Further Results on Some Bi-level Balanced Arrays Using Coincidences

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Dedicated to the memory of Professor Ralph G. Stanton.

Abstract

A bi-level balanced array (B-array) T with parameters (m, N, t) and index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is a matrix with m rows, N columns, and with two elements (say, 0 and 1) such that in every $(t \times N)$-submatrix T^* (clearly, there are $\binom{m}{t}$ such submatrices) of T, the following combinatorial condition is satisfied: every $(t \times 1)$ vector $\underline{\alpha}$ of T^* with i $(0 \leq i \leq t)$ ones in it appears the same number μ_i (say) times. T is called a B-array of strength t. Clearly, an orthogonal array (O-array) is a special case of a B-array. These combinatorial arrays have been extensively used in information theory, coding theory, and design of experiments. In this paper, we restrict ourselves

to arrays with $t = 4$ and $t = 6$. We derive some inequalities involving m and μ_i, using the concept of coincidences amongst the columns of T, which are necessary conditions for B-arrays to exist. We then use these inequalities to study the existence of these arrays and to obtain the bounds on the number of rows (also called constraints) m, for a given value of $\underline{\mu}'$.

1 Introduction and Preliminaries

For the sake of completeness, we first state some basic concepts and definitions.

Definition. A *binary array* T with m rows (constraints, factors), N columns (runs, treatment-combinations) and with two symbols (levels) is merely a matrix T of size $(m \times N)$ with two elements (say, 0 and 1).

A certain combinatorial structure imposed on T leads to the definition of a balanced array (B-array).

Definition. T is called a *B-array* of strength t $(1 \leq t \leq m)$ if it satisfies the following condition: in every t-rowed submatrix T^* (there are $\binom{m}{t}$ such submatrices) of T, each $(t \times 1)$ vector $\underline{\alpha}$ of weight i $(0 \leq i \leq t$; the weight of $\underline{\alpha}$ refers to the number of 1s in it) occurs with the same frequency μ_i (say).

The vector $\underline{\mu}' = (\mu_0, \mu_1, \mu_2, \ldots, \mu_t)$ and m are called the parameters of the array T. For a given $\underline{\mu}'$, the number of runs N is known. Clearly, $N = \sum_{i=0}^{t} \binom{t}{i} \mu_i$.

Definition. If $\mu_i = \mu$ for each i, then the B-array T is called an *orthogonal array* (O-array), and $N = 2^t \cdot \mu$ in this case.

Thus, B-arrays include O-arrays as a special case. Also, the incidence matrix of a balanced incomplete block design (BIBD) is a special case of a B-array of strength $t = 2$. B-arrays have been shown to be related to various other combinatorial structures such as rectangular designs, group divisible designs, nested balanced incomplete block designs, etc. B-arrays and O-arrays of different strengths have been extensively used to construct balanced fractional factorial designs (FFD) of varying resolutions. For example, it has been shown (Chakravarti [5]) that a balanced fractional factorial design T of 2^m (meaning m factors, each at two levels, 0 and 1) of resolution V is identical to a B-array with two symbols and strength $t = 4$. O-arrays have been extensively used in information theory, coding theory, statistical quality control, and were used in disproving Euler's Conjecture on the existence of orthogonal latin squares. Bose [2] points out the connection between information theory and design of experiments. Chakravarti (at the suggestion of C.R. Rao) introduced B-arrays in [5] (under the name of partially balanced arrays). O-arrays clearly do not exist for every value of N (the number of treatment-combinations) where as B-arrays do not have this drawback. For example, to construct an O-array of strength $t = 4$, the total number of treatment-combinations has to be a multiple of 16. Thus for $m = 7$ factors and $t = 4$, there does not exist any O-array for $N = 44$ where as there exists a B-array.

For this paper, we restrict ourselves to B-arrays with $t = 4$, and 6. These arrays, under certain conditions, would allow us to estimate all the effects up to and including two-factor interactions and three-factor interactions (higher order interactions are assumed to be negligible), respectively. To construct such arrays, for a given index set $\underline{\mu}'$ and the maximum possible value of m, is a very important and complex problem in combinatorics and design of experiments. Such problems for O-arrays have been investigated, among others, by Bose and Bush [1], Chopra, Low, and Dios [8], Hedayat, Sloane, Stufken [13], Rao [18, 19, 20], Seiden and/or Zemach [22, 23, 24], and Yamamoto et. al [28]. The corresponding problem for B-arrays has been studied, among others, by Chopra, Low, and Dios [10, 11, 12], Chopra and Bsharat [7], Rafter and Seiden [17], Saha, Mukerjee, and Kayeyama [21], Yamamoto, Kuwada, and Yuan [27], etc. To gain further insight into the importance of O-arrays and balanced arrays, to study their relationships to other combinatorial structures, and to gain understanding of their usefulness to solving problems in design of experiments, the interested reader is referred to the list of references at the end (which by no means, an exhaustive list) of this paper, as well as further references listed therein.

In this paper, we obtain some inequalities involving parameters m and $\underline{\mu}'$ for B-arrays of strength $t = 4$ and 6. These are necessary conditions for the existence of these arrays. Furthermore, we make use of these conditions to obtain, for a given $\underline{\mu}'$, the maximum value of m. We also include some examples to compare the current results with the results published earlier to demonstrate that the results presented here are better. In obtaining these necessary conditions, we make use of the concept of coincidences among columns and the positive semi-definiteness of the moment matrix.

2 Main Results with Illustrative Examples

The following results can be easily established.

Lemma 1. *A B-array T with index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ and $m = t$ always exists.*

Lemma 2. *A B-array T of strength t and index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is also of strength k, where $0 \le k \le t$.*

Remark. It is obvious that the elements of the parameter vector of T (when considered as an array of lower strength k) are linear combinations of the elements of the index set $\underline{\mu}'$. For example, if T is of strength $t = 4$ with $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_4)$, then the index set of T (when considered as an array of strength 3) is $(\mu_0 + \mu_1, \mu_1 + \mu_2, \mu_2 + \mu_3, \mu_3 + \mu_4)$. In general, for an array T of strength t, the jth element $(0 \le j \le k)$ of the parameter vector of T (when viewed as an array of strength k) is given by

$$A(j,k) = \sum_{i=0}^{t-k} \binom{t-k}{i} \mu_{i+j}, \quad \text{where } j = 0, 1, 2, \ldots, k, \ (k \le t). \qquad (2.1)$$

From (2.1), one sees that $A(t,t) = \mu_t$, $A(j,t) = \mu_j$, and $A(j,0) = A(0,0) = N$.

Definition. Two columns of a B-array are said to have i *coincidences* if the symbols appearing in these two columns in i of the rows are the same.

Remark. It is clear that for an m-rowed B-array, the number of coincidences i must satisfy $0 \le i \le m$.

Lemma 3. *Consider an m-rowed B-array T of strength $t = 4$, and suppose there is a column (say, the first one) in T having l ones $(0 \le l \le m)$ in it. Let x_j denote the number of columns in T having j coincidences with the first column, and let $L_k = \sum_{j=0}^{m} j^k x_j$ (where $0 \le k \le 4$). Then, the following results must hold:*

$$L_0 = \sum_{j=0}^{m} x_j = N - 1,$$

$$L_1 = \sum_{j=0}^{m} j x_j = \sum_{i=0}^{1} \binom{l}{i} \binom{m-l}{1-i} [A(i,1) - 1],$$

$$L_2 = \sum_{j=0}^{m} j^2 x_j = L_1 + 2! \sum_{i=0}^{2} \binom{l}{i} \binom{m-l}{2-i} [A(i,2) - 1],$$

$$L_3 = \sum_{j=0}^{m} j^3 x_j = 3L_2 - 2L_1 + 3! \sum_{i=0}^{3} \binom{l}{i} \binom{m-l}{3-i} [A(i,3) - 1],$$

$$L_4 = \sum_{j=0}^{m} j^4 x_j = 6L_3 - 11L_2 + 6L_1 + 4! \sum_{i=0}^{4} \binom{l}{i}\binom{m-l}{4-i}[A(i,4)-1]. \tag{2.2}$$

Proof. (Outline.) Let us consider some column (say, the first one) of T of weight l. Now, let us consider any four rows of T. If the first column contains $(0,0,0,0)^{\mathrm{T}}$, then clearly it would appear $(\mu_0 - 1)$ more times within the remaining columns. Similarly, if it is a 4-vector of weight i $(1 \leq i \leq 4)$, it would appear $(\mu_i - 1)$ more times amongst the other columns. But the first column has l ones in it. Therefore, the number of ways to select four rows such that there is a 4-vector of weight i $(0 \leq i \leq 4)$ in the first column is $\binom{l}{i}\binom{m-l}{4-i}$. Let T^* be the total number of 4-tuples which appear in columns (other than the first) identical with the corresponding 4-tuple in the first column. Then, $T^* = \sum_{i=0}^{4} \binom{l}{i}\binom{m-l}{4-i}(\mu_i - 1)$. Now x_j, the number of columns (other than the first) having j coincidences $(j \geq 4)$, will contribute $\binom{j}{4}$ to T^*. Thus, $T^* = \sum_{j=4}^{m} \binom{j}{4} x_j = \sum_{j=0}^{m} \binom{j}{4} x_j$. Equating the two values of T^* gives us (after some algebraic manipulation) the five results of Lemma 3, for $i = 0, 1, 2, 3$, and 4. □

Let us make some remarks on the non-negative definiteness (n.n.d) of the matrix (even order) of moments. The quantities $L_k = \sum j^k x_j$, $(0 \leq k \leq t)$, are called *moments of order k around zero.* If $t = 2\mu$ (even), then the matrices (for every positive integer μ)

$$M_{2\mu} = \begin{pmatrix} L_0 = N & L_1 & L_2 & \cdots & L_\mu \\ L_1 & L_2 & L_3 & \cdots & L_{\mu+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ L_\mu & L_{\mu+1} & L_{\mu+2} & \cdots & L_{2\mu} \end{pmatrix} \tag{2.3}$$

are called *matrices of moments*, and are non-negative definite. This can be seen by observing the non-negative definiteness of the quadratic form $\sum_{j=0}^{m}(\alpha_0 + \alpha_1 j + \alpha_2 j^2 + \cdots + \alpha_\mu j^\mu)^2 x_j$ in variables $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_\mu$.

We now present the main results for B-arrays with $t = 4$.

Theorem 1. *Consider a B-array T of strength $t = 4$, with index set $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_4)$, and with m constraints. Let l $(0 \leq l \leq m)$ be the number of ones in some column (say, the first) of T, and let x_j be the number of columns having j $(0 \leq j \leq m)$ coincidences with the first column. For T to exist, the following conditions must be satisfied:*

$$L_0 L_2 \geq L_1^2, \tag{2.4}$$

$$L_0 L_2 L_4 + 2L_1 L_2 L_3 \geq L_0 L_3^2 + L_1^2 L_4 + L_2^3, \tag{2.5}$$

where $L_k = \sum_{j=0}^{m} j^k x_j$, $0 \le k \le 4$.

Proof. (Outline.) We obtain M_4 (a 3×3 matrix) from (2.3) by setting $\mu = 2$. This is a n.n.d matrix which means that all of the determinants of its leading principal minors are greater than or equal to 0. Clearly, there are only two such minors: one is a 2×2 and the other is the entire matrix M_4. Conditions (2.4) and (2.5) are obtained by expanding (respectively) the determinants of the following leading principal minors of M_4:

$$\begin{pmatrix} L_0 & L_1 \\ L_1 & L_2 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} L_0 & L_1 & L_2 \\ L_1 & L_2 & L_3 \\ L_2 & L_3 & L_4 \end{pmatrix}.$$

□

Remark. We have already expressed [see (2.2)] L_k in terms of the parameters of array T. Thus, inequalities (2.4) and (2.5) involve only the parameters m and μ_is of array T. Given $\underline{\mu}'$, these inequalities would only involve m.

Next, we present the main results for $t = 6$.

Theorem 2. *Let T be a B-array of strength $t = 6$, with m rows, index set $\underline{\mu}'$, and having a column (say, the first one) with l ones, $(0 \le l \le m)$, in it. For T to exist, the following conditions must be satisfied:*

$$L_0 L_2 \ge L_1^2, \tag{2.6}$$

$$L_0 L_2 L_4 + 2L_1 L_2 L_3 \ge L_0 L_3^2 + L_1^2 L_4 + L_2^3, \tag{2.7}$$

$$\begin{aligned} L_0L_2L_4L_6 &+ 2L_0L_3L_4L_5 + L_1^2L_5^2 + 2L_1L_2L_3L_6 + 2L_1L_3L_4^2 \\ &+2L_2^2L_3L_5 + L_2^2L_4^2 + L_3^4 \ge \\ &L_0L_2L_5^2 + L_0L_3^2L_6 + L_0L_4^3 + L_1^2L_4L_6 + 2L_1L_3^2L_5 \\ &+2L_1L_2L_4L_5 + L_2^3L_6 + 3L_2L_3^2L_4, \end{aligned} \tag{2.8}$$

where $L_k = \sum_{j=0}^{m} j^k x_j$, $0 \le k \le 6$ *and* x_j *being the number of columns of* T *having exactly* j *coincidences with the first column.*

Proof. (Sketch.) We take $\mu = 3$ in (2.3) and use the property that the determinants of each of its leading principal minors (i.e M_2, M_4, and M_6) are greater than or equal to 0. □

In order to express each L_k in terms of the parameters of array T, we merely quote results for $t = 6$, corresponding to the ones given in Lemma 3 for $t = 4$.

Lemma 4. *Let T be a B-array of strength $t = 6$, having m rows, index set $\underline{\mu}'$, and having a column (say, the first one) with l $(0 \le l \le m)$ ones in it. Let $L_k = \sum_{j=0}^{m} j^k x_j$, where $0 \le k \le 6$, and x_j denote the number of columns of T having exactly j coincidences with the first column. Then for T to exist, the following conditions must be satisfied:*

$$
\begin{aligned}
L_0 &= \sum_{j=0}^{m} x_j = N - 1, \\
L_1 &= \sum_{j=0}^{m} j x_j = \sum_{i=0}^{1} \binom{l}{i} \binom{m-l}{1-i} [A(i,1) - 1], \\
L_2 &= \sum_{j=0}^{m} j^2 x_j = L_1 + 2! \sum_{i=0}^{2} \binom{l}{i} \binom{m-l}{2-i} [A(i,2) - 1], \\
L_3 &= \sum_{j=0}^{m} j^3 x_j = 3L_2 - 2L_1 + 3! \sum_{i=0}^{3} \binom{l}{i} \binom{m-l}{3-i} [A(i,3) - 1], \\
L_4 &= \sum_{j=0}^{m} j^4 x_j = 6L_3 - 11L_2 + 6L_1 + 4! \sum_{i=0}^{4} \binom{l}{i} \binom{m-l}{4-i} [A(i,4) - 1], \\
L_5 &= \sum_{j=0}^{m} j^5 x_j = 10L_4 - 35L_3 + 50L_2 - 24L_1 + \\
&\qquad 5! \sum_{i=0}^{5} \binom{l}{i} \binom{m-l}{5-i} [A(i,5) - 1], \\
L_6 &= \sum_{j=0}^{m} j^6 x_j = 15L_5 - 85L_4 + 225L_3 - 274L_2 + 120L_1 + \\
&\qquad 6! \sum_{i=0}^{6} \binom{l}{i} \binom{m-l}{6-i} [A(i,6) - 1].
\end{aligned}
\tag{2.9}
$$

3 Discussion with Illustrative Examples of the Results

In order to check the existence of B-arrays for a given value of m and $\underline{\mu}'$, and/or to find the max(m) for a given $\underline{\mu}'$, a computer program was prepared. We use conditions (2.4) and (2.5) for arrays with $t = 4$, conditions (2.6), (2.7) and (2.8) for $t = 6$. In all instances, we take $l = 0$. The B-array T will not exist if at least one inequality is contradicted. T may exist if all of the inequalities are satisfied. The conditions presented in this paper are necessary conditions for the existence of B-arrays. To obtain max(m) for a given $\underline{\mu}'$, we substitute into each condition, the value of $\underline{\mu}'$ which yield inequalities involving only m. We then check each value of m, starting with $m = t$ (= 4 or 6) and continue until the inequality is not satisfied. Let us suppose the first time contradiction occurs is at $m = k+1$, (say). Then, we list k as the maximum value of m, i.e max(m) $\leq k$. It is clear that finding max(m) for a given $\underline{\mu}'$, is tantamount to accommodating more factors in a experimental situation and thus could lead to cost savings. Next, we give some illustrative examples for $t = 4$ and 6 and compare the present results with those published [7, 9, 10, 12]. In all the cases, we find that (2.5) for $t = 4$ and (2.8) for $t = 6$ give us the best results.

Examples. ($t = 4$): Take, for example, $\underline{\mu}' = (8, 8, 8, 1, 4)$, which was used in [10, 11]. From (2.4) and (2.5), we obtain (respectively) $m \leq 12$ and $m \leq 10$, while results in [10] give us max(m) > 500 and those in [11] give us $m \leq 11$. Thus, $m \leq 10$, obtained by using (2.5), is the best one. Similarly for $\underline{\mu}' = (4, 4, 4, 4, 3)$ and $(4, 1, 1, 5, 1)$, we obtain (respectively), using (2.5), $m \leq 8$ and $m \leq 4$. Corresponding results for these two particular arrays are $m \leq 11$, $m \leq 5$ in [10], and $m \leq 10$, $m \leq 4$ in [11]. Thus, (2.5) is either better or does as good a job compared to earlier published results.

Examples. ($t = 6$): Now, let us consider some arrays found in earlier papers [9] and [11]. Take the following values of $\underline{\mu}'$: $(1, 1, 2, 1, 4, 1, 1)$, $(4, 4, 3, 2, 3, 4, 4)$, $(1, 2, 1, 1, 4, 3, 2)$, $(9, 8, 8, 8, 6, 7, 8)$, and $(8, 7, 7, 5, 6, 6, 8)$. We use (2.6), (2.7), and (2.8) and list the max(m), for each $\underline{\mu}'$. The best values of m are $m \leq 7$, $m \leq 11$, $m \leq 6$, $m \leq 9$ and $m \leq 19$, respectively. In [11], the corresponding results are $m \leq 8$, $m \leq 13$, $m \leq 8$, $m \leq 9$ and $m \leq 23$, respectively. The inequalities in [9] give us the following max(m): $m \leq 9$, $m \leq 18$, $m \leq 10$, $m \leq 12$ and $m \leq 36$. The first set of results is better. These computations clearly demonstrate the improvements on the earlier values of max(m), for some values of $\underline{\mu}'$.

Remark 1. It was observed, while computing, that the values of max(m) for arrays with $\underline{\mu}' = (\mu - a, \mu, \mu, \mu, \mu - b)$ and $(\mu - a, \mu, \mu, \mu, \mu, \mu, \mu - b)$ with $0 \leq a, b \leq 1$, $(a = b \neq 0)$, are quite sharp for several cases. These arrays are very near O-arrays in the sense that we need to attach (to each one) a vector of weight zero and/or a vector of weight m. For example, for $t = 4$, we consider $\underline{\mu}' = (5, 6, 6, 6, 5)$, $(5, 6, 6, 6, 6)$, $(4, 5, 5, 5, 4)$, $(7, 8, 8, 8, 7)$, $(8, 8, 8, 8, 7)$, $(36, 37, 37, 37, 36)$, $(37, 37, 37, 37, 36)$, etc. and see that the values of max(m) are 8, 9, 7, 7, 9, 20 and 24, respectively. For $t = 6$, we take $\underline{\mu}' = (2, 3, 3, 3, 3, 3, 2)$, $(3, 3, 3, 3, 3, 3, 2)$, $(3, 4, 4, 4, 4, 4, 3)$, $(3, 4, 4, 4, 4, 4, 4)$, $(6, 7, 7, 7, 7, 7, 6)$, $(8, 9, 9, 9, 9, 9, 8)$, etc. and see that the values of max(m) are 7, 9, 8, 8, 10 and 13, respectively.

Remark 2. We do not obtain much useful information about max(m) for O-arrays. However, for O-arrays with index set $\mu = 1$, we obtain max(m) ≤ 7 for $t = 6$, and max(m) ≤ 5 for $t = 4$. It is not difficult to check that O-arrays with $t = 6$ for $m = 7$, and $t = 4$ for $m = 5$ do exist and can be easily constructed.

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