Decomposition of λK_v into Multigraphs with Four Vertices and Five Edges

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Abstract

We give necessary and sufficient conditions to decompose λ copies, where necessarily $\lambda \geq 2$, of the complete graph K_v into so called "2-petal", "stem-infinity", "barbell", and "box-edge" graphs, all with four vertices and five edges.

1 Introduction

Definition 1.1 A balanced incomplete black design [BIBD], specifically a (v, k, λ)-BIBD, is a pair (V, B), where V is a set of v elements and B is a collection of subsets, or blocks, of V such that every block contains exactly k points and every pair of distinct elements is contained in exactly λ blocks.

A (v, k, λ)-BIBD can also be considered as a decomposition of λK_v (λ copies of K_v) into K_k's.

For example, a Fano Plane is a combinatorial design with $V = \{1, 2, 3, 4, 5, 6, 7\}$ and set of blocks $B = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{5, 6, 1\}, \{6, 7, 2\}, \{7, 1, 3\}\}$ where any pair of distinct points in V are in exactly one block in B. If we interpret a block {a, b, c} as a complete graph on three

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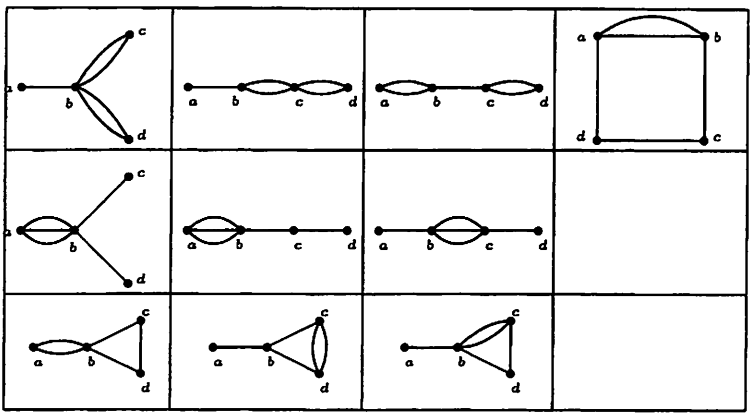
vertices a, b, and c, then one can say that the blocks in B partition the set of edges of K_7 such that each edge of K_7 occurs in exactly one block or K_3. In other words, we get a decomposition of the complete graph K_7 into seven K_3's.

There is much work done on decomposing complete graphs into subgraphs with four vertices and five edges [1]. However, little work has been known to the authors on the decomposition of complete graphs into copies of a multisubgraph except [4], [2], and the work on ternary designs. Since the acceptance of this paper, there is some more work done on multigraph decompositions, see for example, [3], [5] and [6].

In this paper, we decompose complete graphs into four of the possible ten connected multigraphs with four vertices and five edges shown below.

Table of connected multigraphs on four vertices and five edges:



There will be $\frac{\lambda v(v-1)}{2\cdot 5}$ subgraphs in a decomposition of λK_v, if it exists. Hence, the necessary conditions for the existence of a decomposition of λK_v into connected multigraphs with four vertices and five edges are

$\lambda \geq 2$	v
$\lambda \equiv 0 \pmod 5$	all $v \geq 4$
$\lambda \equiv 1,2,3,4 \pmod 5$	$v \equiv 0,1 \pmod 5$

In other words, for such a decomposition to exist, λ must be a multiple of 5 if v is not $\equiv 0,1 \pmod 5$.

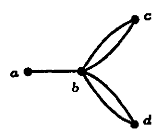
We note that the theory of the PBD closed sets may have provided us with a quick method of such decompositions but unfortunately for $v \equiv$

0, 1(mod 5), the PBD closed sets leave too many exceptions. Hence, our aim is to give general cyclic decompositions with appropriate interpretations of the difference families. If λ is odd, then each edge must occur singly an odd number of times in an odd number of multi-subgraphs. This leads to some interesting anomalies and keeps the the problem of decomposing into different multi-subgraphs intriguing as the necessary conditions may be sufficient for one multigraph but not for another. For example, $5K_4$ cannot be decomposed into 2-petal graphs (see 2.11), whereas $5K_4$ can be decomposed into box-edge subgraphs.

2 Decomposition of λK_v into "2-petal" graphs

First, we would like to decompose λK_v into 2-petal graphs. We call such a decomposition a 2-petal decomposition.

A 2-petal graph with vertices a, b, c, and d:



For convenience, we will denote a 2-petal graph with vertices {a, b, c, d} and the edge set {ab, bc, bc, bd, bd} by <*a*, *b*, *c*, *d*>.

Remark 2.1 Difference sets and difference families provide a powerful tool in the construction of classical combinatorial designs. An interpretation of such difference sets and difference families are useful for these designs as well.

A "difference family" solution of the decomposition of λK_v into 2-petal subgraphs is a set of "difference sets" such that all differences $\{1, 2, \ldots, \frac{v-1}{2}\}$ modulo v, for v odd, and $\{1, 2, \ldots, \frac{v}{2}\}$ for v even, occur the necessary number of times at the "required" locations. By the differences at the "required" locations in the case of 2-petal graphs, we mean the differences between the elements at the first and second locations counted once, the differences between the second and third locations counted twice, and the differences between the second and fourth locations counted twice, so a difference set <*a*, *b*, *c*, *d*> gives the differences between a and b once and differences between b and c as well as between b and d twice.

Throughout this paper similar interpretations will be needed and from the context it will be clear which differences need to be taken into consideration.

2.1 $\lambda = 2$

Recall, for $\lambda = 2$, $v \equiv 0, 1 \pmod 5$ is a necessary condition for a 2-petal decomposition of $2K_v$ to exist.

We note that for $v = 5$ and $v = 6$, difference set solutions do not exist, but a decomposition is given below.

Example 2.2 A 2-petal decomposition of $2K_5$ is
<1, 2, 3, 4>, <2, 1, 4, 5>, <3, 5, 2, 4>, and <5, 3, 1, 4>.

Example 2.3 A 2-petal decomposition of $2K_6$ is
<1, 2, 3, 4>, <2, 1, 4, 5>, <3, 5, 2, 4>, <5, 3, 1, 4>, <5, 6, 1, 2>, and <5, 6, 3, 4>.

Given $\lambda = 2$, $v = 10t$, we can get a 2-petal decomposition of $2K_v$ by a difference family. The difference sets will be constructed modulo $(10t - 1)$ and the size of the difference family is $2t$.

Theorem 2.4 *The difference family for the decomposition of $2K_v$ where $v = 10t$ into 2-petal graphs is given by*
$\{<\infty, 0, t, t+1>, <\infty, 0, t+2, t+3>,$
$<1, 0, t+4, t+5>, <1, 0, t+6, t+7>, \ldots,$
$<t-1, 0, 5t-4, 5t-3>, <t-1, 0, 5t-2, 5t-1>\}.$

This as well as many difference family solutions given in this paper work because each difference occurs the required number of times at the "required" locations (Remark 2.1).

Example 2.5 When $v = 10$, $t = 1$, so there are $2t = 2$ difference sets in the difference family: $<\infty, 0, 1, 2>$, and $<\infty, 0, 3, 4>$.

When $v = 10t + 5$, there are 2t + 1 difference sets, as given below.

Theorem 2.6 *The difference family solutions for the decomposition of $2K_v$ where $v = 10t + 5$ and $v \geq 15$ into 2-petal graphs are:*
$\{<\infty, 0, t, t+1>, <\infty, 0, t+2, t+3>,$
$<1, 0, t+4, t+5>, <1, 0, t+6, t+7>, \ldots,$
$<t-1, 0, 5t-4, 5t-3>, <t-1, 0, 5t-2, 5t-1>\} \cup$
$\{<5t+2, 0, 5t, 5t+1>\}.$

Proof. Note that the last difference set uses the difference $\frac{v-1}{2} = 5t+2$ only once because once the difference set is generated, the pair 0 and 5t+2 will occur again halfway through the generated set, allowing the difference of 5t + 2 to occur twice, as required. ■

Theorem 2.7 *Given $v = 5s$, if the decomposition of $2K_v$ into 2-petal graphs exists, then the decomposition of $2K_{v+1}$ into 2-petal graphs also exists. The decomposition of $2K_{v+1}$ can be obtained by determining the decomposition of $2K_v$ in addition to the following difference sets, which provide the remaining edge set:*
$\{<1,\ v+1,\ s+1,\ s+2>,\ <1,\ v+1,\ s+3,\ s+4>, \ldots,$
$<s,\ v+1,\ 5s-3,\ 5s-2>,\ <s,\ v+1,\ 5s-1,\ \infty>\}$.

Example 2.8 A 2-petal decomposition of $2K_{11}$ is the difference family of $2K_{10}$ together with <1, 11, 3, 4>, <1, 11, 5, 6>, <2, 11, 7, 8>, <2, 11, 0, ∞>.

We note that $v + 1$ is the new point, as ∞ was used in the original difference set.

Lemma 2.9 *The necessary conditions are sufficient for the existence of 2-petal decomposition of $2K_v$.*

2.2 $\lambda = 3$

Theorem 2.10 *$3K_v$'s cannot be decomposed into 2-petal graphs.*

Proof. The number of 2-petal graphs in $3K_v$ is $\frac{3v(v-1)}{10}$, but $\frac{3v(v-1)}{10} < \frac{v(v-1)}{2}$. If λ is odd, then, $\frac{\lambda v(v-1)}{10}$ must be $\geq \binom{v}{2}$, as every edge must come singly at least once. Hence, such a decomposition does not exist. ■

2.3 $\lambda = 5$

Recall, for $\lambda = 5$, $v \geq 4$. However, for the 2-petal graphs in particular, $5K_4$ cannot be decomposed, as shown below.

Theorem 2.11 *$5K_4$ cannot be decomposed into 2-petal graphs.*

Proof. There will be six 2-petal graphs if a decomposition of $5K_4$ exists. Let the vertex set be {1,2,3,4}. As every edge has to come singly at least once, each of the four vertices has to come 3 times as degree one vertex or degree five vertex. So let $x_i + z_i = 3$ for $i = 1, 2, 3$, and 4 where x_i and z_i stand for the number of graphs where the vertex i occurs as degree one vertex and degree 5 vertex respectively. Let y_i be the number of graphs where the vertex i occurs as degree two vertex. Clearly $x_i + y_i + z_i = 6$ and $x_i + 2y_i + 5z_i = 15$. There must be a vertex, WOLG say vertex 1, which comes as degree 1 vertex in at least two of the six graphs in the decomposition or in three graphs as $x_1 + z_1 = 3$. If $x_1 = 2$, then $y_1 + z_1 = 4$ and $2y_1 + 5z_1 = 13$. There is no integral solution for the pair of linear equations, hence x_1 must be 3. This means there is at least one vertex, WOLG say vertex 2, which must come as a degree one vertex in exactly one 2-petal subgraph. Therefore we have $y_2 + z_2 = 5$ and $2y_2 + 5z_2 = 14$. As $x_2 + z_2 = 3$, this system of linear equations does not have an integral solution. ∎

If $\frac{\lambda v(v-1)}{10} \geq \binom{v}{2}$, then $\frac{\lambda}{5} \geq 1$. In other words, if $\lambda \geq 5$, then $5K_v$'s can be decomposed for $v \geq 5$.

Given $\lambda = 5$ and v odd, or $v = 2t + 1$, the number of difference sets modulo $(2t + 1)$ needed is t.

Theorem 2.12 *Given $v = 2t + 1 \geq 5$, the difference family solutions for the decomposition of $5K_v$ into 2-petal subgraphs is:*
$\{< 0, 1, 2, 3 >, < 0, 2, 4, 5 >, < 0, 3, 6, 7 >, \ldots,$
$< 0, t-1, 2t-2, 2t-1 >\} \cup \{< 0, t, 2t, t+1 >\}$.

On the other hand, for $v+1$, the differences $\{1, 2, \ldots, \frac{v-1}{2}\}$ (mod v) can be achieved by taking all difference sets from the solution for v together with a difference set $<0, \infty, 1, 2>$ as noted in the following theorem. Therefore, if a difference family solution for the decomposition of $5K_v$, v odd, is given, then by including the different set, $<0, \infty, 1, 2>$, we get the decomposition of $5K_{v+1}$.

Theorem 2.13 *Suppose we have a difference family solution $\{D_1, D_2, \ldots, D_{\frac{v-1}{2}}\}$ for the decomposition of $5K_v$ for v odd, then the difference family $\{D_1, D_2, \ldots, D_{\frac{v-1}{2}}, D\}$ gives a decomposition of $5K_{v+1}$ where $D = < 0, \infty, 1, 2 >$.*

Proof. Let the point set for $5K_{v+1}$ be $\{\infty, 0, 1, \ldots, v-1\}$. As $\{D_1, \ldots, D_{\frac{v-1}{2}}\}$ is a difference family for $5K_v$, all differences $\{1, \ldots, \frac{v-1}{2}\}$ occur exactly 5 times at the required locations. The "blocks" generated by D account for the pairs with ∞. ∎

Lemma 2.14 *The necessary conditions are sufficient for the existence of 2-petal decomposition of* $5K_v$.

2.4 Any $\lambda \geq 2$

Theorem 2.15 *For any* $\lambda > 1$, *where* $\lambda \neq 3$, λK_v *can be decomposed into 2-petal subgraphs when* $v \equiv 0, 1 \pmod 5$ *and* $v \geq 4$. *Also, when* $\lambda \equiv 0 \pmod 5$, λK_v *can be decomposed for all* $v \geq 5$.

Proof. When $\lambda \equiv 0 \pmod 5$, say $\lambda = 5m$ for any integer $v \geq 5$, λK_v can be decomposed by taking m copies of the decomposition of $5K_v$.
If λ is not $\equiv 0 \pmod 5$ and $\lambda \neq 3$,
If λ is even, say $\lambda = 2m$, we take m copies of the decomposition of $2K_v$.
If λ is odd, λK_v can be decomposed using the decomposition of $5K_v$ and $\frac{\lambda-5}{2}$ copies of $2K_v$. ■

Corollary 2.16 *Necessary conditions are sufficient for the decomposition of* λK_v *into 2-petal graphs.*

3 Decomposition of λK_v into "stem-infinity" graphs

Next, we will decompose λK_v into stem-infinity graphs to obtain a so called stem-infinity decomposition.

A stem-infinity subgraph with vertices a, b, c, and d is given below:

In general, we will denote such a graph by $\subset a, b, c, d\supset$ implying the set of vertices {a, b, c, d} and the edge set {ab, bc, bc, cd, cd}.

3.1 $\lambda = 2$

Recall, for $\lambda = 2$, $v \equiv 0, 1 \pmod 5$ is a necessary condition.

Example 3.1 The decomposition of $2K_5$ into stem-infinity graphs is: $\subset$0, 1, 2, 3$\supset$, $\subset$0, 1, 3, 4$\supset$, $\subset$1, 4, 0, 3$\supset$, $\subset$1, 4, 2, 0$\supset$.

Example 3.2 The decomposition of $2K_6$ into stem-infinity graphs is: $\subset 0, 1, 3, 5 \supset$, $\subset 0, 1, 4, 2 \supset$, $\subset 1, 2, 5, 0 \supset$, $\subset 1, 2, 3, 0 \supset$, $\subset 3, 4, 0, 2 \supset$, $\subset 3, 4, 5, 1 \supset$.

Similarly to the 2-petal decomposition, when $\lambda = 2$ and $v = 10t$, the number of difference sets mod $(10t - 1)$ needed is $2t$.

Theorem 3.3 *The general rule for the decomposition of $2K_v$ where $v = 10t$ into stem-infinity subgraphs is:*
$\{\subset \infty, 0, t, 2t+1 \supset, \subset \infty, 0, t+2, 2t+5 \supset\} \cup$
$\{\subset 1, 0, t+4, 2t+9 \supset, \subset 1, 0, t+6, 2t+13 \supset, \ldots,$
$\subset t-1, 0, 5t-4, 10t-7 \supset, \subset t-1, 0, 5t-2, 10t-3 \supset\}$.

Given $\lambda = 2$ and $v = 10t + 1$, there are 2t difference sets mod $(10t)$ needed.

Theorem 3.4 *The general rule for the decomposition of $2K_v$ where $v = 10t+1$ into stem-infinity subgraphs is:*
$\{\subset 1, 0, t+1, 2t+3 \supset, \subset 1, 0, t+3, 2t+7 \supset,$
$\subset 2, 0, t+5, 2t+11 \supset, \subset 2, 0, t+7, 2t+15 \supset, \ldots,$
$\subset t, 0, 5t-3, 10t-5 \supset, \subset t, 0, 5t-1, 10t-1 \supset\}$.

The number of difference sets mod $(10t + 4)$ needed in a stem-infinity decomposition when $\lambda = 2$ and $v = 10t + 5$ is 2t +1.

Theorem 3.5 *The general rule for the decomposition of $2K_v$ where $v = 10t+5$ and $v \geq 15$ into stem-infinity subgraphs is:*
$\{\subset \infty, 0, t, 2t+1 \supset, \subset \infty, 0, t+2, 2t+5 \supset,$
$\subset 1, 0, t+4, 2t+9 \supset, \subset 1, 0, t+6, 2t+13 \supset, \ldots,$
$\subset t-1, 0, 5t-4, 10t-7 \supset, \subset t-1, 0, 5t-2, 10t-3 \supset, \} \cup$
$\{\subset 5t+2, 0, 5t, 10t+1 \supset\}$.

Proof. The last difference set uses the difference $\frac{v-1}{2} = 5t+2$ only once because after the difference set is generated, the pair 0 and 5t+2 will occur again halfway through the generated set, allowing the difference of 5t + 2 to occur twice, as required. ∎

There are $2t + 1$ difference sets mod $(10t + 6)$ in the stem-infinity decomposition of $2K_v$ where $v = 10t + 6$.

Theorem 3.6 *The general rule for the decomposition of $2K_v$ where $v = 10t+6$ into stem-infinity subgraphs is:*

$\{\subset 1,\ 0,\ t+1,\ 2t+3\supset,\ \subset 1,\ 0,\ t+3,\ 2t+7\supset,$
$\subset 2,\ 0,\ t+5,\ 2t+11\supset,\ \subset 2,\ 0,\ t+7,\ 2t+15\supset,\ \ldots,$
$\subset t,\ 0,\ 5t-3,\ 10t-5\supset,\ \subset t,\ 0,\ 5t-1,\ 10t-1\supset\} \cup$
$\{\subset 5t+3,\ 0,\ 5t+1,\ 10t+3\supset\}.$

Proof. Similarly to the decomposition of $2K_{10t+5}$ into stem-infinity graphs, the decomposition of $2K_{10t+6}$ uses a difference set with the greatest difference occurring singularly. The last difference set uses the difference $\frac{v}{2} = 5t+3$ only once because after the difference set is generated, the pair 0 and 5t+3 will occur again halfway through the generated set, allowing the difference of 5t + 3 to occur twice, as required. ∎

Lemma 3.7 *The necessary conditions are sufficient for the existence of a stem-infinity decomposition of* $2K_v$.

3.2 $\lambda = 3$

Recall the argument as to why $3K_v$ cannot be decomposed into 2-petal graphs. The same argument applies to the stem-infinity decomposition of $3K_v$ because there are 5 edges in both subgraphs.

3.3 $\lambda = 5$

Recall, for $\lambda = 5$, $v \geq 4$.

Example 3.8 $5K_4$ can be decomposed into stem-infinity graphs using the sets:
$\subset 0,\ 1,\ 3,\ 2\supset$, $\subset 0,\ 2,\ 1,\ 3\supset$, $\subset 0,\ 3,\ 2,\ 1\supset$, $\subset 1,\ 2,\ 0,\ 3\supset$, $\subset 2,\ 3,\ 0,\ 1\supset$, and $\subset 3,\ 1,\ 0,\ 2\supset$.

Given $v = 2t+1$, the number of difference sets mod $(2t+1)$ needed to decompose $5K_v$ into stem-infinity subgraphs is t.

Theorem 3.9 *Given v odd, the general rule for the decomposition of* $5K_v$ *where $v = 2t+1$ and is ≥ 5 into stem-infinity subgraphs is:*
$\{\subset 0,\ 1,\ 2,\ 4\supset\} \cup$
$\{\subset 0,\ 2,\ 4,\ 1\supset,\ \subset 0,\ 3,\ 6,\ 2\supset,\ \subset 0,\ 4,\ 8,\ 3\supset,\ \ldots,$
$\subset 0,\ t-2,\ 2t-4,\ t-3\supset,\ \subset 0,\ t-1,\ 2t-2,\ t-2\supset\} \cup$
$\{\subset 0,\ t,\ 2t,\ 2t-1\supset\}.$

If given a difference family solution for the decomposition of $5K_v$ into stem-infinity graphs, where v is odd, then the decomposition of $5K_{v+1}$ also exists. There are $t+1$ difference sets in the decomposition.

Theorem 3.10 *The general rule for the decomposition of* $5K_{v+1}$ *where* $v+1 = 2t+2$ *and is* ≥ 6 *into stem-infinity subgraphs is:*
$\{\subset 0,\ \infty,\ 1,\ 2\supset\} \cup \{\subset 0,\ 1,\ 2,\ 4\supset,\ \subset 0,\ 2,\ 4,\ 7\supset,\ \ldots,$
$\subset 0,\ t-2,\ 2t-4,\ 3t-5\supset,\ \subset 0,\ t-1,\ 2t-2,\ 3t-2\supset\} \cup$
$\{\subset 0,\ t,\ 2t,\ \infty\supset\}.$

Lemma 3.11 *The necessary conditions are sufficient for the existence of a stem-infinity decomposition of* $5K_v$.

3.4 Any $\lambda \geq 2$

Recall the argument as to why any λK_v, where $\lambda \geq 2$ and $\neq 3$, can be decomposed into 2-petal graphs. The same argument applies to the stem-infinity decomposition of λK_v.

Corollary 3.12 *The necessary conditions are sufficient for the decomposition of* λK_v *into stem-infinity graphs.*

4 Decomposition of λK_v into "barbell" graphs

Additionally, we will decompose λK_v into barbell graphs, which we also call a barbell decomposition.

A barbell subgraph with vertices a, b, c, and d:

In general, we will denote such a graph by $[a,\ b,\ c,\ d]$. The vertices are {a, b, c, d} and the edge set is {ab, ab, bc, cd, cd}.

4.1 $\lambda = 2$

Recall, for $\lambda = 2$, $v \equiv 0, 1 (\text{mod } 5)$ is a necessary condition.

Given $\lambda = 2$ and $v = 10t$, the number of difference sets mod $(10t - 1)$ needed for a barbell decomposition is $2t$.

Theorem 4.1 *The general difference set solutions to decompose $2K_v$ into barbell subgraphs, where $v = 10t$, are*
$\{[\infty,\ 0,\ 1,\ t+2]\} \cup \{[t+2,\ 0,\ 1,\ t+4]$,
$[t+4,\ 0,\ 2,\ t+7]$, $[t+6,\ 0,\ 2,\ t+9]$, $\ldots$,
$[5t-8,\ 0,\ t-1,\ 6t-8]$, $[5t-6,\ 0,\ t-1,\ 6t-6]$,
$[5t-4,\ 0,\ t,\ 6t-3]$, $[5t-2,\ 0,\ t,\ 6t-1]\}$.

When $\lambda = 2$ and $v = 10t + 1$, there are $2t$ difference sets mod $(10t + 1)$ needed in a barbell decomposition.

Theorem 4.2 *The difference family solutions needed to decompose $2K_v$ into barbell subgraphs, where $v = 10t + 1$, are*
$\{[t+1,\ 0,\ 1,\ t+3]$, $[t+3,\ 0,\ 1,\ t+5]$,
$[t+5,\ 0,\ 2,\ t+8]$, $[t+7,\ 0,\ 2,\ t+10]$, $\ldots$,
$[5t-7,\ 0,\ t-1,\ 6t-7]$, $[5t-5,\ 0,\ t-1,\ 6t-5]$,
$[5t-3,\ 0,\ t,\ 6t-2]$, $[5t-1,\ 0,\ t,\ 6t]\}$.

Given $\lambda = 2$ and $v = 10t+5$, the number of difference sets mod $(10t+4)$ needed for a barbell decomposition is $2t + 1$.

Theorem 4.3 *The difference family solutions needed to decompose $2K_v$ into barbell subgraphs, where $v = 10t + 5$, are*
$\{[\infty,\ 0,\ 1,\ t+2]\} \cup \{[t+2,\ 0,\ 1,\ t+4]$,
$[t+4,\ 0,\ 2,\ t+7]$, $[t+6,\ 0,\ 2,\ t+9]$, $\ldots$,
$[5t-4,\ 0,\ t,\ 6t-3]$, $[5t-2,\ 0,\ t,\ 6t-1]\} \cup$
$\{[5t,\ 0,\ 5t+2,\ 10t+3]\}$.

Proof. The last set has the pair 0 and $5t+2$ occur once because, as the set is generated, the pair will come a second time. ■

There are $2t + 1$ difference sets mod $(10t + 6)$ needed in the barbell decomposition of $2K_v$ where $v = 10t + 6$.

Theorem 4.4 *The difference family solutions needed to decompose $2K_v$ into barbell subgraphs, where $v = 10t + 6$, are*

$[t+1,\ 0,\ 1,\ t+3]$, $[t+3,\ 0,\ 1,\ t+5]$,
$[t+5,\ 0,\ 2,\ t+8]$, $[t+7,\ 0,\ 2,\ t+10]$, $\ldots$,
$[5t-3,\ 0,\ t,\ 6t-2]$, $[5t-1,\ 0,\ t,\ 6t]$,
$[5t+1,\ 0,\ 5t+3,\ 10t+5]\}$.

Proof. The last set has the pair 0 and $5t+3$ occur once because, as the set is generated, the pair will come a second time. ∎

Lemma 4.5 *The necessary conditions are sufficient for the existence of a barbell decomposition of* $2K_v$.

4.2 $\lambda = 3$

Recall the argument as to why $3K_v$ cannot be decomposed into 2-petal graphs. The same argument applies to the barbell decomposition of $3K_v$ because there are 5 edges in both subgraphs.

4.3 $\lambda = 5$

Recall, for $\lambda = 5$, v can be any integer.

Example 4.6 $5K_4$ can be decomposed into barbell graphs using the sets: [0, 1, 3, 2], [0, 2, 1, 3], [0, 3, 2, 1], [1, 2, 0, 3], [2, 3, 0, 1], and [3, 1, 0, 2].

Given $\lambda = 5$ and $v = 2t+1$, there are t difference sets mod $(2t+1)$ needed.

Theorem 4.7 *Given v odd, the general rule for the decomposition of* $5K_v$ *where* $v = 2t+1$ *and is* ≥ 5 *into barbell subgraphs is:*
$\{[0, 1, 2, 4]\} \cup$
$\{[0, 2, 4, 1], [0, 3, 6, 2], [0, 4, 8, 3], \ldots,$
$[0, t-2, 2t-4, t-3], [0, t-1, 2t-2, t-2]\} \cup$
$\{[0, t, 2t, 2t-1]\}$.

If given a difference family solution for the barbell decomposition of $5K_v$, where v is odd, then the barbell decomposition of $5K_{v+1}$ also exists. There are $t+1$ difference sets in the decomposition of $5K_{v+1}$.

Theorem 4.8 *The general rule for the decomposition of* $5K_{v+1}$ *where* $v+1 = 2t+2$ *and is* ≥ 6 *into barbell subgraphs is:*
$\{[0, \infty, 1, 2]\} \cup$
$\{[0, 1, 2, 4], [0, 2, 4, 7], \ldots,$
$[0, t-2, 2t-4, 3t-5], [0, t-1, 2t-2, 3t-2]\} \cup$
$\{[0, t, 2t, \infty]\}$.

Lemma 4.9 *The necessary conditions are sufficient for the existence of a barbell decomposition of* $5K_v$.

4.4 Any $\lambda \geq 2$

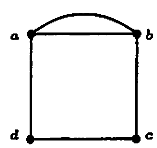
Recall the argument as to why any λK_v, where $\lambda \geq 2$ and $\neq 3$, can be decomposed into 2-petal graphs. The same argument applies to the barbell decomposition of λK_v.

Corollary 4.10 *The necessary conditions are sufficient for the decomposition of λK_v into barbell graphs.*

5 Decomposition into "box-edge" graphs

Finally, we will decompose λK_v into so called "box-edge" graphs, which we also call a box-edge decomposition.

A box-edge subgraph with vertices a, b, c, and d:



In general, we will denote such a graph by $|a,\ b,\ c,\ d|$. The vertices are {a, b, c, d} and the edge set is {ab, ab, bc, cd, ad}.

5.1 $\lambda = 2$

Recall, for $\lambda = 2$, $v \equiv 0, 1 (\text{mod } 5)$ is a necessary condition.

Example 5.1 A decomposition of $2K_5$ into box-edge graphs uses the subgraphs: |0, 3, 1, ∞|, |1, 0, 2, ∞|, |2, 1, 3, ∞|, and |3, 2, 0, ∞|.

Example 5.2 A decomposition of $2K_6$ into box-edge graphs uses the subgraphs: |0, 3, 2, 1|, |1, 3, 4, 0|, |0, 5, 3, 2|, |2, 5, 4, 0|, |4, 2, 1, 5|, and |1, 4, 3, 5|.

There are 2t difference sets mod $(10t-1)$ in a box-edge decomposition of $2K_v$ where $v = 10t$.

Theorem 5.3 *The difference family solutions needed to decompose $2K_v$, where $v = 10t$, into box-edge subgraphs are:*

$\{|0,\ 1,\ 6,\ 2|,\ |0,\ 3,\ 7,\ 2|,$
$|0,\ 6,\ 16,\ 7|,\ |0,\ 8,\ 17,\ 7|,\ \ldots,$
$|0,\ 5t-9,\ 10t-14,\ 5t-8|,\ |0,\ 5t-7,\ 10t-13,\ 5t-8|\ \} \cup$
$\{|0,\ 5t-4,\ 10t-6,\ 5t-3|,\ |0,\ 5t-1,\ \infty,\ 5t-2|\}.$

There are 2t difference sets mod $(10t+1)$ in the box-edge decomposition of $2K_v$ where $v = 10t + 1$.

Theorem 5.4 *The difference family solutions needed to decompose $2K_v$, where $v = 10t + 1$, into box-edge subgraphs are*
$\{|0,\ 1,\ 6,\ 2|,\ |0,\ 3,\ 7,\ 2|,$
$|0,\ 6,\ 16,\ 7|,\ |0,\ 8,\ 17,\ 7|,\ \ldots,$
$|0,\ 5t-9,\ 10t-14,\ 5t-8|,\ |0,\ 5t-7,\ 10t-13,\ 5t-8|,$
$|0,\ 5t-4,\ 10t-4,\ 5t-3|,\ |0,\ 5t-2,\ 10t-3,\ 5t-3|\}.$

There are 2t+1 difference sets mod $(10t+4)$ in a box-edge decomposition of $2K_v$ where $v = 10t + 5$.

Theorem 5.5 *The difference family solutions needed to decompose $2K_v$, where $v = 10t + 5$, into box-edge subgraphs are:*
$\{|0,\ 1,\ 6,\ 2|,\ |0,\ 3,\ 7,\ 2|,$
$|0,\ 6,\ 16,\ 7|,\ |0,\ 8,\ 17,\ 7|,\ \ldots,$
$|0,\ 5t-4,\ 10t-4,\ 5t-3|\ |0,\ 5t-2,\ 10t-3,\ 5t-3|,\} \cup$
$\{|0,\ 5t+1,\ \infty,\ 5t+2|\}.$

There are 2t+1 difference sets mod $(10t + 6)$ for the decomposition of $2K_v$ where $v = 10t + 6$ into box-edge subgraphs.

Theorem 5.6 *The difference family solutions needed to decompose $2K_v$, where $v = 10t + 6$, into box-edge subgraphs are:*
$\{|0,\ 1,\ 6,\ 2|,\ |0,\ 3,\ 7,\ 2|,$
$|0,\ 6,\ 16,\ 7|,\ |0,\ 8,\ 17,\ 7|,\ \ldots,$
$|0,\ 5t-12,\ 10t-20,\ 5t-11|,\ |0,\ 5t-10,\ 10t-19,\ 5t-11|$
$|0,\ 5t-7,\ 10t-10,\ 5t-6|,\ |0,\ 5t-5,\ 10t-9,\ 5t-6|\} \cup$
$\{|0,\ 5t+1,\ 10t+4,\ 5t+2|\}.$

Lemma 5.7 *The necessary conditions are sufficient for the existence of a box-edge decomposition of $2K_v$.*

5.2 $\lambda = 3$

Recall the argument as to why $3K_v$ cannot be decomposed into 2-petal graphs. This argument does not apply to the box-edge decomposition.

This is because the box-edge subgraph has three single edges whereas the 2-petal subgraphs had only one.

$\frac{\binom{v}{2}}{3} \leq \frac{3v(v-1)}{2\cdot 5}$. Simplified, this says that $\binom{v}{2} \leq \frac{9}{5} \cdot \binom{v}{2}$. Hence, it is possible to decompose $3K_v$ into box-edge subgraphs.

Recall, for $\lambda = 3$, $v \equiv 0, 1 (\text{mod } 5)$ is the necessary condition, as the number of box-edge graphs in a possible decomposition must be $\frac{2\cdot v\cdot(v-1)}{2\cdot 5}$.

Example 5.8 $3K_5$ can be decomposed into subgraphs: |0, 1, 3, 2|, |0, 2, 4, 1|, |0, 3, 1, 4|, |0, 4, 2, 3|, |1, 2, 4, 3|, and |3, 4, 1, 2|.

Example 5.9 $3K_6$ can be decomposed into subgraphs: |0, 1, 3, 2|, |1, 2, 4, 3|, |2, 3, 5, 4|, |3, 4, 0, 5|, |4, 5, 1, 0|, |5, 0, 2, 1|, |0, 3, 5, 2|, |1, 4, 0, 3|, and |2, 5, 1, 4|.

To decompose $3K_v$, where $v = 10t$, there are 3t difference sets mod $(10t-1)$.

Theorem 5.10 *The difference family solutions needed to decompose* $3K_v$, *where* $v = 10t$ *into box-edge subgraphs are:*
$\{|0, 1, 4, 3|, |0, 2, 7, 5|, |0, 4, 8, 3|, \ldots,$
$|0, 5t-9, 10t-16, 5t-7|, |0, 5t-8, 10t-13, 5t-5|, |0, 5t-6, 10t-12, 5t-7|\} \cup$
$\{|0, 5t-4, 10t-6, 5t-2|, |0, 5t-3, 10t-4, 5t-1|, |0, \infty, 10t-1, 5t-2|\}$.

To decompose $3K_v$ into box-edge subgraphs, where $v = 10t + 1$, there are 3t difference sets mod $(10t)$.

Theorem 5.11 *The difference family solutions needed to decompose* $3K_v$, *where* $v = 10t + 1$, *into box-edge subgraphs are:*
$\{|0, 1, 4, 3|, |0, 2, 7, 5|, |0, 4, 8, 3|, \ldots,$
$|0, 5t-4, 10t-6, 5t-2|, |0, 5t-3, 10t-3, 5t|, |0, 5t-1, 10t-2, 5t-2|\}$.

To decompose $3K_v$, where $v = 10t + 5$, there are $3t + 1$ difference sets mod $(10t + 4)$, as well as one short set.

Theorem 5.12 *The difference family solutions needed to decompose* $3K_v$, *where* $v = 10t + 5$, *into box-edge subgraphs are:*
$\{|0, 1, 4, 3|, |0, 2, 7, 5|, |0, 4, 8, 3|, \ldots,$
$|0, 5t-4, 10t-6, 5t-2|, |0, 5t-3, 10t-3, 5t|, |0, 5t-1, 10t-2, 5t-2|\}$
$\cup \{|0, \infty, 10t+2, 5t+1|\} \cup$
a short set of $\{|0, 5t+2, 10t+3, 5t+1|\}$.

To decompose $3K_v$, where $v = 10t + 6$, there are $3t + 1$ difference sets mod $(10t + 6)$, as well as one short set.

Theorem 5.13 *The difference family solutions needed to decompose $3K_v$, where $v = 10t + 6$, into box-edge subgraphs are:*
$\{|0, 1, 4, 3|, |0, 2, 7, 5|, |0, 4, 8, 3|, \ldots,$
$|0, 5t-4, 10t-6, 5t-2|, |0, 5t-3, 10t-3, 5t|, |0, 5t-1, 10t-2, 5t-2|\}$
$\cup \{|0, 5t+1, 10t+3, 5t+2|\} \cup$
a short set of $\{|0, 5t+3, 10t+5, 5t+2|\}$.

Lemma 5.14 *The necessary conditions are sufficient for the existence of a box-edge decomposition of $3K_v$.*

5.3 $\lambda = 5$

Recall, for $\lambda = 5$, $v \geq 4$.

Example 5.15 $5K_4$ can be decomposed into box-edge graphs using the sets: |0, 1, 3, 2|, |0, 2, 1, 3|, |0, 3, 2, 1|, |1, 2, 0, 3|, |2, 3, 0, 1|, and |3, 1, 0, 2|.

For the box-edge decomposition of $5K_v$, where $v = 2t + 1$, there are t difference sets mod $(2t + 1)$ needed.

Theorem 5.16 *The difference family solutions needed to decompose $5K_v$, where $v = 2t + 1$, into box-edge subgraphs are*
$\{|0, 1, 3, 2|, |0, 2, 5, 3|, |0, 3, 7, 4|, \ldots,$
$|0, t-1, 2t-1, t|\} \cup \{|0, t, t+1, 1|\}$.

For the box-edge decomposition of $5K_v$, where $v = 2t + 2$, there are $t + 1$ difference sets mod $(2t + 1)$.

Theorem 5.17 *The difference family solutions needed to decompose $5K_v$, where $v = 2t + 2$, into box-edge subgraphs are*
$\{|0, 1, 3, 2|, |0, 2, 5, 3|, |0, 3, 7, 4|, \ldots,$
$|0, t-1, 2t-1, t|\} \cup \{|0, t, t+1, \infty|, |0, \infty, t+1, 1|\}$.

Lemma 5.18 *The necessary conditions are sufficient for the existence of a box-edge decomposition of $5K_v$.*

5.4 Any $\lambda \geq 2$

Recall the argument as to why any λK_v, where $\lambda \geq 2$ and $\neq 3$, can be decomposed into 2-petal graphs. The same argument applies to the box-edge decomposition of λK_v.

Corollary 5.19 *The necessary conditions are sufficient for the decomposition of λK_v into box-edge graphs.*

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