But w cannot be in S. Thus, $w \in V - S$ and w is perfect because $|N[w] \cap S| = |\{u\}| = 1$. This means that w is an external private neighbor of u. Thus, every vertex $u \in S$ has a private neighbor, and hence S is irredundant. Conversely, if S is irredundant, then every vertex $u \in S$ is either its own private neighbor, in which case it is perfect, or has an external private neighbor, say w. But in this case w is perfect and therefore u is adjacent to a perfect vertex. Therefore, S is almost perfect. □

Corollary 3 *For any graph G, $ir(G) = \theta_{ap}(G) \leq \Theta_{ap}(G) = IR(G)$.*

Proposition 5 *If a set S is eap then it is a distance-2 dominating set.*

Proof. If a set S is externally almost perfect, then every vertex $v \in V-S$ is either perfect or adjacent to a perfect vertex. If v is perfect, then it must be adjacent to exactly one vertex in S, and therefore is dominated by S. If v is adjacent to a perfect vertex, say w, then either $w \in S$, in which case v is dominated by S, or $w \in V - S$, in which case w is dominated by S. In either case, v is within distance 2 of at least one vertex in S. □

The following table summarizes the various definitions in this paper.

Type of Perfection	$V-S$ is perfect	$V-S$ is almost perfect	No conditions on $V-S$
S is perfect	efficient dominating set	independent perfect neighborhood set	independent set
S is almost perfect	perfect minimal dominating set	perfect neighborhood set	irredundant set
No conditions on S	perfect dominating set	externally almost perfect set	any set

Table 1. Types of Perfection

The following inequality chain was first introduced in [5] and is now well studied in domination theory [9, 10].

Theorem 1 *For any graph G,*

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta_0(G) \leq \Gamma(G) \leq IR(G).$$

To this inequality chain we can now make a number of additions, all related to the concepts of perfection introduced here.

Theorem 2 *For any graph G, the following system of inequalities holds.*

$$\begin{array}{ccccccc}
 & \theta^{ap}(G) & \leq \theta(G) & \leq \theta_p^{ap}(G) & \leq \Theta_p^{ap}(G) & \leq \Theta(G) & \leq \Theta_{ap}(G) \\
 & & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} & \Vert & \Vert & \Vert \\
\theta_{ap}(G) & = ir(G) & \leq \gamma(G) & \leq i(G) & \leq \beta_0(G) & \leq \Gamma(G) & \leq IR(G) \\
 & \rotatebox{-90}{$\leq$} & \rotatebox{-90}{$\leq$} & \Vert & \Vert & \rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} \\
\theta^{ap}(G) & \leq ir^{ap}(G) & \leq \gamma^{ap}(G) & \leq i^{ap}(G) & \leq \beta^{ap}(G) & \leq \Gamma^{ap}(G) & \leq IR^{ap}(G) \\
\rotatebox{-90}{$\geq$} & \rotatebox{-90}{$\geq$} & & & & & \\
\gamma_{\leq 2}(G) & \theta(G) & & & & &
\end{array}$$

Proof.

(i) $\gamma_{\leq 2}(G) \leq \theta^{ap}(G)$. Proposition 5.

(ii) $\theta^{ap}(G) \leq ir^{ap}(G)$ and $\theta^{ap}(G) \leq \theta(G)$. These two inequalities follow from the definitions, since $\theta^{ap}(G)$ equals the minimum cardinality of a set that is externally almost perfect.

(iii) $ir^{ap}(G) \leq \gamma^{ap}(G) \leq i^{ap}(G) \leq \beta^{ap}(G) \leq \Gamma^{ap}(G) \leq IR^{ap}(G)$. This long inequality chain follows from the fact that every maximal independent set is minimal dominating, and every minimal dominating set is maximal irredundant.

(iv) $ir(G) = \theta_{ap}(G) \leq ir^{ap}(G)$ and $\theta(G) \leq ir^{ap}(G)$. It is interesting to observe that in [8] it was conjectured that for any graph G, $\theta(G) \leq ir(G)$. However, Favaron and Puech [7] subsequently found a counterexample by describing a graph with more than two million vertices for which this inequality does not hold. We can see from these inequalities that $ir(G)$ and $\theta(G)$ should not be comparable, since $ir(G) = \theta_{ap}(G)$ (Proposition 4) and $\theta_{ap}(G) \leq ir^{ap}(G)$. That is, $ir(G)$ is defined by a smallest *maximal* irredundant set in a graph G, while $ir^{ap}(G)$ is defined by a smallest maximal irredundant set whose complement is an ap set. At the same time, $\theta(G) \leq ir^{ap}(G)$, since $\theta(G)$ is defined in terms of a smallest ap set, not necessarily maximal irredundant, whose complement is an ap set.

(v) [8] $\theta(G) \leq \gamma(G)$. Let S be a γ-set of a graph G. Furthermore, let $S = S_1 \cup S_2$, where S_1 is the set of isolates of S in $G[S]$, and S_2 is the set of vertices in S that are adjacent to at least one other vertex in S. Each vertex in S_1 is S-perfect. Each vertex $v \in S_2$ must have at least one external private neighbor in $V - S$, call it v'. Therefore, let $S_2 = \{v_1, v_2, \ldots, v_k\}$ and let $S_2' = \{v_1', v_2', \ldots, v_k'\}$, where v_i' is an external private neighbor of $v_i \in S_2$. It follows that the set $S' = S_1 \cup S_2'$ is an almost completely perfect set having the same cardinality as S. Given S', every vertex in S_1 is perfect, every vertex in S_2 is also perfect, being adjacent to exactly one vertex in S_2', and since S is a γ-set, every vertex in $V - S'$, is adjacent to at least one perfect vertex in S.

(vi) $\gamma(G) \leq \gamma^{ap}(G)$. This follows from the simple observation that every γ^{ap}-set is a dominating set. The fact that this inequality can be strict is illustrated by a graph G consisting of a triangle, two of whose vertices, say u and v, are adjacent to two leaves each. For this graph $\gamma(G) = 2$, yet $\gamma^{ap}(G) = 3$.

(vii) $\theta(G) \leq \theta_p^{ap}(G)$. This inequality is immediate, since every independent perfect neighborhood set is a perfect neighborhood set.

(viii) $\theta_p^{ap}(G) \leq i(G)$. This follows from the observation that every maximal independent set S is both perfect and externally almost perfect.

(ix) $i(G) = i^{ap}(G)$. This is proved in Corollary 1.

(x) $\Theta_p^{ap}(G) = \beta_0(G) = \beta^{ap}(G)$. This is proved in Corollary 2.

(xi) $\Theta_p^{ap}(G) \leq \Theta(G) \leq \Theta_{ap}(G)$. Every Θ_p^{ap}-set S is a set that is perfect and externally almost perfect. Thus a maximum cardinality set that is both almost perfect and externally almost perfect, i.e. a θ-set, is at least this large. Similarly, every Θ-set is a set that is almost perfect, and hence $\Theta(G) \leq \Theta_{ap}$.

(xii) $\Gamma(G) = \Theta(G)$. This is proved in [8].

(xiii) $\Gamma^{ap}(G) \leq \Gamma(G)$. This follows from the observation that Γ^{ap}-sets are always minimal dominating sets, and $\Gamma(G)$ equals the maximum cardinality of a minimal dominating set.

(xiii) $IR(G) = \Theta_{ap}(G)$. This follows from the observation above that every maximal irredundant set is a maximal almost perfect set, and conversely.

(xiv) $IR^{ap}(G) \leq IR(G)$. The notation $IR^{ap}(G)$ describes the maximum cardinality of a set S that is maximal irredundant and whose complement $V - S$ is an ap set, but $IR(G)$ equals the maximum cardinality of an irredundant set without the added requirement that the complement be an ap set. □

It should be mentioned that no inequality holds between the two parameters $\Theta(G)$ and $\Theta^{ap}(G)$, since Θ-sets are eap-sets, but Θ^{ap}-sets must be *minimal* eap-sets.

A similar pair of inequality chains holds when independent sets are considered.

Proposition 6 *For any graph G, the following inequalities hold.*

(i) $\gamma_{\leq 2}(G) \leq \gamma_{\leq 2}^{ap}(G) \leq i_{\leq 2}^{ap}(G) \leq i^{ap}(G) = i(G)$.

(ii) $\gamma_{\leq 2}(G) \leq i_{\leq 2}(G) \leq i_{\leq 2}^{ap}(G) \leq i^{ap}(G) = i(G)$.

One final observation about perfection parameters is this. Define $\gamma_d^{ap}(G)$ to equal the minimum cardinality of a dominating set that is externally almost perfect. This is different than $\gamma^{ap}(G)$, which equals the minimum cardinality of a *minimal* dominating set that is externally almost perfect. In fact the following is true.

Proposition 7 *For any graph G, $\gamma(G) \leq \gamma_d^{ap}(G) \leq \gamma^{ap}(G) \leq i(G)$.*

The fact that each of these inequalities can be strict is illustrated by the graph G in Figure 1. For this graph it can be seen that $\gamma(G) = 4$, let $S_1 = \{1,3,4,6\}$; $\gamma_d^{ap}(G) = 6$, let $S_2 = \{1,2,3,4,5,6\}$; $\gamma^{ap}(G) = 8$, let $S_3 = \{1,6,7,8,9,10,11,12\}$, and $i(G) = 9$, let $S_4 = \{1,7,8,9,4,13,14,15,16\}$.

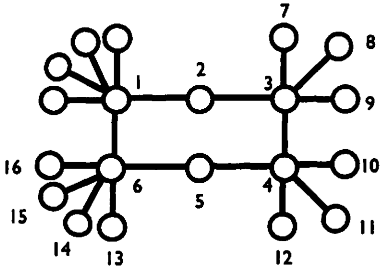

Figure 2: $\gamma < \gamma_d^{ap} < \gamma^{ap} < i$

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