Minimal Pancyclic Graphs

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Abstract

A *pancyclic* graph on v vertices is called *pancyclic* if it contains cycles of every length from 3 to v. In this paper we address the question: what is the minimum number of edges in a pancyclic graph? We present a simple analysis using chord patterns.

1 Introduction

For definitions and theorems involving graph theory, the reader is referred to standard texts on the subject, such as [10]. Graphs are finite, simple and undirected.

A graph with v vertices is called *pancyclic* if it contains cycles of every length from 3 to v. Obviously such graphs exist — the complete graph on v vertices is an example — so it is of interest to know $m(v)$, the smallest possible number of edges in such a graph. A pancyclic graph with this number of edges is called *minimal*. Two related concepts that have been discussed are *uniquely pancyclic* graphs, with exactly one cycle of each order

JCMCC **86** (2013), pp. 125-133

([8, 6]) and *vertex pancyclic* graphs, where every vertex lies in a cycle of each order ([3]).

2 History

Pancyclic graphs were introduced by Bondy [2], although the directed equivalent had been discussed earlier (see [1, 4, 7]). In 1978, Sridharan [9] gave constructions that show:

when:	$v = 3$	$m(v) \leq 3$
	$4 \leq v \leq 5$	$m(v) \leq v + 1$
	$6 \leq v \leq 8$	$m(v) \leq v + 2$
	$9 \leq v \leq 12$	$m(v) \leq v + 3$
	$13 \leq v \leq 20$	$m(v) \leq v + 4$
	$21 \leq v \leq 36$	$m(v) \leq v + 5$.

For larger v, define $\alpha(n) = 2^n + n$. Suppose $n \geq 3$. Then, for each t satisfying $\alpha(n) \leq t \leq \alpha(n+1)$, and for each v such that

$$2^t + \alpha(n+1) \leq v \leq 2^{t+1} + \alpha(n+1),$$

it is shown in [9] that $m(v) \leq v + t + n - 1$. For example,

when:	$21 \leq v \leq 36$	$m(v) \leq v + 5$
	$37 \leq v \leq 52$	$m(v) \leq v + 6$
	$53 \leq v \leq 84$	$m(v) \leq v + 7$
	...	....

In [9], all these inequalities are stated as equalities. However, this is not proven. Moreover, Shi [8] found uniquely pancyclic graphs with 14 vertices and 17 edges, so $m(14) \leq 17$ and it is easy to see that equality holds. We shall also show below that $m(13) = 16$, $m(21) = 25$, and $m(22) = 26$. All of these results contradict [9]. The problem of determining $m(v)$ in general remains open.

3 Establishing lower bounds

Suppose we have a pancyclic graph on v vertices. It must contain at least $v - 2$ cycles — more, if some cycles are of equal length. In particular, a pancyclic graph must contain a Hamilton cycle, and the other cycles will utilize chords in that cycle. So we shall examine the possible patterns of cycles. The basic model is a cycle of length v with k chords, yielding $v + k$ edges in all. We begin with the case $k = 3$. The analysis is based on that in

[6], with some modifications (the distinction we shall make below between types A and B is not drawn there).

In every case, we represent our graph as a circle with the chords as straight lines. The segments of the outer circle may contain a number of vertices, but the chords only have vertices at their ends.

Fewer than two chords

If there are no chords, the graph contains only one (Hamilton) cycle, of length v, and the only pancyclic case is $v = 3$. If there is one chord, there are two further (one-chord) cycles. So, if the graph has only one chord, you obtain exactly three cycles, and if the graph is pancyclic it has $v \leq 5$. These cycles are illustrated in Figure 1 (there are three drawings of the same graph, with the cycles shown in bold).

So $m(v) = v$ if and only if $v = 3$, and $m(v) = v + 1$ if and only if $v = 4$ or 5.

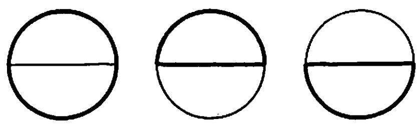


Figure 1: Cycles in the case of one chord

Two chords

If there are two chords, we distinguish three cases:
A. They do not cross, and have no common endpoint;
B. They have one common endpoint;
C. They cross.

In addition to the cycles that contain no chord or one chord, there may be new cycles that contain two chords. Let us count cycles in the three cases:
A. There is one new cycle, so together with the Hamilton cycle and the four one-chord cycles (two per chord) you have six cycles in total;
B. You get one new cycle, for six in total;
C. You get two new cycles, for seven in total.

All the new cycles are shown in Figure 2.

This may allow us to find pancyclic graphs for up to $v = 9$ vertices, with only $v + 2$ edges, and in fact examples exist for $v \leq 8$ (see Figure

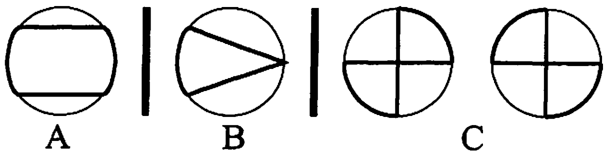
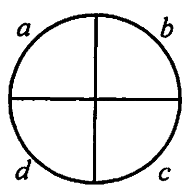


Figure 2: Cases of two chords

5). However, there is no example for $v = 9$, as was shown by Shi [8]. So $m(v) = v + 2$ if and only if $6 \leq v \leq 8$.

A pancyclic graph on 9 vertices needs at least 7 cycles, so such a graph with two chords must be of type C. To see that no such graph is pancyclic, suppose the numbers of edges in the segments of the Hamilton cycle (*lengths* of the segments) are as shown. Then $a + b + c + d = 9$. The lengths of the seven cycles are 9 (containing neither chord), $a + b + 1, c + d + 1$ (horizontal chord), $a + d + 1, b + c + 1$ (vertical chord), $a + c + 2, b + d + 2$ (both chords).



The only possible way to get a cycle of length 3 is if two adjacent segments have length 1, so we assume $a = b = 1$. Then $d = 9 - a - b - c = 7 - c$. The cycles (in the order listed above) have lengths $9, 3, 8, 9 - c$, $c + 2, c + 3, 10 - c$.

In order for there to be cycles of length 4, 5, 6 and 7, it is necessary that

$$\{9 - c, c + 2, c + 3, 10 - c\} = \{4, 5, 6, 7\}.$$

Without loss of generality, $c < d$. No case works (if $c = 1$ then $c + 2 = 3$; if $c = 2$ then $10 - c = 8$; if $c = 3$ then there is no 3–cycle). However, observe that the case $c = 2$ yields cycles of all length except 6, so it is easy to add one more chord and construct a 12-edge pancyclic graph on 9 vertices. Therefore $m(9) = 12$. It is also interesting to note that if we put $a = b = 1, c = 2$ and $d = v - 2 - c$ we obtain a minimal pancyclic graph for $v = 6, 7, 8$.

Three chords

If a graph has three chords, we classify by looking at the three pairs of chords. We refer to the configuration by the string of three letters corresponding to the three types of chord interaction. For example, type AAB is a graph in which two of the pairs of chords are type A (they do not cross, and have no common endpoint) and one pair is type B (they have a common endpoint). There are 14 types of graph: AAAi, AAAii, AABi, AABii, AAC, ABBi, ABBii, ABC, ACC, BBBi, BBBii, BBC, BCC, CCC. (There are two types AAA, two types AAB, two types ABB (one where the three chords form a "C" pattern and one where they form a "Z"), and two types BBB (one where all three chords have a common endpoint and one where they form a triangle).) They are illustrated in Figure 3.

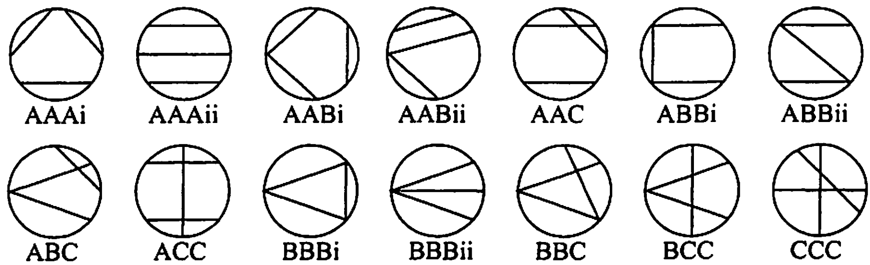


Figure 3: Cases of three chords

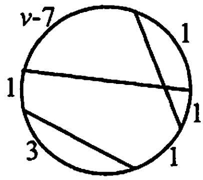
The following table counts the number of cycles in a graph, in each of the 14 cases. C(n) means the number of cycles involving exactly n chords.

	AAAi	AAAii	AABi	AABii	AAC	ABBi	ABBii
C(0)	1	1	1	1	1	1	1
C(1)	6	6	6	6	6	6	6
C(2)	3	3	3	3	4	3	3
C(3)	0	1	1	0	1	1	0
	10	11	11	10	12	11	10

	ABC	ACC	BBBi	BBBii	BBC	BCC	CCC
C(0)	1	1	1	1	1	1	1
C(1)	6	6	6	6	6	6	6
C(2)	4	5	3	3	4	5	6
C(3)	1	2	1	0	1	1	2
	12	14	11	10	12	13	15

From the table, it is clear that one can try to construct pancyclic graphs on v vertices and $v+3$ edges for $10 \leq v \leq 17$. The cases of $v = 10, 11, 12$ were constructed in [9], and 14 was given in [6].

Using the same technique as we did for nine vertices, we can find an example for $v = 13$, and essentially the same method works for all cases with $10 \leq v \leq 14$. The diagram is

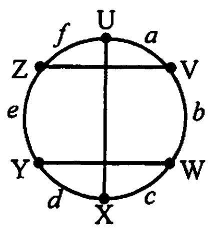


and the graph has cycles of lengths 3, 4, 5, 6, 7, 8, $v-5$, $v-4$, $v-3$, $v-2$, $v-1$ and v.

For $v > 14$, only types BCC, ACC and CCC need be discussed; in fact, type CCC can be eliminated immediately, as no cycle involves fewer than either two chords and two segments or one chord and three segments, so a cycle of length 3 is impossible.

BCC has 13 cycles, so $v = 14$ or 15 could be possible, but the interesting case ($v = 15$) would be uniquely pancyclic, and that is ruled out in [6].

ACC has 14 cycles. We would need 13 different lengths for case $v = 15$, and all 14 different for $v = 16$. In the following diagram, lower-case letters represent the number of edges in a segment, while upper-case letters represent endpoints.



Without loss of generality we can assume $a = f = 1$, in order to get a cycle of length 3. The only possible cycles of length 4 are WXYW (which would imply $c + d = 3$, without loss of generality $c = 2, d = 1$), UVWXU ($b = c = 1$), VWYZV ($b = e = 1$), or UXYZU (mirror image of UVWXU). So we can assume either $b = 1$, and one of $c = 1$ or $e = 1$, or $c = 2$ and $d = 1$.

Assume $b = 1$. If $c = 1$ then VWXYZV and UXYZVU are both length $3 + d + e$, while UVWYZU and UXWYZU are both length $4 + e$; in both cases we have at most 12 different lengths. If $e = 1$ then UVWYXU and UXYZVU are both length $4 + d$, and UXWVZU and UXWYZU are both length $4 + c$; again we have at most 12 different lengths.

Finally, set $c = 2, d = 1$. Then UVWYZU and UXYZVU are both length $4 + e$, and UXYZU and VWYZV are both length $4 + e$; there are at most 12 different lengths.

Therefore cases $v = 15$ and $v = 16$ both require at least four chords. Suitable constructions are found in [9], but see also the following Section.

4 Four chords

There are a large number of possible configurations for four cycles. We have examined several of them.

Figure 4 shows one of the configurations. It represents a graph with $x + 12$ vertices, all on the outer circle. Some vertices are named, while the numbers show the number of edges on the segment between two labeled vertices: for example there are five edges, and consequently four unlabeled vertices, between A and B.

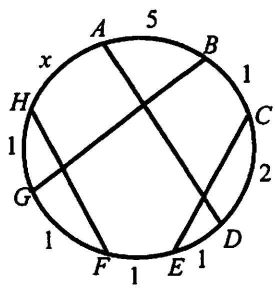


Figure 4: Graph used to construct examples of orders 15 to 22

The graph contains cycles of the following lengths:

length	cycle		length	cycle
3	$FGHF$		v	$ABCDEFGHA$
4	$CDEC$		$v-1$	$ABCDEFHA$
5	$BCEFGB$		$v-2$	$ABCEFGHA$
6	$BCEFHGB$		$v-3$	$ABCEFHA$
7	$BCDEFGB$		$v-4$	$ABGFHA$
8	$BCDEFHGB$		$v-5$	$ABGHA$
9	$ABCDA$		$v-6$	$ADCBGHA$
10	$ADEFGBA$		$v-7$	$ADEFGHA$
11	$ABGHFEDA$		$v-8$	$ADEFHA$
12	$ABGFECDA$		13	$ABGHFECDA$

It will be observed that all lengths from 3 to v inclusive are represented at least once, provided $1 \le x \le 10$. So a minimal pancyclic graph has $v+4$ edges (that is, $m(v) = v + 4$) when $15 \le v \le 22$.

There are cycles of at most 20 different lengths (there are further cycles, but they duplicate lengths already listed), so this construction does not generalize beyond $v = 22$. Therefore the cases $v \ge 23$ remain open.

5 Conclusion

Figure 5 shows examples of minimal pancyclic graphs with v vertices for $3 \le v \le 14$, while examples for $15 \le v \le 22$, may be constructed from Figure 4.

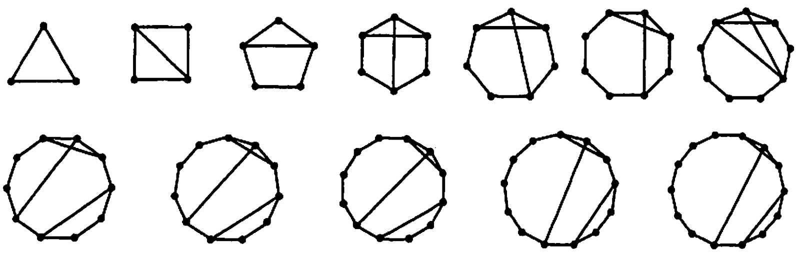


Figure 5: Examples of minimal pancyclic graphs

The sequence $(m(v))$ starts

$$0, 0, 3, 5, 6, 8, 9, 10, 12, 13, 14, 15, 16, 17, 19, 20, 21, 22, 23, 24, 25, 26 \ldots$$

The obvious question is whether $m(23) = 27$ or 28.

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