Cops-and-robbers: remarks and problems

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Abstract

We explore cops-and-robbers games in several directions, giving partial results in each and refuting two reasonable conjectures. We close with some open problems.

Keywords: cops-and-robbers, graph, directed graph, tournament, reflexive graph, graph searching, optimal game, constrained game

1. Introduction and basics

The purpose of this short paper is to collect partial results from several M.Sc. theses, to indicate some unexplored directions of research and to refute several conjectures that seemed reasonable at the time they were made. Further, we suggest a new - and we hope useful - generalization and prove a few results about it. The common beginning is the original cop-and-robber game as defined by Nowakowski and Winkler [31] and, independently, by Quillot [32, 33]. We shall not mention the many ideas, results and variants that appeared since these first papers were written. The interested reader might wish to consult the surveys [1], [20], [22], and the just published book [7]. For undefined terminology and notation see, for example, [2]. A finite graph $G = (V, E)$ will have $|V| = n$.

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[1]The author gratefully acknowledges the support of the Natural Science and Engineering Research Council of Canada.

Let us begin by defining the classical game. We will go through the easy exercise of making definitions very precise so that they appear in print at least once; after all, *intuitively clear* and *obvious* statements sometimes turn out not to be true, often because of intuitive definitions.

The game is played by two players on a reflexive (that is, with a loop at each vertex) graph $G = (V, E)$. For the rest of this section, let us fix a graph G. The players are the *cop* and the *robber* (for future reference we can think of them as tokens to be placed on vertices, say red for the cop and black for the robber). The game is played in *rounds*, each consisting of two *moves*, one by the cop, followed by one by the robber, in that order. The rounds are numbered by natural numbers. At round 0, the cop chooses a vertex u (places a red token on it), then the robber chooses one, v (placing his black token on it). When i rounds have been played, the players (their tokens) are on vertices c_i and r_i, respectively. In round $i+1$, the cop moves to a vertex $c_{i+1} \in N(c_i)$ and then the robber moves to a vertex $r_{i+1} \in N(r_i)$; as usual, $N(u)$ is the neighbourhood of u and we stress the fact that $u \in N(u)$ in a reflexive graph [2]. The cop wins the game if she occupies the same vertex as the robber after a finite number of rounds, otherwise the robber wins. Observe that there are two ways for the cop to win: either the robber (but why?) moves to the cop's vertex, or the cop moves to the robber's vertex. In the latter case the round would be unfinished and we use the convention that in such a case the robber's move is to his current vertex. When the cop wins, the game stops. The game is one of perfect information - each player knows the graph and the positions of both, and the positions define the players' strategies. A *play* is a sequence $\{(c_i, r_i)\}_{i \in I}$ such that $c_i \in N(c_{i-1})$, $r_i \in N(r_{i-1})$, $0 \neq i \in I$ and either (1) $I = [k] = \{0, 1, \ldots, k-1\}$, $c_i \neq r_i, r_{i-1}$ for $0 \neq i < k-1$ and $c_{k-1} = r_{k-2}$, $r_{k-1} = r_{k-2}$ (with our convention), or (2) $I = \mathbb{N}$ and $c_i \in N(c_{i-1})$, $r_i \in N(r_{i-1})$ and $c_i \neq r_i, r_{i-1}$ for $0 \neq i \in I$. Thus the cop wins if $I = [k]$ for some $k \in \mathbb{N}$, otherwise $I = \mathbb{N}$ and the robber wins.

A *strategy* for a player p ($p = c$ or $p = r$ with the obvious interpretation) is a function $\sigma_p : (V \cup \{\varepsilon\}) \times (V \cup \{\varepsilon\}) \longrightarrow V$ that tells the player where to go, based on the current positions, with ε indicating that the player is not on the graph. More precisely, at the beginning, both players are off the graph and the only moves are defined by $\sigma_c(\varepsilon, \varepsilon) = c_0$ and $\sigma_r(c_0, \varepsilon) = r_0$. Subsequent moves are given by $\sigma_c(c_i, r_i) = c_{i+1} \in N(c_i)$ and $\sigma_r(c_{i+1}, r_i) = r_{i+1} \in N(r_i)$. Thus a pair of strategies (σ_c, σ_r) defines a play $P_{(\sigma_c, \sigma_r)}$. A strategy is *winning* for p if its use leads to a win by p no matter what the opponent does. That is, σ_c is winning if there is a $k \in \mathbb{N}$ such that $c_j = r_{j-1}$ for some $j \leq k$ for any σ_r. A play is then a sequence $\{(c_i, r_i)\}_{i \in I}$ defined by σ_c and σ_r: $(c_{i+1}, r_{i+1}) = (\sigma_c(c_i, r_i), \sigma_r(\sigma_c(c_i, r_i), r_i))$. It is clear that while a pair of strategies defines a play, the converse is not necessarily true. The *length* of a play $P = \{c_i, r_i)\}_{i \in I}$ is simply $|I|$, that is, the number of rounds before the game is over (including the initial round of placing the tokens on the graph). We will denote it by $\|P\|$.

[2]some authors prefer to use simple graphs and allow the players to *pass* (stay where they are), but this equivalent definition does not generalize the way we wish.

The *efficiency* of a cop's strategy σ_c is $\|\sigma_c\| = \max\{\|P_{(\sigma_c,\sigma_r)}\| : \sigma_r$ is robber's strategy $\}$. For the robber, define $\|\sigma_r\| = \min\{\|P_{(\sigma_c,\sigma_r)}\| : \sigma_c$ is cop's strategy $\}$ to be the *freedom* of σ_r. Clearly $\|\sigma_c\| \geq \|\sigma_r\|$ for any strategies of the cop and the robber.

We say that a graph is *cop-win* if the cop has a winning strategy. Obviously, a cop-win graph is connected. If a graph is not cop-win, then the robber must have a winning strategy (by von Neumann's theorem, or by direct observation). Let us say that a strategy σ_c is *optimal* for the cop if no cop's strategy has smaller efficiency. Similarly, a strategy σ_r is optimal for the robber if no robber's strategy has greater freedom. A play $P_{(\sigma_c,\sigma_r)}$ on G is optimal if both σ_c and σ_r are. In human language, an optimal play is one where the cop plays to win as fast as possible while the robber plays to remain free as long as possible. A graph is then cop-win if and only if an optimal play on it is finite, that is, the cop's strategy has finite efficiency. In Section 2 we will consider the length of an optimal play on a a cop-win graph.

Since putting a cop on each vertex of a graph guarantees that the robber is caught, it makes sense to define the *cop number* (or *search number*) of a graph as the minimum number of cops that have a winning strategy against one robber on that graph. We wish to make this more precise as there are several interpretations possible. The simplest way is to extend the game even further, to ℓ robbers as well. In what follows, we will need one more piece of notation. For a set X and a natural number $k > 0$, if $(x_1, \ldots, x_k) \in X^k$, write $\sup(x_1, \ldots, x_k) = \{x_1, \ldots, x_k\}$. Let $G = (V, E)$ be the graph on which the two players play. Instead of placing one token on some vertex of G, the cop now places k red tokens on some vertices of G, the robber places ℓ black ones. At each round, each player moves all his tokens along edges incident with the vertices currently occupied by them. The cop wins if all the robber's tokens are on vertices where there are black tokens as well. If no vertex can have more than one token of one colour, the strategies in this case are $\sigma_c : V^k \times V^\ell \longrightarrow V^k$ and $\sigma_r : V^k \times V^\ell \longrightarrow V^\ell$ such that $\sigma_c((u_1, \ldots, u_k), (v_1, \ldots, v_\ell)) = ((u'_1, \ldots, u'_k), (v_1, \ldots, v_\ell))$ only if for each $i = 1, \ldots, k$, $u'_i \in N(u_i)$ and, similarly, for σ_r. A play can then be defined as a sequence $\{(c_i, r_i)\}_{i \in I}$ with each c_i and each r_i being a vector in V^k and V^ℓ, respectively, with the rest as before, mutatis mutandis. It is usual, however, to allow several cop tokens on the same vertex, and similarly for several robber tokens. The functions then become a bit more complicated - one way of doing it is to consider vectors indexed by V whose v-th entry is the number of tokens of the appropriate colour that are on v. We will not give the precise definitions here but will trust the reader can produce them. The k-cop/1-robber games considered here do allow several cops on the same vertex. Note that other rules are possible but we do not consider them here; see Section 7.

The main questions that have been addressed fall into several overlapping classes. We only consider connected graphs since the questions asked can be answered for disconnected ones by looking at the connected components (it is very slightly more complicated for infinite graphs).

- How many cops are sufficient to catch one robber on a finite graph? Can we at least bound the search number for special classes of graphs?

- (*k−cop-win graphs*) Can the graphs on which $k \in \mathbb{N}$ cops are necessary and sufficient to catch one robber be characterized (i.e. graphs with search number k?
- (*complexity*) How hard is it to determine the search number of a graph?
- What kind of rules do we need so that the search number of a graph is equal to some graph width?

We shall not be concerned here with the last two classes and will only touch upon the first two, adding instead some new questions. But let us mention a few important results, if only because they form a basis of much of what has been done later. They are mostly from [31] and [32, 33].

Recall that a homomorphism of a graph $G = (V, E)$ into a graph $H = (U, F)$ is an edge preserving mapping $h : V \longrightarrow U$ and that a retraction is a homomorphism from G into G which is the identity on the image.

Theorem 1. *Any retraction of a cop-win graph is cop-win.*

Corollary 2. *The graph obtained from a finite cop-win graph by retracting a vertex occupied by the robber after the penultimate round onto that occupied by the cop at that round (and leaving all other vertices intact) is cop-win.*

Theorem 3. *A graph on $n \in \mathbb{N}$ vertices is cop-win if and only if its vertices can be enumerated $v_0, \ldots, v_{n-1}$ so that for each $n-1 > i \in [n]$ there is a $i < j \in [n]$ such that $N_i(v_i) \subseteq N_i(v_j)$ (here $N_i(u) = N(u) \cap \{v_i, \ldots, v_{n-1}\}$).*

It is a long-standing open problem to characterize k−cop-win graphs for $k > 1$, in spite of [24] and [13] since the latter papers use auxiliary graphs (the former runs a recognition algorithm on an auxiliary graph, the latter deduces whether or not the original graph is k−cop-win from properties of an auxiliary graph). For infinite graphs, the best characterization of cop-win graphs we have comes from [31] and we describe it now since we use an extension of it in Section 5. The algorithm of [24] is based on the following theorem (even though it was found independently).

Theorem 4. *A reflexive graph G is cop-win if and only if the binary relation R on $V(G)$ defined below is trivial.*

Define R by first defining a sequence of relations R_α for ordinals $\alpha \leq |V(G)|$ (we take the view that a cardinal is the least of all equipotent ordinals). This is done recursively.

1. $R_0 = \{(u, u) : u \in V(G)\}$;
2. for $\alpha > 0$, $(u, v) \in R_\alpha$ if for each $x \in N[u]$ there is a $\beta < \alpha$ and a $y \in N[v]$ with $(x, y) \in R_\beta$.

Since $R_\alpha = R_{|V(G)|}$ for all $\alpha > |V(G)|$, there is a least γ such that $R_\alpha = R_\gamma$ for $\alpha > \gamma$. Observe that $R_\alpha \subseteq R_\beta$ if $\alpha \le \beta$. Set $R = R_\gamma$. Note that $(u, v) \in R_\alpha$ if the cop at v can catch the robber (currently) at u in at most α steps.

For the rest of the paper we will assume that our graphs are reflexive, connected and finite, unless otherwise stated (in particular, in Section 6). In pictures, loops are not drawn for reflexive graphs in any of the figures.

2. Length of the game

All graphs considered in this section are cop-win. The last result of the preceding section brings us to the little studied question of the length of an optimal play in particular, and of properties of an optimal play in general. The problem was studied in [3] and we can use some of its terminology. The *capture time* of a graph G, denoted by capt(G), is the length of an optimal play. The tight upper bound on capture time is $n - 4$ and was obtained in [21], where graphs realizing the bound are also characterized. Capture time is a natural parameter to study and we learnt recently - [28] - that it is of much interest to roboticists. Indeed, the simple algorithm of [24] is used in robotics and we note that it provides optimal strategies for both the cop and the robber in addition to determining the capture time if it determines that the graph is k−cop-win (any $k \in \mathbb{N} \setminus \{0\}$). In studying this parameter, Hahn made two conjectures that could help in analyzing it.

Conjecture 1. In an optimal play, the cop visits each vertex at most once.

Conjecture 2. In an optimal play the distance between the cop and the robber is a non-increasing function of time. That is, $d(c_i, r_i) \le d(c_{i-1}, r_{i-1})$ for $i \in I$.

Unfortunately, both are wrong. Since the conjectures are rather tempting, we will give a counterexample to each in the following subsections. The counterexamples actually disprove even slightly weaker conjectures.

2.1. A vertex must be revisited

We disprove the following conjecture.

Conjecture 3. For each cop-win graph G there is an optimal play during which the cop visits no vertex more than once.

Proposition 5. *There is a cop-win graph such that in any optimal play the cop must visit some vertex more than once.*

Proof. We construct a graph with unique optimal strategies (up to symmetry) and so a unique optimal play during which the cop has to pass over one of the vertices twice. The graph is depicted in Figure 1. Due to its symmetry in c and its shape we will refer to it as two fishes that are connected in vertex c. This graph induces a single optimal strategy for the cop. First, the cop has to choose his initial position. It will be shown below that the unique optimal choice is

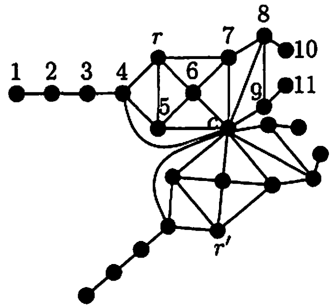


Figure 1: Example graph where the cop revisits vertex c when both players play optimally.

c. Then, the robber chooses his initial position to be either r or r'. Since the graph is symmetric in c we will only consider the play in the upper part i.e., when the robber chooses r. The play requires nine moves with the cop moving first: $(r,c) \stackrel{\text{cop}}{\to} (r,5) \stackrel{\text{robber}}{\to} (7,5) \stackrel{\text{cop}}{\to} (7,6) \stackrel{\text{robber}}{\to} (8,6) \stackrel{\text{cop}}{\to} (8,c) \stackrel{\text{robber}}{\to} (10,c) \stackrel{\text{cop}}{\to} (10,8) \stackrel{\text{robber}}{\to} (10,8) \stackrel{\text{cop}}{\to} (10,10)$.

In the following, we will outline why this play is the unique optimal play (except for the robber's symmetric choice of r or r'). First, consider a different initial position $v \neq c$ for the cop. This vertex has to be in one of the two fishes. By choosing the respective vertex r in the other fish and waiting there until the cop reaches c, the robber can guarantee longer survival. Hence, the cop starts in c.

The robber can choose his initial position in either of the two fishes. Without loss of generality we consider the upper fish only. If he chooses vertices $4,5,6,7,8$ or 9 he will get caught by the cop within one move. If he chooses to start in $1,2$ or 3 the cop will move to 4 in his first move and the play will take at most seven moves. If he chooses 10 or 11 the cop will move to 8 or 9, respectively, yielding a maximum survival of three moves. It remains to show that choosing r guarantees survival for at least nine moves. We leave this as an exercise to the reader. Note that the move back to c is necessary to prevent the robber moving to 9 and delaying capture for two more moves. □

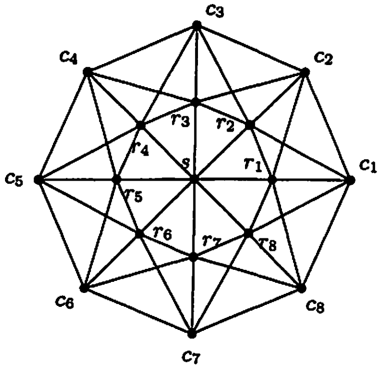
2.2. Distance must increase

The conjecture disproved is this.

Conjecture 4. For each cop-win graph, there is an optimal play such that the distance between the cop and the robber does not increase at any step.

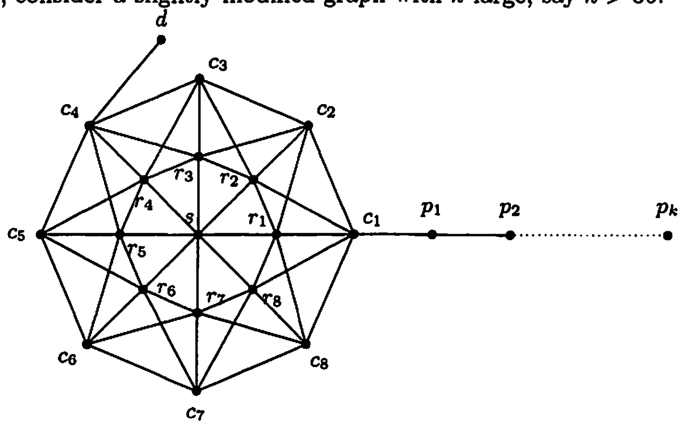
Proposition 6. *There is a graph such that in any optimal play the distance between the cop and the robber is not a non-decreasing function of time.*

Proof. First, consider the following graph:



Notice that if a cop is forced to start at any vertex c_j, she can catch the robber in 4 rounds. To do so, she first moves to s. A smart robber must go to some c_i, otherwise he will be caught immediately. The cop moves to r_i, and no matter what the robber does, he is caught in the next round due to the claw-like edges from r_i. On the other hand, if the cop does not move to s, she can never catch a smart robber.

Now, consider a slightly modified graph with k large, say $k > 30$.



On this graph, the cop would start somewhere near the middle of the long path $p_1 \dots p_k$ to minimize the length of the game. Assume the robber then chooses to start on the side of the path connected to c_1. The cop will move toward c_1. Once she reaches c_1, the only way for her to catch a smart robber is to go to s through r_1, r_2 or r_8. When choosing the path, the cop can prevent the robber from reaching any neighbour of c_1 before the cop reaches s – indeed, the cop can force the robber to move to a distance of at least 3 from c_1 after his choice of r_1, r_2 or r_8. Since the robber cannot reach c_1 in time to escape through the long path, he will be caught in two rounds after the cop reaches s if he stays on any c_i. A smart robber, however, will move to d from c_4 when he sees the cop moving to s in order to maximize his freedom. Notice that the distance between r_2 (where the cop moved to prevent the robber escaping through c_1)

and c_4 is smaller than the distance between s and d, yet both players are playing optimally. □

3. Tournaments

One of the challenges of cops-and-robbers games is the class of games played on directed graphs. Indeed, next to nothing is known and, a fortiori, a characterization of cop-win directed graphs is nowhere in sight. Here, as with k cops, the retraction that is crucial to the proof of the characterization of cop-win graphs is not to be had, at least not in any nice way that anyone has seen. The best we can do at the moment is to investigate the obvious special class of directed graphs, tournaments. Recall that a tournament is simply an oriented complete graph (that is, each of edge is given a direction) Ours have a directed loop at each vertex, contributing 1 both to the in-degree and the out degree. Recall that a dominating vertex in a tournament is a vertex of out-degree n (with the loop contributing 1). Hill [27] looked at tournaments and some of the results here are due to him. The following is easy to see.

Proposition 7. *A tournament is cop-win if and only if it has a dominating vertex.*

Proof. Obviously, if there is a dominating vertex, then a cop stationed at it wins in one round. Conversely, if there is no dominating vertex, then after the cop's choice of c_0, the robber chooses r_0 such that (r_0, c_0) is an arc and then simply follows the cop so that the play becomes $\{(c_i, c_{i-1})\}_{i \in \mathbb{N} \setminus \{0\}}$. □

Nowakowski [29] conjectured that tournaments obtained from Steiner triple systems (see Section 7 for more) are 2−cop-win. This was disproved by Thériault, who programmed the algorithm of [24] and used it to find a (small) counterexample.

There is one advantage to directed graphs - it is possible to use degrees to obtain results, unlike in the case of undirected graphs. Let $\Delta^+(T)$ be the maximum out-degree of a vertex in a tournament $T = (V, A)$ (A is the set of arcs).

Lemma 8. *If a tournament T has $\Delta^+(T) = n - 1$, it is 2−cop-win.*

Proof. Recall that our graphs are reflexive so that the condition means that there is a vertex v dominating $n - 2$ other vertices. We already know that one cop is not sufficient to win on T and we prove that two are. This is clear: one cop goes on v, the other on the unique vertex not dominated by v. The robber is caught in one round. □

This generalizes in a standard way to an easy and well known result.

Lemma 9. *If a tournament T has n vertices then $\lceil \lg n \rceil$ cops can catch the unique robber.*

Proof. As always, $\lg n = \log_2 n$. In $T = T_0$, there is a vertex v_0 of out-degree at least $\frac{n}{2}$. Let T_i be the tournament obtained from T_{i-1} by deleting $N_{i-1}^+(v_{i-1})$ with $N_{i-1}^+(v)$ denoting the out-neighbourhood of v in T_{i-1} (this also deletes v_{i-1}). Then T_i contains a vertex v_i dominating at least half of the vertices in T_i. Clearly T_i is reduced to at most one vertex for $i \leq \lceil \lg n \rceil$. The cops on the vertices v_i dominate T and so win. □

Of course the bound is on the size of a dominating set in a (directed) graph and there is a priori no relation between that and the search number (a k-ary rooted tree is a good example). If, however, the conjecture of [6] is correct and almost all k-cop-win graphs have a k-vertex dominating set then the graphs where the two parameters differ are rare.

Lemma 10. *If a tournament T has $\Delta^+(T) = n - k$ for some $0 \leq k \leq \frac{n}{2}$ then $1 + \lceil \lg k \rceil$ cops win.*

Proof. This is easy to see using the preceding proof on the k vertices not dominated by a vertex of maximum out-degree. □

Sometimes we can do better. To simplify exposition, let us extend the play notation to k cops. In this case a play is a sequence $\{(c_i, r_i)\}_{i \in I}$ with $c_i \in V^k$. We omit the inner parentheses and write, for example, $(c_i, r_i) = (u_1, u_3, u_3, u_7, u_{12})$ to indicate that the cops on the first four vertices and the robber on the last one (this can be extended in the obvious way to more than one robber).

Lemma 11. *A tournament T with $\Delta^+(T) = n - 3$ is 2-cop-win.*

Proof. Let v be a vertex of maximum out-degree and let u_0, u_1 be the two vertices not dominated by v. Without loss of generality assume that $u_0u_1 \in A$. Two cops at v and u_0 dominate all vertices and win. □

Lemma 12. *A tournament T with $\Delta^+(T) = n - 4$ is 2-cop-win.*

Proof. Let v be a vertex of maximum out-degree and let $U = \{u_0, u_1, u_2\}$ be the set of the three vertices not dominated by v. If U does not induce a directed cycle, let u_0 (without loss of generality) dominate the other two. The cops at v and u_0 win. If $u_0u_1u_2$ is a directed cycle, consider a vertex $z \in N^+(v)$ such that $|N^+(z) \cap U|$ is maximum. We have three cases to consider. If $U \subseteq N^+(z)$, then $\{v, z\}$ is a dominating set and the cops there win. If $|U \cap N^+(z)| = 2$, assume, without loss of generality, that $zu_0, zu_1 \in A$. The cop's winning play then begins $\{(v, u_0, u_1), (z, v, x)\}$ and the robber is caught on the next move no matter what. In the last case, $|U \cap N^+(z)| = 1$ and the cop wins by beginning $\{(u_0, u_1, x)\}$ with $x \in N^+(v)$. Since one of the two cops now controls the unique (if any) vertex of U accessible from x, the other can move to v and the cops win on the next move at the latest. □

A similar lemma can be shown for tournaments with maximum out-degree $n-5$. We spare the reader the case-by-case proof (first observe that the four vertices not dominated by a vertex v of maximum out-degree induce one of two non-isomorphic tournaments and analyze the situation for each of them when the maximum number of vertices dominated by a vertex in $N^+(v)$ is one, two, three or four; it cannot be zero and the last case is almost trivial.)

Lemma 13. *A tournament T with $\Delta^+(T) = n-5$ is 2−cop-win.*

Given a Steiner triple system on a set V, one can construct a tournament by putting a directed cycle on each of the triples. Nowakowski (private communication) conjectured that such a tournament has search number 2. This was disproved by a computer search, see [34] but leaves an open question 11.

4. Non-reflexive graphs

The classic game is played on graphs with loops, or, alternately, on simple graphs where the players can "pass", that is, stay where they are. It seems reasonable to re-interpret the rules: on a reflexive graph, each player MUST move on his turn by going to some neighbour of the vertex currently occupied. This is how the game is played and so it makes sense to play with the same rules on graphs with some, but not necessarily all, loops. This is a different game, as the following simple examples show.

Example 1. The house. It is not cop win if it is reflexive, but becomes cop-win with a loop only on the roof. The house graph consists of a four-cycle together with a fifth vertex joined to two adjacent vertices of the cycle; this latter triangle is the roof.

Example 2. The graph consisting of a four-cycle and a triangle that share exactly one vertex is robber-win if it is reflexive and cop-win with no loops or if a loop is added to just the vertex on the four-cycle that is at distance two from the vertex shared by the triangle and the four-cycle.

On the other hand, the two games are obviously related, as the following observations show. Let G be a graph some of whose vertices have loops. Let G° be the graph obtained from G by adding a loop to each vertex that does not have one. Let $\tilde{G}$ be the graph obtained from G by removing all the loops.

Lemma 14. *Let $G = (V, E)$ be a graph.*

1. *If G° is cop-win then two cops can win on G.*
2. *If G is cop-win then two cops can win on $\tilde{G}$.*
3. *If G° is cop-win then two cops can win on $\tilde{G}$.*

Proof. Proofs of all three statements are essentially the same. The two cops move together on the second graph in such a way that they are always on adjacent vertices after their move, following the winning strategy on the first graph and exchanging positions if the winning strategy has the cop move along a loop. To be more precise, if the cop's strategy on the first graph is σ then the two cops' strategy σ' on the second graph is defined by setting $\sigma'(\varepsilon,\varepsilon,\varepsilon) = (\sigma(\varepsilon,\varepsilon), v_0, \varepsilon)$ for a randomly chosen $v_0 \in N(\sigma(\varepsilon,\varepsilon))$, and $\sigma'(u,v,z) = (u',u,z)$ with $u' = \sigma(u,z)$ if $\sigma(u,z) \neq u$ and $v' = v$ otherwise. This is clearly a winning strategy for two cops. □

Lemma 15. *If $\tilde{G}$ is $k-$cop-win then so is G°.*

Proof. On G°, the k cops use the winning strategy they have for $\tilde{G}$. □

The above result depends on the precise definition of "strategy" we gave here. If the usual intuitive "definition" is used, the claim might be false. Indeed, we often think of a strategy as a partial function that tells us what to do in all *expected* cases. With this notion, the cops would not know what to do in G° if the robber decides to stay where he is and the Lemma would not be true. Of course, we can deal with this.

Lemma 16. *If $\tilde{G}$ is $k-$cop-win then $k+1$ cops can win on G°.*

Proof. On G°, k cops use the winning strategy they have for $\tilde{G}$ as long as the robber does not move along a loop, one stays at some random vertex. If the robber moves along a loop, the k cops do as well while the $(k+1)-st$ cop moves toward the robber and eventually forces him to move to a new vertex. Then the original strategy is applied again. □

It is obvious that a complete graph is always cop-win, no matter how many vertices have loops. It is very easy to see that the same is true for any tree. Consider now a tournament without loops (irreflexive tournament). Proposition 7 applies here as well. Further, Lemmas 11, 12 and 13 extend to non-reflexive tournaments.

Lemma 17. *If T is an irreflexive tournament on n vertices with $\Delta^+(T) \geq n-4$ then T is either cop-win or $2-$cop-win.*

Proof. If $T = (V, A)$ has a dominating vertex, it is cop-win. If $n-1 > \Delta^+(T) > n-4$, consider a vertex v with $d^+(v) = \Delta^+(T)$ and $R = V \setminus N^+[v]$ (recall that $N^+[v] = N^+(v) \cup \{v\}$). By the assumption, $|R| \leq 2$. without loss of generality, $R = \{u_0\}$, or $R = \{u_0, u_1\}$ with $u_0u_1 \in A$. The cops begin the play by (v, x, ε) and wherever the robber goes, he is caught. If $\Delta^+(T) = n-4$, the proof of Lemma 12 can be used mutatis mutandis. □

As in the case of reflexive tournaments, there is more, and as there, we omit the case-by-case proof.

Lemma 18. *A tournament T with $\Delta^+(T) = n-5$ is $2-$cop-win.*

5. Constrained game

One very natural extension of the game is to separate the rules for the cop(s) and the robber(s). One way to do this is to give them different means - different speeds, or helicopters for the cops, etc. This has been exploited to great advantage in many papers, among other things to prove the connection to various graph widths. See [8] and [20] for more and for references. Another way is to allow the players different edges. This has been studied, for example, in [30]. Clarke explored the idea of forcing two cops to move in a tandem, that is, to be on distinct adjacent vertices after each move, in [9, 14, 15]. Her thesis title led to the idea of generalizing further: we can require that the set of k cops occupy a certain configuration in the graph and that the cops and the robber move on distinct (but not necessarily disjoint) sets of vertices. This can be generalized to more than one robber (see also Section 7). Here we show how a common generalization can be formalized. A graph in this section is not necessarily reflexive or irreflexive, that is, the graphs are as in Section 4.

Let $G_c = (V_c, E_c)$ and $G_r = (V_r, E_r)$ be (not necessarily connected) graphs and let $\leq_0 \subseteq V_r \times V_c$ be a *capture relation*. The cop plays on G_c, the robber on G_r, with a play $\{(c_i, r_i)\}_{i \in I}$ defined as in Section 1; of course a strategy for the player p is now a function $\sigma_p : V_c \times V_r \longrightarrow V_p$. The robber is captured if $r_i \leq_0 c_i$ for some $i \in \mathbb{N}$. As in [31], we can define a relation $\leq_k$ for $0 < k \in \mathbb{N}$ recursively. For vertices $u \in V_r$ and $v \in V_c$, $u \leq_k v$ if, and only if, for each $x \in N_{G_r}(u)$ there is a $y \in N_{G_c}(v)$ and a $k > j \in \mathbb{N}$ such that $x \leq_j y$. We can think of the $u \leq_k v$ as *the cop at v can capture the robber at u in at most k rounds.* Clearly $\leq_k \subseteq \leq_{k+1}$ for $k \in \mathbb{N}$ and since the graphs are finite, there is a least $k \in \mathbb{N}$ such that $\leq_k = \leq_{k+1}$. Set $\preceq = \leq_k$. We will need the following obvious lemma.

Lemma 19. *If $u \not\preceq v$ for some $u \in V_r$, $v \in V_c$, then there is an $x \in N_{G_r}(u)$ such that $x \not\preceq y$ for any $y \in N_{G_c}(v)$.*

Observe also that by the rules of the game, a play started in a connected component C of G_c and a connected component R of G_r never leaves the respective components. Thus only connected graphs need be considered, as before. A *game* is a triple $(G_c, G_r, \leq_0)$. It is *cop-win* if the cop has a winning strategy, otherwise it is robber-win. We can now prove a theorem analogous to that of [31]. As there, the proof works for infinite graphs with very slight modifications, see Section 6.

Theorem 20. *A game $(G_c, G_r, \leq_0)$ is cop-win if, and only if, the relation $\preceq$ is trivial and non-empty.*

Proof. Recall that "trivial" means that $u \preceq v$ for any $u \in V_r$ and any $v \in V_c$ and that our graphs are connected. Assume first that the relation is trivial. For $(u, v) \in V_r \times V_c$, let $k_{u,v} \in \mathbb{N}$ be the least such that $u \leq_{k_{u,v}} v$. The cop's strategy is to start at an arbitrary vertex c_0 of her graph and when the robber chooses r_0 in his graph, to go to $\sigma_c(c_i, r_i) = c_{i+1}$ such that $k_{c_{i+1}, r_i} < k_{c_i, r_i}$; such a $c_{i+1} \neq c_i$ exists by the definition of $\preceq$. Since the sequence $\{k_{c_i, r_i}\}_{i \in \mathbb{N}}$ is strictly decreasing, it eventually reaches zero and stops, and the robber is captured. Conversely, let $(u, v) \in V_r \times V_c$ be such that $u \not\preceq v$. Then by Lemma 19, the robber clearly has a winning strategy if the cop starts at v. If the cop had a winning strategy σ_c with $\sigma_c(\epsilon, \epsilon) = c_0$, she would also have one from v. Indeed, define $\sigma'_c(\epsilon, \epsilon) = v = v_0$ (start at v), $\sigma'_c(v_i, r_i) = v_{i+1}$ for $i = 0, \ldots, \ell$ and $v = v_0 \ldots v_\ell = c_0$ (migrate to c_0). Now by the definition of $\preceq$ and the assumption that the cop wins starting at c_0, for any $x \in N_{G_r}(r_\ell)$ there is a $y \in N_{G_c}(c_0)$ such that $x \leq_k y$ for some $k < k_{r_\ell, c_0}$. This contradicts Lemma 19 and the assumption that $u \not\preceq v$. □

Let $(G_c, G_r, \leq_0)$ and $(H_c, G_r, \sqsubseteq_0)$ be games. Let $\rho : V(G_c) \longrightarrow V(H_c)$ be a surjective graph homomorphism such that for $u \in V_r$ and $v \in V(H_c)$, $u \sqsubseteq_0 \rho(v)$ whenever $u \leq_0 v$. Call such a homomorphism *capture-preserving*.

Lemma 21. *Let $(G_c, G_r, \leq_0)$ be a cop-win game and let $(H_c, G_r, \sqsubseteq_0)$ be a game. Let $\rho : V(G_c) \longrightarrow V(H_c)$ be a capture-preserving graph homomorphism. If $(G_c, G_r, \leq_0)$ is cop-win then so is $(H_c, G_r, \sqsubseteq_0)$.*

Proof. Half of the work is done in the remark preceding the lemma. Since ρ is onto, when $\leq$ is trivial, so is $\sqsubseteq$. □

Corollary 22. *Let $(G_c, G_r, \leq_0)$ be a cop-win game and let H_c be a retract of G_c such that some retraction is capture-preserving. Then $(H_c, G_r, \sqsubseteq_0)$ is also cop-win, with $\sqsubseteq_0$ being the relation induced on $V_r \times V(H_c)$ by restricting $\leq_0$.*

Sometimes we can *lift* a strategy from a homomorphic image of a graph to its pre-image.

Definition 23. Let G be a graph and let $\simeq$ be an equivalence relation on $V(G)$. We say that $\simeq$ has the *lifting property* if for any $u \simeq u'$ and any $v \in N(u)$ there is a $v' \in N(u')$ such that $v \simeq v'$.

Let us recall and slightly extend some standard ideas on homomorphism (see [25] and/or [26] for more). Given a graph G and an equivalence relation $\simeq$ on $V(G)$, we can define a new graph on $G/\simeq$ whose vertices are the equivalence classes of $\simeq$, joined if some vertex of one class is adjacent to some vertex of the second class. More precisely, $V(G/\simeq) = V/\simeq = \{[u] : u \in V\}$ and $E(G/\simeq) = E/\simeq = \{[u][v] : \exists x \in [u], y \in [v], xy \in E\}$. Let $\rho_\simeq : V(G) \longrightarrow V/\simeq$ be the canonical projection defined by $\rho_\simeq(u) = [u]$. Clearly $\rho_\simeq$ is a surjective homomorphism which is also surjective on the edges[3]. This can be reversed: given graphs G and H and a graph homomorphism $\rho : V(G) \longrightarrow V(H)$ that is onto both the vertices and the edges, we can define (induce) an equivalence relation $\simeq_\rho$ on $V(G)$ by setting $u \simeq_\rho v$ if $\rho(u) = \rho(v)$. We will say that a homomorphism ρ has the lifting property if either there is a $\simeq$ such that $\rho = \rho_\simeq$, or if $\simeq_\rho$ has the property (note that since $\rho_{\simeq_\rho} = \rho$ and $\simeq_{\rho_\simeq} = \simeq$, the two possibilities are equivalent).

[3]More precisely, this means that the mapping $\bar{\rho} : E \longrightarrow E/\simeq$ defined by $\bar{\rho}(uv) = [u][v]$ is onto.

Lemma 24. *Let $(G_c, G_r, \leq_0)$ be a game and let $\rho : G_c \longrightarrow H_c$ be a homomorphism surjective on edges and vertices that has the lifting property. Then for any strategy $\sigma_c : V(H_c) \times V_r \longrightarrow V(H_c)$ there is a strategy $\sigma_c' : V_c \times V_r \longrightarrow V_c$ such that $\rho(\sigma_c(u,v)) = \rho(\sigma_c(u), v)$.*

Proof. To define σ_c', let $\simeq_\rho$ be the equivalence relation induced by ρ on V_c and let $\sigma_c(u,v) = u'$. Since $\simeq_\rho$ has the lifting property, for any $x \in N(u)$ there is a $y \in N(u')$ such that $x \simeq y$. Thus $\rho(x) = \rho(y)$. If $\sigma_c(u,v) = x$, we can define $\sigma_c'(\rho(u), v) = y$; this choice satisfies the conditions of the lemma. □

Definition 25. Let $(G_c, G_r, \leq_0)$ be a game and let $\simeq$ be an equivalence relation on V_c. The relation $\simeq$ is *compatible with* $\leq_0$ if $r \leq_0 u$ if, and only if, $r \leq_0 u'$ for $u \simeq u'$.

The relation $\leq_0 \subseteq V_r \times V_r$ induces a relation $\sqsubseteq_0 \subset V_r \times (V_c/\simeq)$ in the obvious manner: $r \sqsubseteq_0 [u]$ if, and only if, $r \leq_0 u$.

The following lemma is easy to prove by induction.

Lemma 26. *Let $(G_c, G_r, \leq_0)$ be a game and let $\simeq$ be a an equivalence relation on V_c compatible with $\leq_0$. If $\simeq$ has the lifting property then for all $k \in \mathbb{N}$, if $u \leq_k v$ and $v \simeq v'$ then $u \leq_k v'$.*

Proof. The property holds for $k = 0$. Assume it holds for all $j < k$. If $u \leq_k v$ then for all $x \in N(u)$ there is $y \in N(v)$ and $j < k$ such that $x \leq_j y$. By the lifting property, there is $y' \simeq y$ such that $y' \in N(v')$; by the induction hypothesis $x \leq_j y'$ and thus $u \leq_k v'$. □

Theorem 27. *Let $(G_c, G_r, \leq_0)$ be a game, $(H_c, G_r, \sqsubseteq_0)$ a cop-win game and $\rho : V_c \longrightarrow V(H_c)$ a homomorphism with the lifting property such that such that $u \leq_0 v$ whenever $u \sqsubseteq_0 \rho(v)$. Then $(G_c, G_r, \leq_0)$ is also cop-win.*

Proof. Let σ_c be a winning strategy for the cop for $(H_c, G_r, \sqsubseteq_0)$. By lemma 24, σ_c lifts to a strategy $\sigma_c' : V_c \times V_r \to V_c$. For any play $\{(h_i, r_i)\}_{i\in I}$ according to σ_c there is a play $\{(c_i, r_i)\}_{i\in I}$ according to σ_c' such that $\rho(c_i) = h_i$ for $i \in I$ and the last condition guarantees that $\{(c_i, r_i)\}_{i\in I}$ is wining as soon as $\{(h_i, r_i)\}_{i\in I}$ is. □

We now briefly consider another restriction on the game. Let $G_p = (V_p, E_p)$ be a graph and let p be a player with k tokens. At each move, the vertices on which the tokens are placed must induce a graph (isomorphic to a graph) in a set specified at the beginning. Such a set $\mathcal{C}_p$ is called a *constraint* for p. For example, the tandem game defined by Clarke is constrained for the cop by the constraint $\{K_2\}$. A graph is $(k, \ell, \mathcal{C}_c, \mathcal{C}_r)$−cop-win if k cops have a winning strategy against ℓ robbers. We could simply say $(\mathcal{C}_c, \mathcal{C}_r)$−cop-win and let k be the maximum order of a graph in $\mathcal{C}_c$, and similarly for ℓ, but this does not carry through to infinite graphs and, further, our definition allows us to specify that at least some vertices receive more than one token. With the machinery described above, we have a simple way to characterize $(k, \ell, \mathcal{C}_c, \mathcal{C}_r)$−cop-win graphs. If the player plays on a graph G_p, the constrained play will be on the directed graph whose vertices are the induced subgraphs isomorphic to graphs in $\mathcal{C}_p$, with an arc from s_1 to s_2 if the tokens that induce s_1 in G_p can each be moved to a neighbour so that the vertices then occupy induce s_2. Call the new graph $\mathcal{G}_p$. If we use the capture rule that has the cops win when each robber's vertex is covered by a cop's vertex by the relation $\leq_0 \subseteq V_r \times V_c$, Theorem 20 now applies to the game $(\mathcal{G}_c, \mathcal{G}_r, \sqsubseteq_0)$ where $\sqsubseteq_0$ is defined coordinate-wise.

To make the above discussion more precise and more general, we can use the definitions from the introduction. A constrained play on graphs G_c and G_r with k cops and ℓ robbers is a sequence $\{(c_i, r_i)\}_{i \in I}$ with each c_i and each r_i being a vector in V^k and V^ℓ and the additional condition that for each $i \in I$, $\sup(c_i)$ induces a graph in $\mathcal{C}_c$ and $\sup(r_i)$ induces a graph in $\mathcal{C}_r$. If the capture relation is $\leq_0 \subseteq V_r \times V_c$ is defined, it can be extended to subsets of V_c and V_r by defining, for $X \subseteq V_r$ and $Y \subseteq V_c$, $X \sqsubseteq_0 Y$ if for each $x \in X$ there is a $y \in Y$ such that $x \leq_0 y$. The graphs $\mathcal{G}_c$ and $\mathcal{G}_r$ are defined as in the preceding paragraph, but based on the respective graphs. With this, we have a theorem whose proof is by now standard.

Theorem 28. *Let G_c and G_r be graphs, $0 \leq k, \ell \in \mathbb{N}$, and let $\leq_0 \subseteq V_r \times V_c$ be a capture relation. Let also $\mathcal{C}_c$ and $\mathcal{C}_r$ be constraints for the cops and the robbers, respectively. The game $(\mathcal{G}_c, \mathcal{G}_r, \sqsubseteq_0)$ is cop-win if, and only if, the relation $\sqsubseteq$ is nonempty and trivial.*

If the game is played on only one graph with k cops and one robber and a constraint set $\mathcal{C}$ for the cops, one natural question to ask is which constraints are stronger than others. More precisely, suppose that on a graph G the k cops can play with a constraint $\mathcal{C}_0$ and a constraint $\mathcal{C}_1$. The constraint $\mathcal{C}_0$ is *stronger* than $\mathcal{C}_1$ if for every winning strategy for the cops under $\mathcal{C}_0$ there is a winning strategy for them under $\mathcal{C}_1$. For a trivial example, if $\mathcal{C}_0 = \{K_2\}$ and $\mathcal{C}_1 = \{K_1, K_2\}$, then $\mathcal{C}_0$ is stronger than $\mathcal{C}_1$. This defines an order relations $\preceq$ on the set of constraints: $\mathcal{C} \preceq \mathcal{C}'$ if for any graph $\mathcal{C}'$ is stronger than $\mathcal{C}$. The resulting partial order can be called the *constraint poset.* What can be said about it (see Section 7)

6. Infinite graphs

For infinite graphs, not much is known beyond some work on the random (Rado, homogeneous) graph (which is not cop-win) and some density results on the search number and the capture time, see [5, 3]. Unlike in the finite case, there are vertex-transitive cop-win graphs. In fact, for each cardinal α, there are are 2^α non-isomorphic ones, see [4]. Here we mention how to extend the results of the preceding section to infinite connected graphs, the only ones we consider in the present section. Again, the first theorem is essentially the same as that of [31].

Theorem 29. *A game $(G_c, G_r, \leq_0)$ is cop-win if, and only if, the relation $\preceq$ is trivial.*

Proof. First, we need to modify the definition of $\preceq$. This is easy if we replace $\mathbb{N}$ by Ord, then the class of ordinals. We define $\leq_0$ as before and replace $k, j \in \mathbb{N}$ by ordinals α, β. Next, we observe that $\leq_\mu = \leq_{\mu+1}$ with $\mu = |V_c \times V_r|$ and so there is a least κ such that $\leq_\kappa = \leq_{\kappa+1}$. We set $\preceq = \leq_\kappa$. The rest of the proof is the same, recalling that there is no infinite descending sequence of ordinals. □

The rest of the section goes through essentially unchanged and we will not go through the exercise of writing down the details.

Since for some people an infinite graph is a locally finite one, and so countable if it is connected, it is worthwile to state the following proposition.

Proposition 30. *A localy finite infinite graph is not cop-win.*

Proof. If the graph is not connected, this is obvious, Otherwise the graph is connected and has a spanning tree. In particular, for any vertex u there is a breadth-first search spanning tree T_u routed at u. By König's lemma (or directly), any vertex is the start of an infinite path in a breadth-first tree rooted at that vertex. Thus if the cop chooses a vertex v_0 as her intial vertex, the robber will choose a vertex u_0 at distance at least 2 on an infinite path P starting at v_0. Now the best the cop can do is follow the robber on the path and the robber can sit at his current vertex until the cop is at distance two from him, at which point he moves away from her along P. Since the tree is breadth-first, the cop cannot get ahead of the robber, and since the path is infinite, the robber always has a place to go. □

7. Open problems

We close with some open problems. Some may be trivial, others not at all.

1. We have defined a strategy for k cops and one robber as a function $\sigma : V^k \times V \longrightarrow$. This allowed for a simple proof of Lemma 15. But a strategy need not be a total function (indeed, it mostly isn't when we think about a game). Is Lemma 15 still valid?
2. Is the cop number of (reflexive) tournaments obtained from Steiner Triple Systems by orienting each triple cyclically bounded? If so, what is the best bound?
3. Characterize cop-win graphs in which each vertex may or may not have a loop.
4. What does the constraint poset look like?
5. In spite of the Clarke-MacGillivray result, we still do not have a characterization of $k-$cop-win graphs in terms of their structure. Is there one (or is there one for each k)?
6. Can cop-win directed graphs be characterized? How about $k-$cop-win?
7. Is the length of the game directly related to some other known graph parameter? Some obvious ones are independent of the length of the game, see the last part of [23].

8. It is of interest to roboticists to know what happens if the robber is invisible. Some work in this direction has been done, but much is still open. See [10], [11], [12], [16], [17].
9. The basic game can be extended, as we have seen, to $k > 1$ cops. and $\ell > 1$ robbers and the rules amended so that the cops win if each robber's vertex is also occupied by a cop. But there are other ways we could make the cops win.
   (a) There could be a prison and as a robber is caught, he could be taken to the prison vertex by one of the cops while the rest of them hunt for other robbers. The cop could return to his colleagues after delivering the robber to the prison.
   (b) The transfer to prison could be instantaneous, i.e. the robber is simply taken out of the game.
   (c) A robber could simply remain in the custody of the cop while other cops try to catch the rest of the robbers.

   The variations certainly lead to different games and different capture times and are somehow related. How? What about constrained games in these settings? How can these be made precise?
10. What if in a $k-$cop game only one cop can move at a time?
11. Is there a bound on the search number of tournaments obtained from Steiner triple systems as described at the end os Section 3?

8. Acknowledgments

The authors would like to thank the referees for helpful suggestions. They also suggested credit be given to the students where it is due: optimal plays and non-reflexive graphs were looked at by El Ouarari, tournaments by Thériault, constrained games by El Harti. The counterexamples to the conjectures are due to Ganian, Moldenauer, Rutter, Vatshelle. Lemma 30 is due to Gavenčiak.

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