# Computation and Simulation of Langevin Stochastic Differential Equation

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ABSTRACT: We will study the random perturbation on a linear differential equation as a nowhere differentiable function. The noise in the historical Langevin stochastic differential equation will be treated as a nowhere differentiable model for Brownian motion. A short introduction of Wiener process leading to Ito's calculus will be used in derivation of the mean and variance of the solutions to the Langevin Equation. Computational algorithms were developed and applied to study the numerical solutions to linear stochastic differential equations. Symbolic computation and simulation of a computer algebra system will be used to demonstrate the behavior of the solution to the Langevin Stochastic Differential Equation when the perturbation is density independent.

## 1 Historical Background:

**Brownian Motion:** Brownian motion, named after Robert Brown who first observed the motion in 1827, was eventually analyzed by Albert Einstein [15]. Brown observed that pollen grains immersed in water are randomly bombarded by the molecules of the surrounding medium. Einstein pointed out that this motion is caused by random bombardment of heat excited water molecules on the pollen. The mathematical model of Brownian motion has several real-world applications. An often quoted example is stock market fluctuations. However, movements in share prices may arise due to unforeseen events which do not repeat themselves, so physical and economic phenomena are not comparable.

Louis Bachelier, a Ph.D. student of Henri Poincare, introduced **Brownian Motion in 1900** as a model for the dynamic behavior of the Paris stock market [5]. Notice that this took place 5 years before **Albert Einstein** developed a physical model of Brownian motion to describe small particles suspended in a liquid and 23 years before **Norbert Wiener** gave the first rigorous mathematical construction of Brownian motion. For that reason, Bachelier is now considered by many as the founder of modern Mathematical Finance. See the article by Robert Jarrow and Philip Protter for the historical summary ([15]).

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Brownian motion is among the simplest of the continuous-time stochastic (or probabilistic) processes and in mathematical language is called stochastic process, whose **time derivative is everywhere infinite. Random Walk is a good example** of a two dimensional discrete Brownian motion that can be considered as a "drunken man wandering around the road to his home". More precisely, each of his steps (in both x- and y-directions) are independent normal random variables.

**The origin of the Langevin Equation:** Langevin observed random movement of particles in fluid due to collisions between molecules of the fluid. Brownian motion was described by Langevin in 1908 [8] through his famous equation

$$m\frac{d^2\overrightarrow{r}}{dt^2} = -\lambda\frac{d\overrightarrow{r}}{dt} + \overrightarrow{\eta}(t) \tag{1}$$

where $\overrightarrow{r}$ represents the position of the particle and m denotes the particle's mass. The force acting on the particle is written as a sum of a viscous force proportional to the particle's velocity (in Physics called Stokes law), and noise term $\eta(t)$ representing the effect of the collisions between the molecules of the fluid.

Readers who are computational expert may find interesting to see some inter-related of the Langevin's equation to other disciplines. The organization of the paper begins with the derivation of the Langevin equation, presentation of some examples, nowhere-differentiability, examples of nowhere differentiable noise, introduction to stochastic calculus, Brownian motion, Ito's calculus, computation of the solution- mean - variance, and upper-lower fluctuations.

## 1.1 Derivation of Langevin Equation:

In statistical physics, a Langevin equation is a stochastic differential equation describing Brownian motion using potential theory. The first physical use of the Langevin equation: potential is constant, so that the acceleration of

i) a Brownian particle of mass m is expressed as the sum of a viscous force which is proportional to the particle velocity (Stokes' law),

ii) a noise term representing the effect of a continuous series of collisions between the atoms of the underlying fluid (systematic interaction force due to the intermolecular interactions).Consider a colloidal particle suspended

in a liquid.

On its path through the liquid it will continuously collide with the liquid molecules. It will experience i) a systematic resisting force $\overrightarrow{R(t)}$ proportional to its velocity, and directed opposite to its velocity. ii) In addition, the particles will experience random forces with the resultant $\overrightarrow{F(t)}$.

The equation (1) can be translated to a linear system of differential equations

$$\frac{d\overrightarrow{r(t)}}{dt} = \overrightarrow{v(t)}, \quad \frac{d\overrightarrow{v(t)}}{dt} = \vec{R} + \overrightarrow{F(t)} = -\beta.\overrightarrow{v(t)} + \overrightarrow{F(t)} \qquad (2)$$

In hydrodynamics the constant of the friction force is given by $\beta = \frac{6\pi\eta a}{m}$ where $\eta$ is the viscosity of the solvent. The random force $\overrightarrow{F(t)} = w(t)$ is the average resultant force of the collision of millions of particles. Using standard differential and integral calculus a general solution for $\overrightarrow{v(t)}$ is

$$\overrightarrow{v(t)} = \vec{v_0} e^{-\beta t} + \int_0^t e^{-\beta(t-s)} \overrightarrow{F(s)} ds. \qquad (3)$$

One can integrate the first equation in this system to evaluate the position vector $\overrightarrow{r(t)}$. This will be a deterministic solution of the system if we are certain about the average force function $\overrightarrow{F(t)}$. Due to the uncertainty nature of the random forces generated by the collisions of particles, the Riemann integral on the right hand side will not be well-defined.
We will call the random force $\overrightarrow{F(t)} = W_t$ a noise or perturbation which causes the solution integral to be undefined in Lebesgue-Steiltjes sense.

## 1.2 Motivation and Approach:

Many aspects of the Langevin equation and its difficulties in using the Lebesgue - Stieltjes integral have been studied (see reference Schuss, Z. 1980, p.61, [17]). We would like to use a computational approach to simulate the solution of the following stochastic differential equation

$$d(y(t), t) = -\beta(t) \cdot y(t)dt + g(t) \cdot dw(t). \qquad (4)$$

After a short review of the materials related to noise, nowhere differentiability, integrability, and stochastic calculus, we will use a computer algebra system to demonstrate and simulate the solution.

We are assuming that the general solution of the Langevin SDE will be created by two forces i) deterministic force that can be predicted by Newton's law or any set of mathematical modeling postulates. ii) the noise created by the stochastic force which represents the fluctuation. In other words, the SDE model has a **superposition property**. It, is the sum of the deterministic and the stochastic solution.

Let us call the deterministic solution Y(t) and the non-deterministic solution X(t). A central notion for stochastic calculus is that of a continuous random process Z(t) that can be written as the sum of a local nowhere differentiable function (for example, Brownian motion) and a drift process Y (a continuous process of locally bounded variation, typically the solution of some conventional differential equation). The decomposition Z(t) = Z(0) + Y(t) + X(t) is unique and can be thought of as a decomposition of Z into Y-signal Y plus X- noise [15].

## 2 Nowhere Differentiability and Integrability:

It is natural to ask the question of under what conditions the Lebesgue-Stieltjes integral ( LS-integral) exists. In a simpler case, the following theorem is a criteria for existence of the Steiltjes integral (S-integral). The integral $\int_a^b g(t)dw(t)$ exists if the function g(t) is continuous on [a,b] and w(t) is of finite variation on [a,b]. (see p.230 [4]).

The simplest existence theorem states that if the function f is continuous and w is of **bounded variation** on [a, b], then the integral exists. A function w is of bounded variation if and only if it is the difference between two monotone functions. If w is not of bounded variation, then there will be continuous functions which cannot be integrated with respect to w. In general, the integral is not well-defined if f and w must share any points of discontinuity, however this sufficient condition is not necessary. How do we integrate $\int_a^b g(t)dw(t)$ when the function w(t) is not of finite variation?
Integrability in Riemann- Steiltjes Sense: We apply the basic ideas behind the Fundamental Theorem of differential and integral calculus in all areas of computational science, engineering, and mathematics. Under what conditions are these two expressions equivalent ?

$$dy(t) = f(t, y(t)dt + g(t)dw(s) \qquad \Leftrightarrow \tag{5}$$

$$y(t) = y(t_0) + \int_{t_0}^{t} f(s, y(s)ds + \int_{t_0}^{t} g(s)dw(s) \tag{6}$$

In the sense of Lebesgue-Stieltjes, the integrands g should be continuous and w(t) should be absolutely continuous (**bounded variation in S-integral).** We conclude that the following is a relation which may be used in the algorithm for computation ([2],[3],[12]):

$$\int_a^b g(t) \cdot dw(t) = g(b)w(b) - g(a)w(a) - \int_a^b w(t) \cdot dg(t).$$

**Important Note:** One can demonstrate that *if w is differentiable* then the Lebesgue-Stieltjes integral can be expressed by pure Riemann integration in the following form, that is

$$\int_{[a,b]} g(t) \cdot dw(t) = \int_{[a,b]} g(t) \cdot w'(t)dt \tag{7}$$

Let us assume that the symbol ND represents a class of functions continuous and nowhere differentiable. It can be verified that

i) if a function $w \in ND, and$ g is differentiable on R, then g+w $\in ND.$

ii) If a function $w \in ND$, and f is differentiable on R, and $g(x) \neq 0, \forall x \in R$, then $g \cdot w \in ND.$

*Thus the integrand in the second integral of the integration by parts formula, will not be differentiable. This should not prevent us to realize that the integral does exist.*

### 2.1 Nowhere Differentiable Perturbations and Noise:

The concept of the continuous nowhere differentiable function was first explored by Andre Marie Ampere in 1806 (. [15 ]). He was unsuccessful in his attempt to demonstrate by example. The first example was presented by Weirestrass fifty years later. In the practical application of SDE, one of the characteristics of the random perturbation is the nowhere differentiability of the noise. We would like to present a few examples.

**Example 1- (Weirestrass Function)** On the advanced calculus level, it can be verified that a function

$$w(t) = \sum_{k=1}^{\infty} a^k cos(b^k \pi t) \tag{8}$$

is nowhere differentiable but continuous everywhere, where a and b satisfy certain relations $(0 < a < 1, b \in Z^+$, and $a \cdot b > 1 + 3 * \pi/(2)$ (see: [13 ], page 38-41).

**Example 2 (van der Waerden function):** The following function which is continuous and nowhere differentiable, is known as van der Waerden's function (see [1] for van der Waerden 1930) $w(t) = \sum_{n=1}^{\infty} \frac{\sin(t \cdot n^2 \cdot \pi)}{n^2}$

## 2.2 Stochastic Integral Calculus:

Suppose that f(t) is a stochastic process and $W_t$ is a **Wiener process,** then the stochastic integral of f(t) with respect to a process $W_t$ is a random variable defined as

$$I = \int_{[a,b]} f(t)dW_t = \lim \sum_{i=1}^{n} f(t_{i-1}) \Delta W(t_i)$$

where $\Delta W_{t_i} = W(t_i) - W(t_{i-1})$.

**Stochastic Integral is not consistent with classical integral Calculus:**

It makes a difference in how the independent variable $\tau_i$ in $[t_{i-1}, t_i]$ is selected. We will show the differences through the following example where the result will not be consistent with traditional integral calculus. If this independent variable is selected at the midpoint $\tau_i = (t_{i-1} + t_i)/2$ the integral is called **Strotonovich integral,** which will be consistent with the regular differential and integral calculus ([2],[11]).

**Example 3**: Assume $g(t) = W_t$ to be the Wiener Process (standard Brownian motion). Compute $\int_0^1 W_s dW_s$ using stochastic integral.

-Using left hand point:

$$I_1 = \int_{[0,1]} g(t) dW_t = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} g(t_{i-1}) \Delta W(t_i) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} W_{t_{i-1}} \cdot \Delta W(t_i)$$

-Using right hand end point:

$$I_2 = \int_{[0,1]} g(t) dW_t = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} g(t_i) \Delta W(t_i) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} W_{t_i} \cdot \Delta W(t_i)$$

It can be proved by indirect computation that

$$I_2 - I_1 = \lim_{n \longrightarrow \infty} (\sum_{i=1}^{n} W_{t_{i-1}} \cdot \Delta W(t_i) - \sum_{i=1}^{n} W_{t_i} \cdot \Delta W(t_i)) = \lim_{n \longrightarrow \infty} (\sum_{i=1}^{n} (W_{t_i} - W_{t_{i-1}}) \cdot \Delta W(t_i) = \lim_{n \longrightarrow \infty} (\sum_{i=1}^{n} (\Delta W_{t_i})^2 =$$

$I_1(t) = \frac{1}{2}(W_t^2 - t)$ and $I_2(t) = \frac{1}{2}(W_t^2 + t)$. This computation shows that the stochastic integral is inconsistent with the classical integral calculus where $\int_0^t x dx = \frac{1}{2} x^2$. Thus, these two computations do not lead to the same result but $I_2 = I_1 + t$. using the telescope law and gives the following result:

$\int_0^1 W_s dW_s = \frac{1}{2} \sum_{i=1}^{n} [(W^2(b) - W^2(a)) - \frac{1}{2}[b - a]^2$

# 3 Brownian Motion and Wiener Process:

In mathematics, Brownian motion is described by the Wiener process; a continuous-time stochastic process named in honor of Norbert Wiener. Brownian motion is one of the best known Lévy processes (càdlàg stochastic processes with stationary independent increments) and occurs frequently in pure and applied mathematics, economics and physics. Let X(t) be the coordinate of a free particle on a real line. In modeling the stochastic process Einstein was able to show the following properties for Brownian motion: (i) The increment $X(t_1) - X(t_2)$ has a normal distribution for every $t_1$ and $t_2$ on the real line with (ii) the expectation $E\{X(t_1) - X(t_2)\} = 0$ and (iii) variance $E\{[X(t_1) - X(t_2)]^2\} = 2D \cdot |t_1 - t_2|$, where D is a physical constant. iv) Two consecutive events $X(t_{i-1}, t_i)$ and $X(t_i, t_{i+1})$ are statistically independent.

Following is a sample path or trajectories of Wiener process, demonstrated and development by P. Levy 1948, that are continuous but almost all nondifferentiable functions [9].

$$(\Delta t)^2 \rightarrow 0, \quad \Delta t \cdot \Delta W = \Delta W \cdot \Delta t \rightarrow 0, \quad \Delta W \cdot \Delta W \rightarrow dt \tag{9}$$

Our objective is to use the Riemann -Stieltjes integral to solve stochastic differential equations. It will be interesting to examine SDE with nowhere differentiable perturbation.

**Ito's calculus:** According to Ito, for the Brownian motion $W_t$ the Langevin Stochastic Differential equation will be used symbolically

$$d(y(t), t) = -b(t) * y(t) * dt + g(t) * dw(t); \tag{10}$$

if the following integral exists.

$$y(t) = y(t_0) + \int_{t_0}^{t} -b(s) * y(s)ds + \int_{t_0}^{t} g(s)dw_s; \tag{11}$$

## 3.1 Ito's Chain Rule Formula:

Taylor's expansion of a multivariable function $Y = f(t, X)$ considered in deterministic calculus and expanded to $Y_2 = f(t_2, X_2)$ about a point $(t_1, X_1)$ will be

$f(t_2, X_2) = f(t_1, X_1) + \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial X}\Delta X + \frac{1}{2}[\frac{\partial^2 f}{\partial t^2}(\Delta t)^2 + \frac{\partial^2 f}{\partial t \partial X}\Delta t \cdot \Delta X + \frac{\partial^2 f}{\partial X^2}(\Delta X)^2] + ...$

Assume $\Delta t \rightarrow 0$, thus, the Taylor's expansion will be

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}[\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial X}dt \cdot dX + \frac{\partial^2 f}{\partial X^2}(dX)^2] + ...$$

Apply the relation (13) as a principle of Brownian motion X in the Taylor's formula

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}[\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial t \partial X}dt \cdot dX + \frac{\partial^2 f}{\partial X^2}(dX)^2] + ...$$

and use $(dt)^2 \rightarrow 0$ , $dt \cdot dX \rightarrow 0$, and $(dX)^2 \rightarrow dt :$

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X}dX + \frac{1}{2}[\frac{\partial^2 f}{\partial X^2}(dX)^2] + ...$$

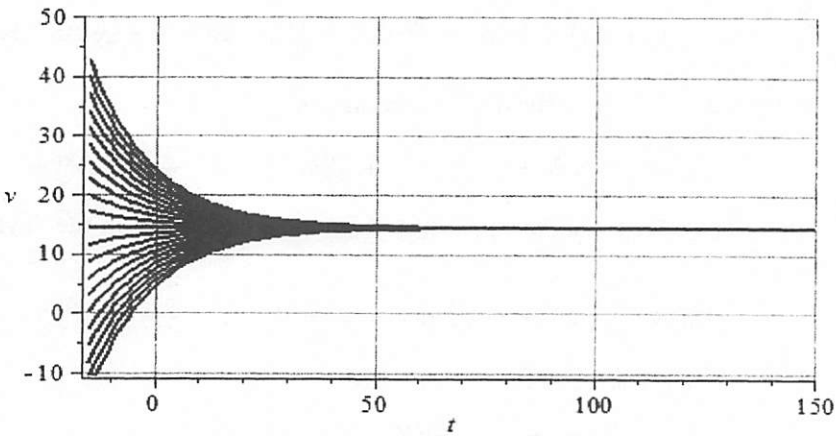
This relation is known as **Ito's chain rule formula for stochastic differential equations.**

$$dY_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_t}dX_t + \frac{1}{2}[\frac{\partial^2 f}{\partial X_t^2}(dX_t)^2] \tag{12}$$

## 3.2 Density Independent Random Perturbation :

The following Langevin DE with random coefficients b and c will be examined by random perturbation function w(t) at time $t$. In the following Maple codes, random initial values, drift coefficient, and diffusion will be assigned at the beginning of the simulation. The solution produced can be considered a deterministic trajectory, with all initial parameters selected randomly. In the following program we can observe simulations for the trajectories and their asymptotic limits.

```
> with(plots);
> SDE1 := diff(v(t), t) = -b*v(t)+c*w(t);
> soln1 := dsolve(SDE1, v(t));
> b := (1/100)*(rand(1 .. 9))(); c := (1/10)*(rand(1 .. 6))();
> w := proc (t) options operator, arrow; (rand(1 .. 6))() end proc;
> SDE2 := diff(v(t), t) = -b*v(t)+c*w(t);
> soln2 := dsolve(SDE2, v(t));
> soln3 := subs(_C1 = m, soln2);
> myplot1 := {seq(subs(m = i, rhs(soln3)), i = -10 .. 10)};
> plot(myplot1, t = -15 .. 150, v = -10 .. 50, color = blue);
```




**Simulation of the trajectories of deterministic Langevin Equation.**

Fig. (1)

## 3.3 Random Solution, Mean, and Variance

**Density Independent Model:** Assume that the random force function $g(t)$ =c ( in the relation $\overrightarrow{R(t)} = c \cdot dW_t$) is not proportional to the density function Y. That is

$$dX_t = -bX_t dt + c \cdot dW_t, \qquad X_0 = X(t_0) \tag{13}$$

This is a symbolic form only in the Ito sense. Using a deterministic linear differential equation the solution will be

$$X_t = X_0 e^{-bt} + \int_0^t c \cdot e^{-b(t-s)} dW_s, \qquad \text{for all } t \geq 0 \tag{14}$$

Since $E(W_t) = 0$ at any moment t, then $E(\int_0^t e^{-b(t-s)} dW_s) = 0$. As a result

$$E(X_t) = E(X_0) \cdot e^{-bt} \tag{15}$$

To find the variance of the solutions, we will use the variance properties:
$Var(X_t) = E(X_t^2) - [E(X_t)]^2 = E\{X_0 e^{-bt} + \int_0^t c \cdot e^{-b(t-s)} dW_s\}^2 - \{E(X_0) \cdot e^{-bt}\}^2$
$= E\{X_0^2 e^{-2bt} + 2 \cdot X_0 e^{-bt} \cdot \int_0^t c \cdot e^{-b(t-s)} dW_s + [c \int_0^t e^{-b(t-s)} dW_s]^2\} - [E(X_0)]^2 \cdot e^{-2bt}\}$
$= Var(X_0) \cdot e^{-2bt} + \frac{c^2}{2b}[1 - e^{2bt}]$

$$Var(X_t) = Var(X_0) \cdot e^{-2bt} + \frac{c^2}{2b}[1 - e^{2bt}] \tag{16}$$

(for further information see [6],[7],[16]).

# 4 Numerical Approximation to the Nowhere Differentiable Perturbed Langevin Equation:

We presented nowhere differentiability of the noise in the Langevin differential equation. It was also demonstrated that the fundamental theorem of calculus fails in Lebesgue-Steiltjes sense. Imposing a Wiener process can help us to introduce Ito's integral calculus but it is inconsistent with the existing calculus (see [10], [12]). The following Computational Algorithm is developed to satisfy the stochastic process. In the initial step, parameters are selected randomly and will not stay constant for the next step of time increment. In fact, the position at the end of each step will be considered an initial position for the next step. We call this algorithm a dynamic random algorithm [14]. The following Maple program was used to approximate the solution to the random perturbed differential equations.

## 4.1 Simulation of the Solution of the Langevin Equation:

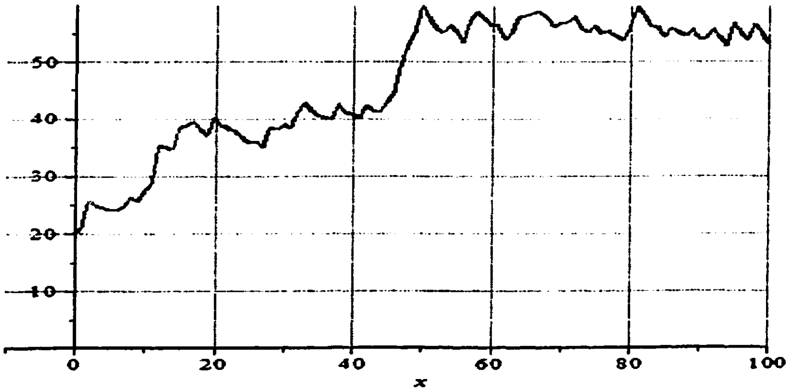
For a deterministic initial value problem the necessary conditions for existence and uniqueness of a solution to the differential equation has been studied extensively. For a stochastic differential equation the approach will be different. In fact for any certain initial conditions, there will be infinite possibilities for the random choice of trajectories. Thus, we see the solutions with their expectation and variances.

To create an algorithm representing a Markovian phenomena, we used a Maple procedure in order to have random choices for fluctuations in a certain time interval $[t_i, t_{i+1}]$. As a Markovian process, in every step, the system does not remember the past information and chooses new random values for w and sigma=c for the Langevin Equation:

$$\frac{dy(x)}{dx} = -b \cdot y(x) + \sigma \cdot w(x). \tag{17}$$

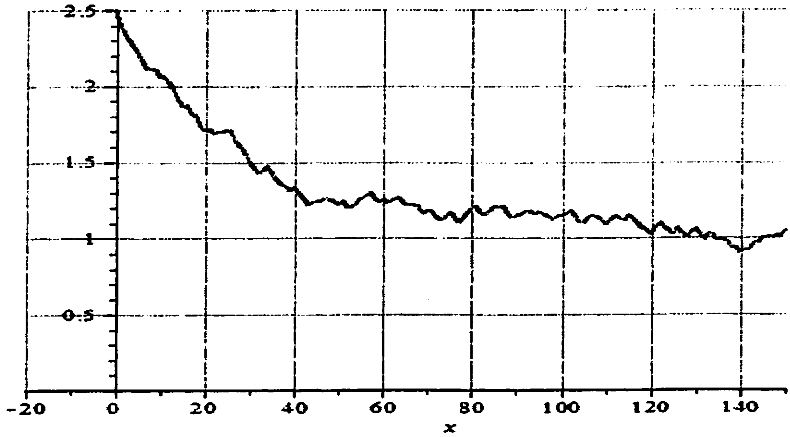
In the following programs, the function g(t) in (4) is considered as a random constant real number c=sig. Maple reserved the symbol sigma for other applications.

```
> Langevin1 := proc (ic1, b, n)
local i, w, sig, eq, s, c, ic, f, g;
c[1] := 0; ic[1] := y(c[1]) = ic1;
for i to n do
w := (1/10)*(rand(1 .. 9))(); sig := (rand(1 .. 9))();
eq := diff(y(x), x) = -b*y(x)+sig*w;
s[i] := rhs(dsolve(ic[i], eq, y(x)));
c[i+1] := 100*i/n; ic[i+1] := y(c[i+1]) = evalf(subs(x = c[i+1], s[i]));
f[i] := s[i]*Heaviside(x-c[i])*(1-Heaviside(x-c[i+1]))
end do;
g := seq(f[i], i = 1 .. n) end proc;
> plot([Langevin1(20, 0.45e-1, 100)], x = -10 .. 100, discont = true);
```




The solution to the Langevin Equation with density independent fluctuations. Random perturbations signa and w are generated inside the program loop.

Fig.(2)

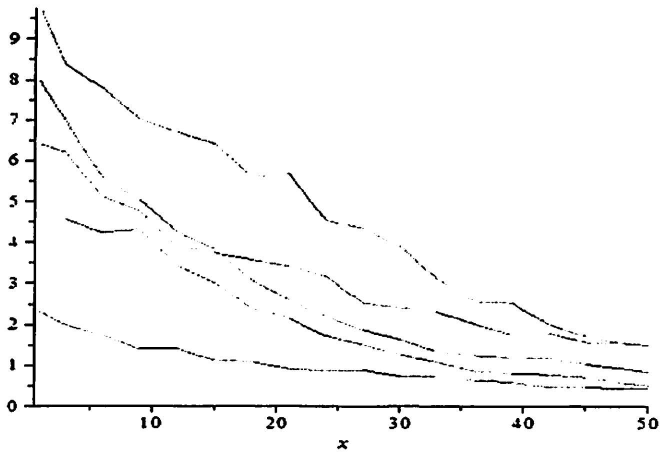



One path for a random trajectory of solution to Langevin Equation when the perturbation w=rand(1..9)()/100 is density independent

Fig. (3)

**Simulation on the initial conditions when the drift coefficient b=.085.**

```
> c[1] := 2.50;
> for i to 5 do c[i+1] := c[i]+2;
f[i] := Langevin(0.85e-1, c[i], 50) end do;
> g := seq(f[i], i = 1 .. 5);
> plot([g], x = 1 .. 50, discont = true);
```



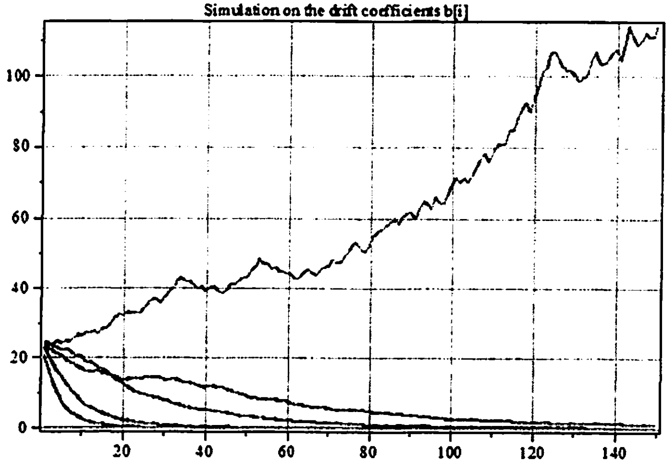
Simulation on initial conditions in Langevin Equation with drift b=.085.

Fig. (4)

**Simulation on the drift coefficient b:**

```
> b[1] := 0.4e-2;
> for i to 5 do b[i+1] = b[i]+0.25e-1;
f[i] := Langevin(b[i], 25, 150)
end do;
> g := seq(f[i], i = 1 .. 5);
> plot([g], x = 1 .. 150, discont = true);
```

Simulation on the drift coefficients b[i]



Drift coefficients b[i] begin from .004 and change with the increment of .025 to create sample path

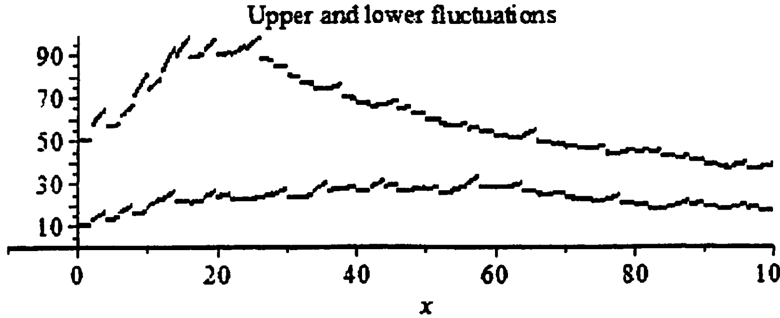
Fig. (5)

## 4.2 Upper and Lower Fluctuations:

The solution mean and variance which were demonstrated in previous sections by formula (15) and (16) respectively will be used in the following computation algorithm. The idea is to find the interval $(\mu - k \cdot \sigma, \mu + k \cdot \sigma)$ for special case k=1. We will evaluate mean and variance in every short time interval to present and demonstrate the upper and lower fluctuations.

If we choose m0 (initial mean) and v0 (initial variance) as the random initial solution mean and random initial variance respectively in every step then the upper and lower fluctuations will not be a continuous process due to presence of fluctuations. To demonstrate the result, recall the program (see appendix) for upper and lower fluctuations.

```
> p2 := plot([Lang1(50, 0.95e-2, 50)], x = -10 .. 100, color = red, discont
= true);
> p3 := plot([Lang2(10, 0.95e-2, 50)], x = -10 .. 100, color = blue, discont
= true);
> display({p2, p3})
```




These are the graph of upper (μ+σ) and lower (μ-σ) fluctuations for two initial values of ic=50 and 10. The procedure is computed with constant parametres b=.0095 and n=50.

Fig. (6)

## 4.3 Discussion:

A quick review of the history of research on the evolution of the stochastic differential equations will guide us through a variety of views and application of Langevin's equation which may be considered a simplest form of the stochastic differential equations. In addition this rich history will show how this problem posed many challenges from integration theory, analysis, and probability theory for many brilliant mathematician for centuries. These problems are also linked to many disciplines in physics, mathematics, business, and economics. The theoretical nature of nowhere differentiability and integrability of this phenomenon might not be possible to explain for lay people. However it is possible to present and demonstrate the solution to them by computational approach or simulation. Mathematically, we created, imposed, and added a nowhere differentiable perturbation on a differential equation in an arbitrary small subinterval. We solved the differential equation in that subinterval and continued this process in the next time interval. The algorithm designed and presented in the article connects all piecewise solutions for random initial points, random parameters, and random perturbations. Numerical computations can be achieved by computer algebra system (CAS) or any spreadsheet. We presented our approach in MAPLE.

## 4.4 Further Research:

- Notice that the random noise is not selected from a certain random probability distribution. To meet the conditions of Wiener process. The noine could be selected from a normal distribution N(0,1).

- The original Langevin equation has a perturbation which is density independent noise. To study a perturbation, the function g in the Langevin differential equation (10) may be selected $g(t, y(t))$as a function of t and density y(t).

- According to the Chebychev's theorem, for some positive k:

$$p[\mu + k \cdot \sigma \leq Y \leq \mu - k \cdot \sigma] \leq 1 - \frac{1}{k^2}. \tag{18}$$

Further study may be useful to demonstrate the computation within k - sigma standard deviation ($k \neq 1$).

- Readers who are interested in developing the research further may apply the algorithm to other linear or nonlinear perturbed differential equations.

**Acknowledgement:** The following algorithm originally was developed using computer algebra system Maple (Appendix (I) and (II)) It was used also for a random perturbation in logistic differential equations [14]. This latest modification was applied to find the mean solution, variance, upper/lower fluctuations in the Langevin equation.

**Appendix:**

I- Program for upper fluctuations:

```
> restart;
> Lang1 := proc (ic1, b, n)
local i, w, sigma, eq, s, m, m0, v, v0, c, ic, f, f1,
g; c[1] := 0; ic[1] := y(c[1]) = ic1; m0 := ic1;
for i to n do
w := (1/10)*(rand(1 .. 9))(); sigma := (rand(1 .. 9))();
v0 := (1/100)*(rand(1 .. 9))();
eq := diff(y(x), x) = -b*y(x)+sigma*w;
s[i] := rhs(dsolve({ic[i], eq}, y(x)));
m[i] := evalf(m0*exp(-2*b*c[i]));
v[i] := (v0*exp(-2*b*c[i])+(1/2)*(1-exp(-2*b*c[i]))/b)^(1/2);
f1[i] := m[i]+v[i];
c[i+1] := 100*i/n; ic[i+1] := y(c[i+1]) = evalf(subs(x = c[i+1], s[i]));
f[i] := f1[i]*Heaviside(x-c[i])*(1-Heaviside(x-c[i+1]));
m0 := s[i]
end do;
g := seq(f[i], i = 1 .. n)
end proc;
```

**II- Program for Lower Fluctuations:**

```
> restart;
> Lang2 := proc (ic1, b, n)
local i, w, sigma, eq, s, m, m0, v, v0, c, ic,f2, p, h;
c[1] := 0; ic[1] := y(c[1]) = ic1; m0 := ic1;
for i to n do
w := (1/10)*(rand(1 .. 9))(); sigma := (rand(1 .. 9))();
v0 := (1/100)*(rand(1 .. 9))();
eq := diff(y(x), x) = -b*y(x)+sigma*w;
s[i] := rhs(dsolve({ic[i], eq}, y(x)));
m[i] := evalf(m0*exp(-2*b*c[i]));
v[i] := (v0*exp(-2*b*c[i])+(1/2)*(1-exp(-2*b*c[i]))/b)^(1/2);
f2[i] := m[i]-v[i];
c[i+1] := 100*i/n; ic[i+1] := y(c[i+1]) = evalf(subs(x = c[i+1], s[i]));
p[i] := f2[i]*Heaviside(x-c[i])*(1-Heaviside(x-c[i+1]));
m0 := s[i]
end do;
h := seq(p[i], i = 1 .. n)
end proc;
```

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