Preservers of cover numbers and independence numbers of undirected graphs.*†

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Abstract

Let $\mathcal{G}_n$ be the set of all simple loopless undirected graphs on n vertices. Let T be a linear mapping, $T : \mathcal{G}_n \to \mathcal{G}_n$ for which the independence number of $T(G)$ is the same as the independence number for G for any $G \in \mathcal{G}_n$. We show that T is necessarily a vertex permutation. Similar results are obtained for mappings preserving the matching number of bipartite graphs, the vertex cover number of undirected graphs, and the edge independence number of undirected graphs.

1 Introduction

Let $\mathcal{G}_n$ denote the set of all simple loopless undirected graphs on n vertices. In this paper we will investigate transformations of $\mathcal{G}_n$ which preserve of some functions mapping $\mathcal{G}_n$ into the nonnegative integers. These functions include the independence number, the vertex cover number, the max matching number which is the edge independence number, the edge cover number and others. For example, we address questions like, if $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear map such that the vertex covering number of an element $G \in \mathcal{G}_n$

* Key words and phrases : Independence numbers, cover numbers, linear transformation, preserver.
† AMS Subject Classification: 15A86, 05C69

is the same as the vertex covering number of $T(G)$ for every $G \in \mathcal{G}_n$, what can be said of the structure of T?

In section 2 we give necessary definitions and notation; in section 3 we will investigate matching number preservers in bipartite graphs; in section 4 we will discuss the structure of independence and cover number preservers in undirected graphs.

2 Preliminaries

We will assume that the reader is familiar with the basic concepts of graph theory and matrix theory. See [5, 8, 9, 11] for basic definitions. We call a graph on n vertices an *edge graph* if the cardinality of the edge set is one, that is if the edge set of a graph is $\{ab\}$ where a and b are vertices of G and ab is the edge joining vertex a to vertex b, then the graph is an edge graph and is denoted $E_{a,b}$. A *star graph* is a graph all of whose edges are incident with a single vertex. If this vertex is the vertex a and there are $n-1$ edges in the graph we call it a full star and is denoted S_a. If the vertex set of G is $V = \{v_1, v_2, \cdots, v_n\}$ we shorten the notation to $V = \{1, 2, \cdots, n\}$ and use the notation S_i to be the full star centered at v_i and $E_{i,j}$ to denote the edge graph whose edge set is $\{v_i v_j\}$.

Let $G \in \mathcal{G}_n$. There are several numbers associated with G:

- The *(vertex) independence number of* G, $\alpha(G)$, is the size of a largest subset of the vertex set of G which induces an edgeless graph.
- The *vertex cover number of* G, $\beta(G)$, is the size of a smallest set of vertices which are incident with every edge in the graph,
- The *edge independence number of* G, $\alpha'(G)$, is the size of a largest set of edges of G no two of which are adjacent.
- The *edge cover number of* G, $\beta'(G)$, is the size of a smallest set of edges that are incident with every vertex of G.

This list is obviously very incomplete, however it includes most of the numbers associated with G that we will be investigating.

Note that some of these numbers are known by other names and descriptions. In particular, the edge independence number, $\alpha'(G)$, is also the size of the largest matching in the graph G. In [1], the *star cover number* was defined to be the smallest number of full star graphs the union of whose edge sets contains (perhaps properly) all the edges of the graph. The star cover number is the same as the vertex cover number, $\beta(G)$.

Let $\mathbb{B}$ be the set $\{0, 1\}$ with Boolean arithmetic, that is all the arithmetic is the same as for real numbers except that $1 + 1 = 1$. Let $\mathcal{M}_{m,n}(\mathbb{B})$ be the set of all $m \times n$ $(0,1)$-matrices, and if $m = n$ we write $\mathcal{M}_n(\mathbb{B})$. Also, we let $\mathcal{S}_n^{(0)}(\mathbb{B})$ denote the set of all $A \in \mathcal{M}_n(\mathbb{B})$ which are symmetric with all diagonal entries equal zero. A *line* of a matrix is a row or column of that matrix. The *term rank* of a matrix $A \in \mathcal{M}_{m,n}(\mathbb{B})$, $tr(A)$, is the minimum number of lines that contain all the nonzero entries of A.

Let $G \in \mathcal{G}_n$ and $A(G) \in \mathcal{M}_n(\mathbb{B})$ be the adjacency matrix of G. Then, for $H, G \in \mathcal{G}_n$, $A(H \cup G) = A(G) + A(H)$, where $G \cup H$ is the graph on the same vertex set as G and H and whose edge set is the union of the edge set of G with the edge set of H. We use the term sum of graphs to mean the union and write $G + H = G \cup H$. Since the graphs in $\mathcal{G}_n$ are undirected loopless simple graphs, $\mathcal{G}_n \cong \mathcal{S}_n^{(0)}(\mathbb{B})$. Because of this, when we are interested in a property of graphs as it applies to the adjacency matrix, we do not distinguish between the notion of graph and matrix, so that the independence number of a matrix, $\alpha(A)$ is the same as the independence number, $\alpha(G)$, of the graph, G, where $A(G) = A$. Likewise, the term rank of a graph is the term rank of the adjacency matrix. Similarly, $E_{i,j}$ denotes not only the edge graph but also the matrix whose only nonzero entry is one in the (i, j) location.

Let G and H be graphs with the same vertex set. We say that G *dominates* H, written $G \geq H$, if the edge set of H is a subset of the edge set of G. Similarly, if H and K are two $m \times n$ matrices we say H dominates K, written $H \geq K$ if $k_{i,j} \neq 0$ implies that $h_{i,j} \neq 0$.

The following is a theorem of Gallai, (1959) See [9, Theorem 8.17].

Theorem 2.1 *Let $G \in \mathcal{G}_n$ with no isolated vertices, then*

$$\alpha(G) + \beta(G) = n \textit{ and } \alpha'(G) + \beta'(G) = n.$$

An easy observation is that if G has isolated vertices, say p vertices of G are isolated, then $\alpha'(G) + \beta'(G) = n - p$, and the other equation is unchanged.

A transformation on $\mathcal{M}_{m,n}(\mathbb{B})$ is linear if it is additive and $T(O) = O$. Also, a transformation on $\mathcal{G}_n$ is linear if the image of the union of two graphs is the union of the images of the two graphs and the image of the edgeless graph is empty. Let $T_G : \mathcal{G}_n \to \mathcal{G}_n$ be a linear transformation on $\mathcal{G}_n$. Then, define the transformation $T_M : \mathcal{S}_n^{(0)}(\mathbb{B}) \to \mathcal{S}_n^{(0)}(\mathbb{B})$ by $T_M(A(G)) = A(T_G(G))$. Similarly, for a transformation $T_M : \mathcal{S}_n^{(0)}(\mathbb{B}) \to \mathcal{S}_n^{(0)}(\mathbb{B})$ define $T_G : \mathcal{G}_n \to \mathcal{G}_n$ by $T_G(G) = T_M(A(G))$ where $A(G)$ is the adjacency matrix of G. In this way we see that the set of linear transformations on $\mathcal{G}_n$ is isomorphic, in a natural way, to the set of linear transformations on $\mathcal{S}_n^{(0)}(\mathbb{B})$. Thus, we will not distinguish between transformations on $\mathcal{S}_n^{(0)}(\mathbb{B})$ from transformations on $\mathcal{G}_n$ We write $T(G)$ or $T(A)$ as appropriate.

Let $W \subseteq \mathcal{G}_n$. A linear transformation $T : \mathcal{G}_n \to \mathcal{G}_n$ is said to *preserve* the set W if $G \in W$ implies that $T(G) \in W$. We say that T *strongly* preserves the set W if, $G \in W$ if and only if $T(G) \in W$. We similarly define preservers of sets of matrices. Let $\varphi : \mathcal{G}_n \to \mathbb{Z}_+$ where $\mathbb{Z}_+$ is the set of nonnegative integers. Then we say that T preserves (strongly) preserves φ if T(strongly) preserves all the sets $W_i = \{G \in \mathcal{G}_n | \varphi(G) = i\}$. Further, for some fixed i, we say that T (strongly) preserves $\varphi = i$ if T (strongly) preserves W_i.

The investigation of preservers of sets and functions has been an active area of research in the past few years. The study of linear preservers began with Frobenius in 1896 and for most of a century, all of the problems considered were preservers of sets and functions of matrices over fields or rings. In 1984, Beasley and Pullman [2] came out with the first article on linear preservers of sets of matrices over a semiring, specifically over $\mathbb{B}$. Results of Beasley and Pullman that we will use in the sequel are summarized in:

Theorem 2.2 [3, 4] *Let $T : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathcal{M}_{m,n}(\mathbb{B})$ be a linear operator. Then the following are equivalent:*

1. *T preserves term rank,*
2. *T preserves term ranks 1 and 2,*
3. *T strongly preserves term rank k for some $1 \leq k \leq n$,*
4. *There are permutation matrices P and Q such that $T(X) = QXP$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$; or, $m = n$ and $T(X) = QX^tP$ for all $X \in \mathcal{M}_{m,n}(\mathbb{B})$ where X^t denotes the transpose of X.*

Note that in graph theoretic terms the characterization of T in part 4, when $m = n$ and $Q = P^t$, is that T is a vertex permutation.

Note that preservers of a single term rank were not characterized in Theorem 2.2. Let W be any subset of $\mathcal{M}_{m,n}(\mathbb{B})$ and let T be any transformation on $\mathcal{M}_{m,n}(\mathbb{B})$ whose image is a subset of W. Then T preserves the set W. In the investigation of preservers of sets of matrices over the Boolean algebra $\mathbb{B}$, an additional condition has to be added to T to have any hope of characterizing T. This condition is usually that T is bijective (or equivalently surjective or injective), that T strongly preserves the set or that T preserves two or more (usually disjoint) sets. Of these conditions, the condition that T be bijective is the most restrictive, and that T preserve two sets is the least. If $W \subseteq \mathcal{M}_{m,n}(\mathbb{B})$ and $W \neq \mathcal{M}_{m,n}(\mathbb{B})$ then, if the image of T is a subset of W, T is not bijective, T does not strongly preserve W and T cannot preserve two disjoint sets unless they are both in W, however, T preserves W. If T preserves a function, like term rank, then clearly T preserves two disjoint sets.

3 The Bipartite Case

In this section we will investigate linear preservers of numbers associated with bipartite graphs. A graph is *bipartite* if the vertex set of the graph can be partitioned into two sets X and Y such that every edge in the graph has one incident vertex in X and the other in Y. We will let $\mathcal{B}(X, Y)$ be the set of all bipartite graphs with bipartition X, Y. Since the orders of the sets X and Y is the only consideration for our study, we let $\mathcal{B}_{m,n} = \mathcal{B}(X, Y)$ if $|X| = m$ and $|Y| = n$. Note that $\mathcal{B}_{m,n} \subset \mathcal{G}_{m+n}$.

Let $G \in \mathcal{B}_{m,n} = \mathcal{B}(X, Y)$, where $X = \{v_1, v_2, \cdots, v_m\}$ and $Y = \{v_{m+1}, v_{m+2}, \cdots, v_{m+n}\}$. Then, the adjacency matrix of G is $A(G) = \begin{bmatrix} O & B \\ B^t & O \end{bmatrix}$ for some matrix $B \in \mathcal{M}_{m,n}(\mathbb{B})$. B is called the reduced adjacency matrix of G. Notice that for $G \in \mathcal{B}_{m,n}$, $\alpha'(G)$ is the size of the largest matching in G. Transformations that preserve perfect matchings were investigated in [7]. In that article, the transformation was required to be bijective, but not necessarily linear.

A classical theorem of D. König, 1936, see [6, page 164, Theorem 8.2.2], is:

Theorem 3.1 *Let G be a bipartite grapk with reduced adjacency matrix B, then the term rank of B is the size of a largest matching in G, that is $tr(B) = \alpha'(G)$.*

We now characterize the linear preservers of α':

Theorem 3.2 *Let $T : \mathcal{B}_{m,n} \to \mathcal{B}_{m,n}$ be a linear transformation. Then the following are equivalent:*

1. *T preserves α';*
2. *T strongly preserves $\alpha' = k$ for some k, $1 \le k \le min\{m, n\}$;*
3. *T preserves $\alpha' = 1$ and $\alpha' = 2$;*
4. *T is a vertex permutation.*

Proof. If $G \in \mathcal{B}_{m,n}$, and the reduced adjacency matrix of G is B then $\alpha'(G) = tr(B)$ by Theorem 3.1, thus, the theorem follows by applying Theorem 2.2. ■

4 Preservers of Undirected Graph Numbers

In this section we will investigate the preservers of the vertex independence number, the vertex cover number, the edge independence number, and the edge covering number of an undirected graphs in $\mathcal{G}_n$. We let K_n denote the complete loopless simple graph in $\mathcal{G}_n$ and also the matrix in $\mathcal{S}_n^{(0)}(\mathbb{B})$ with all entries one except the diagonal entries which of course are all 0. Let $O = \overline{K}_n$ be the edgeless graph or the zero matrix, the matrix all of whose entries are 0.

We begin with a result recently obtained by Beasley, Kang and Song [1]:

Theorem 4.1 *If $T : \mathcal{S}_n^{(0)}(\mathbb{B}) \to \mathcal{S}_n^{(0)}(\mathbb{B})$ is a liner operator then the following are equivalent:*

(i) *T preserves the star cover number;*

(ii) *T preserves star cover number 1 and $T(K_n) = K_n$;*

(iii) *T preserves star cover numbers 1 and 2;*

(iv) *There exists a permutation matrix P such that $T(X) = P^t X P$ for every $X \in \mathcal{S}_n^{(0)}(\mathbb{B})$.*

4.1 The vertex cover number and vertex independence number.

We observe that the star cover number of a graph is precisely the vertex cover number of that graph. so a corollary to the above theorem is:

Corollary 4.1.1 *If $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator then the following are equivalent:*

(i) *T preserves the vertex cover number, β*

(ii) *T preserves $\beta = 1$ and $T(K_n) = K_n$;*

(iii) *T preserves $\beta = 1$ and $\beta = 2$;*

(iv) *T is a vertex permutation.*

Theorem 4.2 *If $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator then the following are equivalent:*

(i) *T preserves the (vertex) independence number, α*

(ii) *T preserves $\alpha = n - 1$ and $T(K_n) = K_n$;*

(iii) *T preserves $\alpha = n - 1$ and $\alpha = n - 2$;*

(iv) *T is a vertex permutation.*

Proof. By Theorem 2.1, $\alpha(G) + \beta(G) = n$, so if T preserves $\alpha = k$, T preserves the $\beta = n - k$. By Corollary 4.1.1 the theorem follows. ■

4.2 The edge independence number.

We now characterize the edge independence number, α', also known as the max matching number. We will use without reference that the set of graphs of edge independence number 1 are precisely the set of star graphs together with the set of three cycles. If $n = 2$, There are only two graphs in $\mathcal{G}_2$, and hence, If T preserves α', T is the identity.

Recall that a mapping is nonsingular if $T(X) = O$ implies $X = O$, where O is either the zero matrix or the edgeless graph. Note that for graphs or for matrices over $\mathbb{B}$, nonsingularity is not equivalent to invertibility. For example, if G is any nonempty graph in $\mathcal{G}_n$ and $T(X) = G$ for every $X \in \mathcal{G}_n, X \neq O$, and $T(O) = O$, then T is a nonsingular linear operator on $\mathcal{G}_n$, but clearly, T is not invertible.

Theorem 4.3 *Let $T : \mathcal{G}_3 \to \mathcal{G}_3$ be a liner operator. Then, T preserves α' if and only if T preserves the set of graphs whose edge independence number is 1 if and only if T is nonsingular.*

Proof. Since every nonempty graph in $\mathcal{G}_3$ has edge independence number 1, the theorem follows by the definition of nonsingular. ■

Define the operator $L_1 : \mathcal{G}_4 \to \mathcal{G}_4$ by $L_1(E_{1,4}) = E_{2,3}$, $L_1(E_{2,3}) = E_{1,4}$ and $L_1(E_{i,j}) = E_{i,j}$ for all $\{i,j\} \neq \{1,4\}, \{2,3\}$. Then it is easily checked that L_1 preserves α'.

Theorem 4.4 *Let $T : \mathcal{G}_4 \to \mathcal{G}_4$ be a liner operator. The following are equivalent:*

1. *T preserves the edge independence number, α'*
2. *T preserves $\alpha' = 1$ and $\alpha' = 2$;*
3. *T is a vertex permutation, $T = L_1$, or T is a composition of these two.*

Proof. Since all nonempty graphs in $\mathcal{G}_4$ have independence number 1 or 2, clearly (*1.*) is equivalent to (*2.*). Also, we clearly have that (*3.*) implies (*1.*). We now show that (*1.*) implies (*3.*).

Suppose that T preserves the edge independence number but does not preserve the set of star graphs, then, since every star has edge independence number 1, the image of some star is a 3 cycle. Without loss of generality we may assume that $T(S_1) = C_3(1,2,3)$, where $C_3(1,2,3)$ is the three cycle $E_{1,2} + E_{1,3} + E_{2,3}$.

Suppose that the image of an edge graph dominated by S_1 is not an edge graph, but the sum of at least 2 edge graphs.

Suppose that $T(E_{1,2}) = C_3(1,2,3)$. Then, since $\alpha'(E_{1,2} + E_{2,4}) = 1$, we have that $\alpha'(T(E_{1,2} + E_{2,4})) = 1$. That is, $\alpha'(C_3(1,2,3) + T(E_{2.4})) = 1$. Thus, $T(E_{2,4})$ is dominated by $C_3(1,2,3)$. But then, since $\alpha'(E_{1,3} + E_{2,4}) = 2$, we must have that $T(E_{2,4})$ contains an edge not in $C_3(1,2,3)$, a contradiction. Thus, $T(E_{1,2}) \neq C_3(1,2,3)$.

Suppose that $T(E_{1,2})$ contains two edges, then, without loss of generality, we may assume that $T(E_{1,2}) = E_{1,2} + E_{1,3}$ and that $T(E_{1,3}) \geq E_{2,3}$. Since $T(S_1) = C_3(1,2,3)$, we have that $T(E_{1,4}) \leq T(E_{1,2} + E_{1,3})$. Since $\alpha'(E_{1,2} + E_{1,3} + E_{2,3}) = 1$, we have that $\alpha'(T(E_{1,2} + E_{1,3} + E_{2,3})) = 1$. But, since $T(E_{1,4} + E_{2,3}) \leq T(E_{1,2} + E_{1,3} + E_{2,3})$ and $\alpha'(E_{1,4} + E_{2,3}) = 2$, we must have that $\alpha'(T(E_{1,2} + E_{1,3} + E_{2,3}))$ is at least 2, a contradiction. Thus $T(E_{1,2})$ is an edge graph. It follows that the image of any edge graph dominated by S_1 is an edge graph.

Thus, we may assume that $T(E_{1,2}) = E_{1,2}$, $T(E_{1,3}) = E_{1,3}$ and $T(E_{1,4}) = E_{2,3}$.

Now, $\alpha'(E_{1,3}+E_{2,4}) = 2$, thus, $\alpha'(T(E_{1,3}+E_{2,4})) = \alpha'(E_{1,3}+T(E_{2,4})) = 2$, so $T(E_{2,4}) \geq E_{2,4}$. That is, $T(E_{2,4}) = E_{2,4} + X$. for some $X \in \mathcal{G}_4$. Similarly, $\alpha'(E_{1,4}+E_{2,3}) = 2$, thus, $\alpha'(T(E_{1,4}+E_{2,3})) = \alpha'(E_{2,3}+T(E_{2,3})) = 2$, so $T(E_{2,3}) \geq E_{1,4}$. That is, $T(E_{2,3}) = E_{1,4} + Y$. for some $Y \in \mathcal{G}_4$. Also $\alpha'(E_{1,2} + E_{3,4}) = 2$, thus, $\alpha'(T(E_{1,2} + E_{3,4})) = \alpha'(E_{1,2} + T(E_{3,4})) = 2$, so $T(E_{3,4}) \geq E_{3,4}$. That is, $T(E_{3,4}) = E_{3,4} + Z$. for some $Z \in \mathcal{G}_4$.

It follows that $T(E_{1,4} + E_{2,4} + E_{3,4}) = E_{2,3} + E_{2,4} + E_{3,4} + X + Y + Z$. That is $T(S_4) = C_3(1,3,4) + X + Y + Z$ where $C_3(1,3,4)$ is the three cycle on vertices v_1, v_3, and v_4. It follows that $T(S_4) = C_3(1,3,4)$ and as above, we have $T(E_{2,4}) = E_{2,4}$ and $T(E_{3,4}) = E_{3,4}$.

Similarly, $T(E_{2,3}) = E_{1,4}$. That is T is the operator L_1.

We now assume that for each $i = 1, \cdots, 4$ there is some j such that $T(S_i) \leq S_j$. Now if two stars are mapped into the same star, say $T(S_1 + S_2) \leq S_1$ with out loss of generality, then we have a contradiction since $\alpha'(S_1 + S_2) = 2$ while $\alpha'(S_1) = 1$. That is T permutes the stars, or, T is a vertex permutation. ∎

Theorem 4.5 *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ If $n \geq 5$, then the following are equivalent:*

1. *T preserves the edge independence number, α'*
2. *T preserves $\alpha' = 1$ and $T(K_n) = K_n$;*
3. *T preserves $\alpha' = 1$ and $\alpha' = 2$;*
4. *T strongly preserves $\alpha' = 1$;*
5. *T is a vertex permutation.*

Proof. Clearly (*1.*) implies (*3.*), (*3.*) is equivalent to (*4.*), (*5.*) implies (*1.*), and (*5.*) implies (*2.*). We begin by showing that (*2.*) implies (*5.*).

Suppose that T preserves $\alpha' = 1$ and $T(K_n) = K_n$. Then, $T(S_1)$ is either a star or a three cycle. Suppose with out loss of generality that $T(S_1) = C_3(1,2,3)$. Since $T(K_n) = K_n$, there is some (r, s) such that $T(E_{r,s}) \geq E_{4,5}$. But since $\alpha'(E_{1,r} + E_{r.s}) = 1$, $\alpha'(T(E_{1,r} + E_{r.s})) = 1$. But, $T(E_{1,r}) \leq C_3(1,2,3)$ and is nonzero, and $T(E_{r.s}) = E_{4,5}$. It follows that $\alpha'(T(E_{1,r} + E_{r.s})) \geq 2$, a contradiction. Thus the image of a star must be a star and since the edge independence number of the sum of two full stars is two, T maps distinct stars into distinct stars, that is T is a vertex permutation. That is, (*2.*) implies (*5.*).

We now show that (*3.*) implies (*5.*).

Suppose that T does not preserve stars. Then we may assume as in the proof of Theorem 4.4 that $T(S_1) = C_3(1,2,3)$.

Suppose that the image of an edge graph dominated by S_1 is not an edge, but the sum of at least 2 edges.

Further, suppose that $T(E_{1,2}) = C_3(1,2,3)$. Then, since $\alpha'(E_{1,2} + E_{2,4}) = 1$, we have that $\alpha'(T(E_{1,2} + E_{2,4})) = 1$. That is, $\alpha'(C_3(1,2,3) + T(E_{2.4})) = 1$. Thus, $T(E_{2,4})$ is dominated by $C_3(1,2,3)$. But then, since $\alpha'(E_{1,3} + E_{2,4}) = 2$, we must have that $T(E_{2,4})$ dominates an edge not in $C_3(1,2,3)$, a contradiction, thus, $T(E_{1,2}) \neq C_3(1,2,3)$.

Suppose that $T(E_{1,2})$ dominates two edges, then without loss of generality we may assume that $T(E_{1,2}) = E_{1,2} + E_{1,3}$ and that $T(E_{1,3}) \geq E_{2,3}$.

Since $T(S_1) = C_3(1,2,3)$, we have that $T(E_{1,4}) \leq T(E_{1,2} + E_{1,3})$. Since $\alpha'(E_{1,2}+E_{1,3}+E_{2,3}) = 1$, we have that $\alpha'(T(E_{1,2}+E_{1,3}+E_{2,3})) = 1$. But since $T(E_{1,4} + E_{2,3}) \leq T(E_{1,2} + E_{1,3} + E_{2,3})$ and $\alpha'(E_{1,4} + E_{2,3}) = 2$, we must have that $\alpha'(T(E_{1,2}+E_{1,3}+E_{2,3}))$ is at least 2, a contradiction. Thus $T(E_{1,2})$ is an edge. It follows that the image of any edge graph dominated by S_1 is an edge.

Thus, we may assume that $T(E_{1,2}) = E_{1,2}$, $T(E_{1,3}) = E_{1,3}$ and $T(E_{1,4}) = E_{2,3}$. Suppose with out loss of generality, that $T(E_{1,5}) = E_{1,2}$. Then $\alpha'(E_{1,5} + E_{3,5}) = 1$ and $\alpha'(E_{1,2} + E_{3,5}) = 2$. But $T(E_{1,5} + E_{3,5}) = T(E_{1,2} + E_{3,5})$, a contradiction, Thus, T maps stars to stars.

We now have that for each $i = 1, \cdots, n$ there is some j such that $T(S_i) \leq S_j$. Now if two stars are mapped into the same star, say without loss of generality that $T(S_1 + S_2) \leq S_1$, then we have a contradiction since $\alpha'(S_1 + S_2) = 2$ while $\alpha'(S_1) = 1$. That is T permutes the stars, or, T is a vertex permutation. ■

4.3 The edge covering number.

We now turn our attention to the preservers of the edge cover number of G, $\beta'(G)$.

Note that there are only two graphs in $\mathcal{G}_2$, the edgeless graph and the graph with one edge. Thus, T preserves β' if and only if T is the identity transformation. We thus assume that $n \geq 3$. Observe that the set of graphs whose edge cover number is 1 is precisely the set of edge graphs. Thus, if T preserves edge cover number 1, then the image of any edge graph is an edge graph. Let $\mathcal{E}$ be the set of edge graphs in $\mathcal{G}_n$.

Lemma 4.5.1 *Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. If $n \geq 3$ and T preserves $\beta' = 1$ and $\beta' = 2$ then T maps $\mathcal{E}$ bijectively onto itself. Further, T is a bijection on $\mathcal{G}_n$.*

Proof. Since T preserves the set of edge graphs, we only need show that T is $1-1$ on $\mathcal{E}$. Suppose that two distinct edge graphs are mapped to a single edge graph, say $T(E) = T(F)$ where E and F are edge graphs. Then $T(E+F) = T(E)$ is an edge graph, but $\beta'(E+F) = 2$ while $\beta'(T(E+F) = 1$, a contradiction. Thus T is a bijection on $\mathcal{E}$. Because T is a bijection on $\mathcal{E}$, the number of distinct edges in the image of a graph is the same as the numbers of distinct edges in the graph, Thus T is a bijection on $\mathcal{G}_n$. ■

Lemma 4.5.2 *Let $T : \mathcal{G}_n \to \mathcal{G}_n$, $n \geq 4$. Then, if T maps the set of full stars bijectively onto the set of full stars, T is a vertex permutation.*

Proof. If $T(S_i) = S_{j_i}$, then the mapping $i \longrightarrow j_i$ is a permutation, otherwise the union of two full stars would be mapped into one, a contradiction. ■

In th following theorem we will make use of these observations on subsets of $\mathcal{G}_n$:

- $\beta' = 1$ is the set $\mathcal{E}$;
- $\beta' = 2$ is the set of graphs with exactly two edges;
- $\beta' = n - 1$ is the set of full stars.

Theorem 4.6 *Let $T : \mathcal{G}_3 \to \mathcal{G}_3$ be a linear operator. Then the following are equivalent:*

1. *T preserves the edge cover number, β';*
2. *T strongly preserves the set of edge graphs;*
3. *T is a vertex permutation.*

Proof. Since the set of graphs with edge cover number 1 is precisely $\mathcal{E}$, we see that (1) is equivalent to (2). Further by Lemma 4.5.1, if T preserves β', then T is a bijection on $\mathcal{E}$, and for $n = 3$, any bijection on $\mathcal{E}$ is a vertex permutation. ■

Theorem 4.7 *Let $T : \mathcal{G}_4 \to \mathcal{G}_4$ be a linear operator. Then the following are equivalent:*

1. *T preserves the edge cover number, β';*
2. *T preserves $\beta' = 1$ and $\beta' = 2$;*
3. *T preserves $\beta' = 2$ and $\beta' = 3$;*
4. *T strongly preserves $\beta' = 3$;*
5. *T is a vertex permutation.*

Proof. It is easily shown that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) implies all the others.

Suppose (2) that T preserves edge cover numbers one and two. Then by Lemma 4.5.1, T is a bijection on the set of edge graphs, and hence, T preserves $\beta' = 3$. That is (1) $\Leftrightarrow$ (2). Since T maps full stars, the only graphs with edge cover number 3, to full stars, T must be a bijection on the set of full stars, that is T is a vertex permutation by Lemma 4.5.2.

Suppose (4) that T strongly preserves $\beta' = 3$, then full stars are mapped to full stars, bijectively. Again, by Lemma 4.5.2, T is a vertex permutation. ■

Theorem 4.8 *If $n \geq 5$ and $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator, then the following are equivalent:*

1. *T preserves the edge cover number, β';*
2. *T preserves $\beta' = 1$, $\beta' = 2$ and $\beta' = 4$;*
3. *T preserves $\beta' = n - 1$, and $\beta' = n - 2$;*
4. *T strongly preserves $\beta' = n - 1$;*
5. *T is a vertex permutation.*

Proof. It is easily shown that (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) implies all the others.

Now, suppose (2), that T preserves edge cover numbers 1, 2, and 4. By Lemma 4.5.1 T is a bijection on the set of edges. Suppose that T does not preserve full stars. Suppose without loss of generality that $T(S_1) = G$ and G is not a full star. Then there are three edges in G which do not form a three star. These three edges must be 1) a 3-path, 2) a 2-path and a disjoint edge, or 3) three disjoint edges. So the image of a three star in S_1, say $E_{1,2} + E_{1,3} + E_{1,4}$, is one of these three possibilities. Say without loss of generality $T(E_{1,2}) = E_{1,2}$.

If $T(E_{1,2} + E_{1,3} + E_{1,4})$ is a 3-path, which must have edge cover number two, the the image of any additional edge can increase the edge cover number at most one, but the addition of any other edge graph dominated by S_1, say $E_{1,5}$, to $E_{1,2} + E_{1,3} + E_{1,4}$ gives a graph with edge cover number 4, and the image must have edge cove number at most 3, a contradiction since T preserves $\beta' = 4$. Thus, the image of $E_{1,2} + E_{1,3} + E_{1,4}$ is not a three path. In either of the other two cases, the addition of any edge to the image must have edge cover number at least 3, but $E_{1,2} + E_{1,3} + E_{1,4} + E_{2,3}$ has edge cover number 2 and its image must have edge cover number a least three, contradicting that T preserves $\beta' = 2$. Thus, T is a bijection on the set of full stars and by Lemma 4.5.2, T is a vertex permutation. That is (2) $\Rightarrow$ (5).

Suppose (4) that T strongly preserves $\beta' = n - 1$, then T is a bijection on the set of full stars, and by Lemma 4.5.2, T is a vertex permutation. ■

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