An Edge Bicoloring View of Edge Independence and Edge Domination

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ABSTRACT

A red-blue coloring of a graph G is an edge coloring of G in which every edge is colored red or blue. For a connected graph H of size at least 2, a color frame F of H is obtained from a red-blue coloring of H having at least one edge of each color and in which a blue edge is designated as the root edge. An F-coloring of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G and the F-chromatic index of G is the minimum number of red edges in an F-coloring of G. An F-coloring of G is minimal if whenever any red edge of G is changed to blue, then the resulting red-blue coloring of G is not an F-coloring of G. The maximum number of red edges in a minimal F-coloring of G is the upper F-chromatic index of G. In this paper, we investigate F-colorings and F-chromatic indexes of graphs for all color frames F of paths of orders 3 and 4.

1 Introduction

The subject of edge colorings is one of the major areas of graph theory. While there are many concepts and problems in graph theory dealing with edge colorings, the best known and most studied is that of *proper edge colorings* of a graph G where each edge of G is assigned one color from a given set of colors and adjacent edges are colored differently. That is, an edge coloring of G is proper if the two edges in every copy of P_3 in G are colored differently. The fundamental problem here is determining the minimum number of colors needed in a proper edge coloring of G. This number is called the *chromatic index* of G and is denoted by $\chi'(G)$. The classic theorem in this connection is due to Vadim Vizing [16] who proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$ for every nonempty graph G. A graph G is said to be of *Class* 1 if $\chi'(G) = \Delta(G)$ and of *Class* 2 if $\chi'(G) = \Delta(G) + 1$. In particular, a regular graph G is of Class 1 if and only if G is 1-factorable.

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Determining which graphs belong to which class is a major problem of study in this area.

There are edge colorings of a graph G where adjacent edges may be colored the same but containing a certain subgraph where no two edges are colored the same. A subgraph of G all of whose edges are colored differently is called a *rainbow subgraph* of G. For a graph F of order p without isolated vertices and a given integer $n \geq p$, the *rainbow number* $\mathrm{rb}_n(F)$ of F is the smallest positive integer k such that every edge coloring of K_n with k colors in which each color is assigned to at least one edge results in a rainbow F. One result in this connection involves the Turán graph $T_{n,k-1}$, which is the $(k-1)$-partite graph of order n, the cardinalities of whose partite sets differ by at most 1. The size of $T_{n,k-1}$ is denoted by $t_{n,k-1}$. This Turán number $t_{n,k-1}$ is the maximum size of a graph of order n containing no complete subgraph of order k. Montellano-Ballesteros and Neumann-Lara [15] proved for $3 \leq k < n$ that $rb_n(K_{k+1}) = t_{n,k-1} + 2$.

A *rainbow coloring* of a connected graph G is an edge coloring of G such that every two vertices of G are connected by a rainbow path. The minimum number of colors used in a rainbow coloring of a connected graph G is the *rainbow connection number* of G and is denoted by $\mathrm{rc}(G)$. It was shown in [3] for integers s and t with $2 \leq s \leq t$ that $\mathrm{rc}(K_{s,t}) = \min\left\{\lceil \sqrt[s]{t}\, \rceil,\ 4\right\}$.

Among the most famous problems in graph theory are those concerning edge colorings of complete graphs with two colors. By a *red-blue coloring* of a graph G is meant an edge coloring of G in which every edge is colored red or blue. For given graphs F and H, the *Ramsey number* $R(F, H)$ of two graphs F and H is the minimum positive integer n for which every red-blue coloring of K_n results in either a red F (a copy of F where each edge is colored red) or a blue H. It is consequence of a theorem of Ramsey that the Ramsey number $R(F, H)$ exists for every pair F, H of graphs.

A related Ramsey number is the *rainbow Ramsey number* $RR(F, H)$ of two graphs F and H, defined as the minimum positive integer n for which every edge coloring of K_n using any number of colors results in either a monochromatic copy of F (where all edges are colored the same) or a rainbow copy of H. As a consequence of a result of Erdös and Rado [7], the rainbow Ramsey number $RR(F, H)$ exists if and only if F is a star or H is a forest. On the other hand, if H has size m and k is an integer with $k \geq m$, then for every pair F, H of graphs, there is always a smallest positive integer n such that any edge coloring of K_n using no more than k colors always results in a monochromatic F or rainbow H (see [5, p. 319]). This number is denoted by $RR_k(F, H)$. In particular, $RR_3(K_3, K_3) = 11$.

While proper edge colorings, monochromatic subgraphs, rainbow subgraphs and rainbow colorings have been the subject of many studies, there are also numerous other red-blue colorings of graphs whose definitions depend on a fixed graph H, certain red-blue colorings of H and a specified blue edge of the resulting edge-colored graph F of H. This gives rise to the concepts of color frames F of a given graph H and red-blue colorings of graphs called F-colorings. We study such F-colorings for all color frames F of the paths of orders 3 and 4 and show that these F-colorings provide a new framework for edge independence and various types of edge domination in graphs. We refer to the book [4, 6] for graph theory notation and terminology not described in this paper.

2 F-Colorings of Graphs

As noted earlier, in a red-blue coloring of a graph G, every edge of G is colored red or blue (where adjacent edges may be colored the same). Also, all edges of G may be colored red or all edges may be colored blue. Let F be a connected graph of size 2 or more with a red-blue coloring in which at least one edge is colored red and at least one edge of F is colored blue. One of the blue edges of F is designated as the *root edge* of F. The *underlying graph* of F is the graph H obtained by removing the colors assigned to the edges of F. In this case, F is called a *color frame* of H. The simplest example of this is the unique color frame F_0 of the path P_3 of order 3 (shown in Figure 1). The five (distinct) color frames $F_1, F_2, \ldots, F_5$ of the path P_4 of order 4 are also shown in Figure 1, where each root edge is indicated by a bold line.

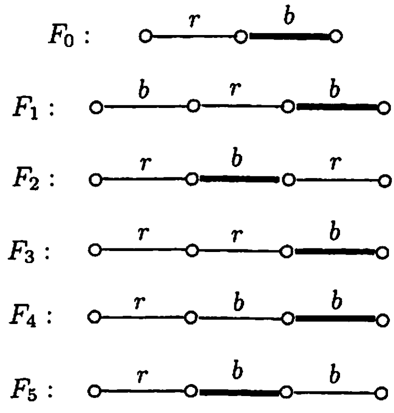

Figure 1: Color frames of P_3 and P_4

For a color frame F, an *F-coloring* of a graph G is a red-blue coloring of G in which every blue edge of G is the root edge of a copy of F in G. If G contains no subgraph isomorphic to F, then the only F-coloring of G is that in which every edge of G is red. The *F-chromatic index* $\chi'_F(G)$

of G is the minimum number of red edges in an F-coloring of G. Since the edge coloring of G that assigns red to every edge is an F-coloring of G, the number $\chi'_F(G)$ exists for every color frame F and every graph G. An F-coloring of G having exactly $\chi'_F(G)$ red edges is called a *minimum F-coloring* of G. These concepts were introduced by Chartrand and Zhang and first studied in [12] by Johnston, Kratky and Mashni. A vertex version of this concept was introduced in [2], which provided a generalization of the area of domination, and studied further by many (see [1, 8, 9] for example). Although these concepts are related through the line graph of a graph, this fact, as with proper colorings, has shown no benefit.

As an illustration, consider the five red-blue colorings c_i $(1 \leq i \leq 5)$ of Q_3 shown in Figure 2, where the solid lines are red edges and the dashed lines are blue edges. The coloring c_1 is an F_i-coloring for each color frame F_i $(1 \leq i \leq 5)$ shown in Figure 1. For each j with $2 \leq j \leq 5$, the coloring c_j is a minimum F_j-coloring of Q_3. Thus $\chi'_{F_2}(Q_3) = \chi'_{F_3}(Q_3) = 4$, $\chi'_{F_4}(Q_3) = 2$ and $\chi'_{F_5}(Q_3) = 3$. The coloring c_5 is a minimum F_j-coloring $(j = 0, 1)$ and so $\chi'_{F_0}(Q_3) = \chi'_{F_1}(Q_3) = 3$.

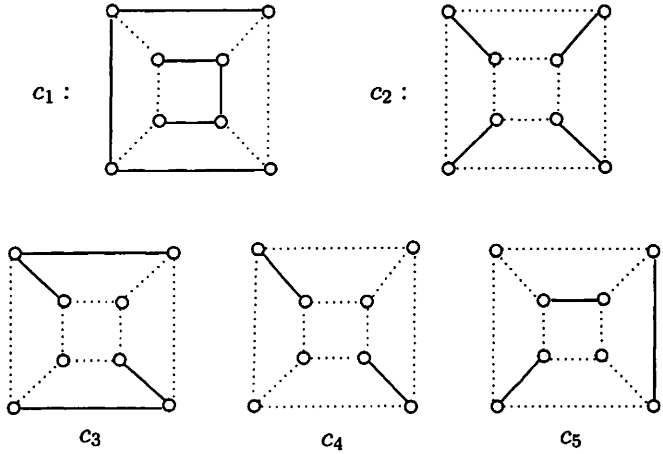

Figure 2: F_i-colorings of Q_3 for $0 \leq i \leq 5$

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$ where $k \geq 2$, then $\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_k)$. Thus, it suffices to consider only connected graphs. For an F-coloring c of a graph G, let $E_{c,r}$ denote the set of red edges of G and $E_{c,b}$ the set of blue edges of G. (We also use E_r and E_b for $E_{c,r}$ and $E_{c,b}$, respectively, when the coloring c under consideration is clear.) Thus $\{E_r, E_b\}$ is a partition of the edge set $E(G)$ of G. Furthermore, let $G_r = G[E_r]$ denote the *red subgraph* induced by E_r and $G_b = G[E_b]$ the *blue subgraph* induced by E_b.

For a given color frame F, a *minimal F-coloring* of a graph G is an F-coloring with the property that if any red edge of G is re-colored blue, then the resulting red-blue coloring of G is not an F-coloring of G. Obviously, every minimum F-coloring is minimal but the converse is not true in general (as we will soon see). The maximum number of red edges in a minimal F-coloring of G is the *upper F-chromatic index* $\chi''_F(G)$ of G. Since every minimum F-coloring of G is minimal, $\chi'_F(G) \leq \chi''_F(G)$. For example, consider the two red-blue colorings c_2 and c_5 of Q_3 shown in Figure 2. The coloring c_5 is a minimum F_0-coloring and so c_5 is also minimal. On the other hand, the coloring c_2 is a minimal F_0-coloring that is not minimum. In fact, $\chi''_{F_0}(Q_3) = 4$, while $\chi'_{F_0}(Q_3) = 3$ as we saw earlier.

In the next two sections, we show that F_0-colorings of graphs (and the F_0-chromatic index) have connections with two well-known concepts in graph theory.

3 A Bichromatic View of Matchings

A central topic in graph theory is that of matchings. In fact, Lovász and Plummer have written a book [14] on the theory of matchings. A set of edges in a graph G is *independent* if no two edges in the set are adjacent in G. The edges in an independent set of edges of G form a *matching* in G. If M is a matching in a graph G with the property that every vertex of G is incident with an edge of M, then M is a *perfect matching* in G. Clearly, if G has a perfect matching M, then G has even order and the subgraph induced by M is a *1-factor* of G.

A matching of maximum size in G is a *maximum matching*. Thus every perfect matching is a maximum matching but the converse is not true. In particular, if the order of G is odd, then G cannot have a perfect matching. The *edge independence number* $\alpha'(G)$ of G is the number of edges in a maximum matching of G. The number $\alpha'(G)$ is also referred to as the *matching number* of G.

A matching M in a graph G is a *maximal matching* of G if M is not a proper subset of any other matching in G. While every maximum matching is maximal, a maximal matching need not be a maximum matching. The minimum number of edges in a maximal matching of G is called the *lower edge independence number* or *lower matching number* $\alpha''(G)$ of G. Necessarily, $\alpha''(G) \leq \alpha'(G)$. We will see that matchings in graphs can be looked at in terms of F-colorings for a specific color frame F. In particular, we show for the unique color frame F_0 of P_3 that the F_0-chromatic index $\chi'_{F_0}(G)$ is in fact the lower edge independence number of G. We begin with a lemma.

Lemma 3.1 *Let G be a connected graph of size 2 or more and F_0 the color frame of P_3. If G has a minimum F_0-coloring c with $|E_{c,r}| = k$, then G has a minimum F_0-coloring c' with $|E_{c',r}| = k$ such that $E_{c',r}$ is a matching of G.*

Proof. Let $F = F_0$. Among all minimum F-colorings of G with exactly k red edges, let c' be one such that the red subgraph $G_r = G[E_{c',r}]$ has the maximum matching number. We claim that $E_{c',r}$ is a matching, for suppose that $E_{c',r}$ contains two adjacent edges, say uv and vw where $u, v, w \in V(G)$. If either u or w is an end-vertex of G, say the former, then the red-blue coloring obtained from c' by changing the color of uv to blue is also an F-coloring of G with fewer red edges, which contradicts c' being a minimum F-coloring of G. Thus neither u nor w is an end-vertex of G. Suppose that $w_1, w_2, \ldots, w_a$ are the vertices distinct from v that are adjacent to w. If some edge ww_i $(1 \leq i \leq a)$ is red, then the red-blue coloring obtained from c' by changing the color of vw to blue is also an F-coloring of G with fewer red edges, a contradiction. Thus all edges ww_i $(1 \leq i \leq a)$ are blue. If there exists some vertex w_i $(1 \leq i \leq a)$ such that w_i is not incident to a red edge, then we can change the color of vw to blue and the color of ww_i to red, producing a minimum F-coloring in which the matching number of the red subgraph is larger than that of c'. This contradicts the defining property of c'. Thus every vertex w_i $(1 \leq i \leq a)$ is incident to at least one red edge. However then, the red-blue coloring obtained from c' by changing the color of vw to blue is again an F-coloring of G containing a smaller number of red edges, a contradiction. Therefore, $E_{c',r}$ is a matching, as claimed. ∎

Theorem 3.2 *Let G be a connected graph of size 2 or more. If F_0 is the color frame of P_3, then $\chi'_{F_0}(G) = \alpha''(G)$ and $\chi''_{F_0}(G) \geq \alpha'(G)$.*

Proof. Let $F = F_0$. We first show that $\chi'_F(G) = \alpha''(G)$. To verify that $\chi'_F(G) \leq \alpha''(G)$, let M be a maximal matching of G with $|M| = \alpha''(G)$. Since the red-blue coloring of G in which M is the set of red edges is an F-coloring of G, it follows that $\chi'_F(G) \leq |M| = \alpha''(G)$. Next, we verify that $\alpha''(G) \leq \chi'_F(G)$. By Lemma 3.1, G has a minimum F-coloring c such that $E_{c,r}$ is a matching of G. We show that $E_{c,r}$ is maximal. Let $e \in E(G) - E_{c,r}$ be a blue edge. Since c is an F-coloring of G, it follows that e is adjacent to some edge in $E_{c,r}$. This implies that $E_{c,r} \cup \{e\}$ is not a matching and so $E_{c,r}$ is a maximal matching of G. Therefore, $\alpha''(G) \leq |E_{c,r}| = \chi'_F(G)$ and so $\chi'_F(G) = \alpha''(G)$.

To show that $\chi''_F(G) \geq \alpha'(G)$, let M be a maximum matching of G and so $|M| = \alpha'(G)$. Then the red-blue coloring c of G in which $E_{c,r} = M$ is an F-coloring of G. It remains to show that c is a minimal F-coloring of G. Assume, to the contrary, that c is not minimal. Then there is a red edge e such that the red-blue coloring c' obtained from c by changing the color of e to blue is also an F-coloring of G. Since $E_{c',r} = M - \{e\}$ and the blue edge e is not adjacent to any red edge in $E_{c',r}$, it follows that c' is not F-coloring of G, a contradiction. Therefore, $\chi''_F(G) \geq |M| = \alpha'(G)$ ■

Although there are many connected graphs G for which $\chi''_{F_0}(G) = \alpha'(G)$, it is also possible that $\chi''_{F_0}(G) > \alpha'(G)$. For example, Figure 3 shows a connected graph G of order 7 such that $\chi''_{F_0}(G) = 4$ and $\alpha'(G) = 3$ together with a minimal F_0-coloring with 4 red edges (indicated by bold lines).

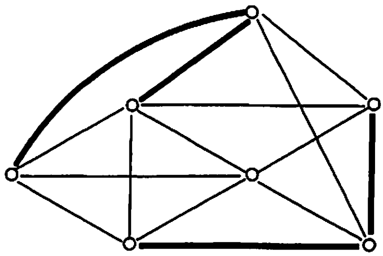

Figure 3: A graph G with $\chi''_{F_0}(G) = 4$ and $\alpha'(G) = 3$

As a result of Theorem 3.2, we see that the lower matching number of G is the minimum number of red edges in an F_0-coloring of G, where F_0 is the color frame of P_3. Theorem 3.2 therefore provides a new setting for the lower edge independence number (the lower matching number) $\alpha''(G)$ of a graph G. Furthermore, if G is a connected graph of size 2 or more, then maximal matchings of G and minimal F_0-colorings of G are closely related, as we now show.

Theorem 3.3 *Let G be a connected graph of size 2 or more, F_0 the color frame of P_3 and M a matching of G. Then M is a maximal matching of G if and only if G has a minimal F_0-coloring whose set of red edges is M.*

Proof. First, suppose that M is a maximal matching of G. As we saw in the proof of Theorem 3.2, the red-blue coloring having M as the set of red edges is a minimal F_0-coloring of G. It remains to verify the converse. Assume that c is a minimal F_0-coloring of G such that $E_{c,r} = M$. We claim that M is maximal, for otherwise, there is $e \notin M$ such that $M \cup \{e\}$ is a matching. This, however, implies that there is the blue edge e that is not adjacent to any red edge in M and so c is not an F_0-coloring of G, which is a contradiction. ■

It was shown in [13] that if G is a graph and k is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then G contains a maximal matching with k edges. The following is then a consequence of this result and Theorem 3.3.

Corollary 3.4 *Let G be a connected graph of size 2 or more and F_0 the color frame of P_3. If k is an integer with $\alpha''(G) \leq k \leq \alpha'(G)$, then there is a minimal F_0-coloring of G with exactly k red edges.*

We now describe the structure of the red subgraph produced by a minimal F_0-coloring of a graph. A graph H is a *galaxy* if each component of H is a star of order at least 2.

Theorem 3.5 *Let G be a connected graph of size 2 or more and F_0 the color frame of P_3. If c is a minimal F_0-coloring of G, then the red subgraph induced by the set of red edges in G is a galaxy.*

Proof. Let c be a minimal F_0-coloring of G and let S be a component of the corresponding red subgraph G_r of G. If S contains a path or triangle (u, v, w, x) of length 3 as a subgraph, then the red-blue coloring obtained from c by changing the color of vw to blue is also an F_0-coloring of G with fewer red edges, which contradicts that c is minimal. Hence S contains no cycle and the diameter of S is at most 2, which implies that S is a star. Therefore, G_r is a galaxy. ■

4 A Bichromatic View of Edge Domination

An area of graph theory that has received increased attention during recent decades is that of domination. Two books [10, 11] by Haynes, Hedetniemi and Slater were devoted to this subject. An edge e in a graph G is said to *dominate* itself and all edges adjacent to e. A set S of edges of G is an *edge dominating set* of G if every edge of G is dominated by some edge in S. The minimum size of an edge dominating set of G is the *edge domination number* of G and is denoted by $\gamma'(G)$. Moreover, $\gamma'(G)$ is the domination number of the line graph of G. An edge dominating set of size $\gamma'(G)$ is called a *minimum edge dominating set* of G, while an edge dominating set S of a graph G is a *minimal edge dominating set* if no proper subset of S is also an edge dominating set of G. While a minimum edge dominating set is minimal, the converse is not true. The maximum size of a minimal edge dominating set in G is the *upper edge domination number* of G and is denoted by $\gamma''(G)$.

Theorem 4.1 *Let G be a connected graph of size 2 or more. If F_0 is the color frame of P_3, then $\gamma'(G) = \chi'_{F_0}(G)$ and $\gamma''(G) = \chi''_{F_0}(G)$.*

Proof. Let $F = F_0$. We first show that $\gamma'(G) = \chi'_F(G)$. Let X be an edge dominating set with $|X| = \gamma'(G)$. Define a red-blue coloring c of G such that $E_{c,r} = X$. Let e be a blue edge of G. Since X is an edge dominating set of G, it follows that e is adjacent to at least one red edge in $X = E_{c,r}$. Thus c is an F-coloring of G and so $\chi'_F(G) \leq |X| = \gamma'(G)$. Next, let c' be an F-coloring of G with $|E_{c',r}| = \chi'_F(G)$. Thus every edge not in $E_{c',r}$ is adjacent to at least one edge in $E_{c',r}$, which implies that $E_{c',r}$ is an edge dominating set of G. Thus $\gamma'(G) \leq |E_{c',r}| = \chi'_F(G)$.

Next, we show that $\gamma''(G) = \chi''_F(G)$. Let Y be a minimal edge dominating set in G with $|Y| = \gamma''(G)$. Define a red-blue coloring c of G such that $E_{c,r} = Y$. Let e be a blue edge of G. Since Y is an edge dominating set of G, it follows that e is adjacent to at least one red edge in $Y = E_{c,r}$. Thus c is an F-coloring of G. We claim that c is minimal, for suppose that c is not is minimal. Then there is $e \in E_{c,r}$ such that the red-blue coloring c_0 obtained from c by changing the color of e to blue is also an F-coloring of G. This implies that each blue edge in the coloring c_0 is adjacent to at least one edge in $E_{c_0,r} = E_{c,r} - \{e\}$ and so the proper subset $E_{c_0,r}$ of $E_{c,r}$ is an edge dominating set of G, a contradiction. Thus, as claimed, c is minimal and so $\chi''_F(G) \geq |Y| = \gamma''(G)$. Next, let c' be a minimal F-coloring of G with $|E_{c',r}| = \chi''_F(G)$. We claim that $E_{c',r}$ is a minimal edge dominating set in G. Since c' is an F-coloring, every edge not in $E_{c',r}$ is adjacent to at least one edge in $E_{c',r}$ and so $E_{c',r}$ is an edge dominating set of G. Since c' is minimal, if the color of any edge e' in $E_{c',r}$ is changed to blue, then the resulting red-blue coloring is not an F-coloring. Thus some blue edge not in $E_{c',r} - \{e'\}$ is not adjacent to any edge in $E_{c',r} - \{e'\}$ and so $E_{c',r} - \{e'\}$ is not an edge dominating set of G. This implies that no proper subset of $E_{c',r}$ is an edge dominating set in G. Hence, $E_{c',r}$ is minimal. Thus $\gamma''(G) \geq |E_{c',r}| = \chi''_F(G)$ and so $\gamma''(G) = \chi''_F(G)$. ∎

As a result of Theorem 4.1, the edge domination number of a connected graph G of size 2 or more is the minimum number of red edges in an F_0-coloring of G and the upper edge domination number is the maximum number of red edges in a minimal F_0-coloring of G. Therefore, Theorem 4.1 provides a new setting for the two edge domination numbers $\gamma'(G)$ and $\gamma''(G)$ of a graph G. Furthermore, the proof of Theorem 4.1 shows that (1) if c is a minimal F_0-coloring of G, then $E_{c,r}$ is a minimal edge dominating set of G and (2) if S is a minimal edge dominating set of G, then the red-blue coloring c' of G with $E_{c',r} = S$ is a minimal F_0-coloring. Therefore, there is a one-to-one correspondence between the set of all minimal edge dominating sets of G and the set of all minimal F_0-colorings of G. Hence the following is a consequence of the proof of Theorem 4.1.

Corollary 4.2 *Let G be a connected graph of size 2 or more and F_0 the color frame of P_3. Then S is a minimal edge dominating set of G if and only if G has a minimal F_0-coloring whose set of red edges is S.*

In view of Theorems 3.2 and 4.1, F_0-colorings of graphs provide a new framework for both edge independence and edge domination and lead us to consider F-colorings of graphs for other choices of color frames F.

5 The Color Frames of P_4

In this section, we turn our attention to F-colorings of connected graphs of size at least 3, where F is one of the five color frames $F_1, F_2, \ldots, F_5$ of P_4 shown in Figure 1 and the five parameters $\chi'_{F_i}(G)$ $(1 \leq i \leq 5)$ of a graph G. We begin with the color frame F_1, which we also refer to as the blue-red-blue color frame of P_4.

5.1 The Blue-Red-Blue Color Frame F_1 of P_4

Because of the symmetry of the blue-red-blue color frame F_1 of P_4 shown in Figure 1, it doesn't matter which of the two blue edges is chosen as the root edge of F_1. Since every F_1-coloring is also an F_0-coloring, it follows that

$$\chi'_{F_0}(G) \leq \chi'_{F_1}(G) \text{ for every connected graph } G \text{ of size at least 3.} \qquad (1)$$

There are conditions under which equality in (1) holds. Let $\delta(G)$ denote the minimum degree of a graph G.

Theorem 5.1 *Let G be a connected graph of size at least 3 and F_1 the blue-red-blue color frame of P_4. If G is (i) triangle-free and $\delta(G) \geq 2$ or (ii) $\delta(G) \geq 3$, then $\chi'_{F_1}(G) = \chi'_{F_0}(G)$.*

Proof. By (1), it remains only to show that $\chi'_{F_1}(G) \leq \chi'_{F_0}(G)$. Let M be a maximal matching of G such that $|M| = \alpha''(G)$. By Theorem 3.2, $\chi'_{F_0}(G) = \alpha''(G)$. Let c be a minimum F_0-coloring of G such that $E_{c,r} = M$. We claim that c is an F_1-coloring of G. Let e_b be a blue edge of G. Since $E_{c,r}$ is a maximal matching, $E_{c,r} \cup \{e_b\}$ is not a matching and so e_b is adjacent to a red edge e_r in $E_{c,r}$. We may assume that $e_r = uv$ and $e_b = vw$. If G is triangle-free and $\delta(G) \geq 2$ or $\delta(G) \geq 3$, then u is adjacent to some vertex u' distinct from v and w. Since $e_r \in E_{c,r}$ and $E_{c,r}$ is a matching, uu' cannot be red and so is blue. Thus e_b belongs to the copy of F with $E(F) = \{u'u, uv, vw\}$ rooted at e_b in G and so c is an F_1-coloring of G, as claimed. Hence $\chi'_{F_1}(G) \leq |E_{c,r}| = \alpha''(G)$. Therefore, $\chi'_{F_1}(G) = \alpha''(G) = \chi'_{F_0}(G)$. ■

The conditions (i) and (ii) in Theorem 5.1 are only sufficient for equality to hold in (1). For example, the graph P_4 in Figure 4 is triangle-free and has minimum degree 1 but $\chi'_{F_0}(P_4) = \chi'_{F_1}(P_4) = 1$. Furthermore, the graphs H_1 and H_2, also shown in Figure 4, contain triangles but have minimum degree 1 and 2, respectively; yet $\chi'_{F_0}(H_1) = \chi'_{F_1}(H_1) = 2$ and $\chi'_{F_0}(H_2) = \chi'_{F_1}(H_2) = 1$.

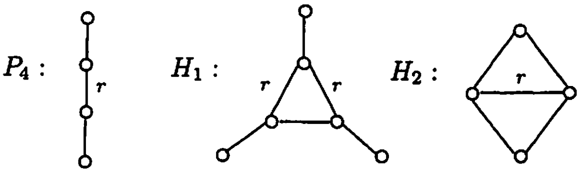

Figure 4: Graphs G with $\chi'_{F_0}(G) = \chi'_{F_1}(G)$

There are graphs G not satisfying the conditions stated in Theorem 5.1 such that $\chi'_{F_0}(G) \neq \chi'_{F_1}(G)$. For example, consider the graphs G_1, G_2 and G_3 in Figure 5.

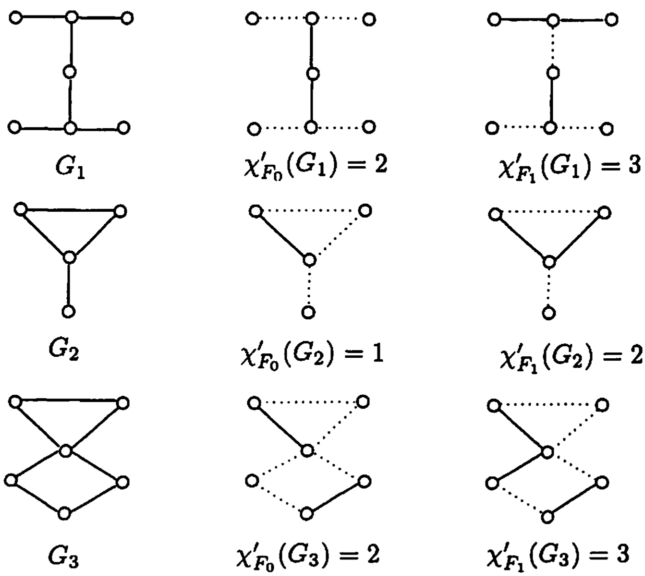

Figure 5: Graphs G_i with $\chi'_{F_0}(G_i) < \chi'_{F_1}(G_i)$ $(1 \leq i \leq 3)$

The graph G_1 is triangle-free and $\delta(G_1) = 1$, G_2 contains a triangle and $\delta(G_2) = 1$ and G_3 contains a triangle and $\delta(G_3) = 2$. For each graph G_i $(i = 1, 2, 3)$, $\chi'_{F_0}(G_i) < \chi'_{F_1}(G_i)$. A minimum F_0-coloring and a minimum F_1-coloring for each of these graphs are shown Figure 5. In each red-blue coloring, the solid edges are red edges and the dashed lines are blue edges. For $i = 1, 2, 3$, $\chi'_{F_1}(G_i) = \chi'_{F_0}(G_i) + 1$. The difference $\chi'_{F_1}(G) - \chi'_{F_0}(G)$

can not only be arbitrarily large, there is essentially no restrictions on the values of $\chi'_{F_0}(G)$ and $\chi'_{F_1}(G)$ of a graph G.

Theorem 5.2 *For each pair a, b of positive integers $a \leq b$, there is a connected graph G such that $\chi'_{F_0}(G) = a$ and $\chi'_{F_1}(G) = b$.*

Proof. First, assume that $a = b$. If $a = b = 1$, then $\chi'_{F_0}(P_4) = \chi'_{F_1}(P_4) = 1$; while if $a = b \geq 2$, then $\chi'_{F_1}(K_{2a}) = \chi'_{F_0}(K_{2a}) = a$. Thus, we may assume that $1 \leq a < b$. For $a = 1$, the star $K_{1,b}$ has $\chi'_{F_0}(K_{1,b}) = 1$ and $\chi'_{F_1}(K_{1,b}) = b$ (as $K_{1,b}$ contains no P_4 as a subgraph). Thus, we may assume that $a \geq 2$. First, let $G_1, G_2, \ldots, G_a$ be a copies of the star $K_{1,b-a+1}$ of order $b - a + 2$, where

$$V(G_i) = \{v_i, v_{i,1}, v_{i,2}, \ldots, v_{i,b-a+2}\}$$

and v_i is the central vertex of G_i for $1 \leq i \leq a$. The graph G is obtained from these a graphs by adding a new vertex v and joining v to each vertex v_i in G_i for $1 \leq i \leq a$. Then $\chi'_{F_0}(G) = \alpha''(G) = a$. It remains to show that $\chi'_{F_1}(G) = b$. Since the red-blue coloring c with

$$E_{c,r} = E(G_1) \cup \{vv_i : \ 2 \leq i \leq a\}$$

is an F_1-coloring of G, it follows that $\chi'_{F_1}(G) \leq |E_{c,r}| = b$. Next, we show that $\chi'_{F_1}(G) \geq b$. Let c' be a minimum F_1-coloring of G. For each integer i with $1 \leq i \leq a$, if c' assigns blue to an edge in $E(G_i)$, then $c'(vv_i)$ must be red and some $c'(vv_j)$ must be blue where $j \neq i$. This in turn implies that c' must assign red to all edges in $E(G_j)$. Furthermore, c' must assign blue to at least one edge in $E(G_1) \cup E(G_2) \cup \cdots \cup E(G_a)$ (for otherwise, all edges in G must be colored red). We may assume that $E(G_1) \cup \{vv_2\} \subset E_{c',r}$. Also, c' must assign red to at least one edge in $E(G_i) \cup \{vv_i\}$ for $3 \leq i \leq a$, it follows that $|E_{c',r}| \geq (b - a + 2) + (a - 2) = b$. Therefore, $\chi'_{F_1}(G) = b$, as claimed. ■

5.2 The Red-Blue-Red Color Frame F_2 of P_4

Consider the red-blue-red color frame F_2 of P_4 in Figure 1 in which the only blue edge is the root edge of F. Since every F_2-coloring is also an F_0-coloring, $\chi'_{F_0}(G) \leq \chi'_{F_2}(G)$ for every connected graph G of size at least 3.

A set S of edges of G is a *k-edge dominating set* of G if every edge in $E(G) - S$ is dominated by at least k edges in S. Since $E(G)$ is such a set, every graph G has a k-edge dominating set. The minimum size of a k-edge dominating set of G is the *k-edge domination number* of G and is denoted by $\gamma'_k(G)$. A k-edge dominating set of size $\gamma'_k(G)$ is called a *minimum k-edge dominating set* of G. Observe that if c is an F_2 coloring of a graph G, then each blue edge is adjacent to two independent red edges. Thus $E_{c,r}$ is a 2-edge dominating set of G. This implies that

$$\gamma_2'(G) \le \chi_{F_2}'(G) \quad \text{for every connected graph } G \text{ of size at least 3.} \qquad (2)$$

Furthermore, no blue edges can be pendant edges; that is, every pendant edge must be colored red in any F_2-coloring of G. Hence if G has p pendant edges, then $\chi_{F_2}'(G) \ge p$. For example, if G is a double star (a tree of diameter 3) of size $m \ge 5$, then $\chi_{F_2}'(G) = m - 1$. Since $\gamma_2'(G) = 3$, it follows that $\chi_{F_2}'(G) - \gamma_2'(G) = m - 4$ which can be arbitrarily large. In fact, more can be said.

Proposition 5.3 *For each pair a, b of integers with $2 \le a \le b$, there is a connected graph G such that $\gamma_2'(G) = a$ and $\chi_{F_2}'(G) = b$.*

Proof. We consider two cases, according to $a = b$ or $a < b$.

Case 1. $a = b$. If a is even, say $a = 2k$ where $k \ge 1$, then let G be the graph obtained from kP_3 by adding a new vertex v and joining v to each end-vertex of kP_3. Then $\gamma_2'(G) = \chi_{F_2}'(G) = a$. If a is odd, say $a = 2k + 1$ where $k \ge 1$, then let G be the graph obtained from the union $kP_3 \cup P_2$ of kP_3 and P_2 by adding a new vertex v and joining v to each end-vertex of $kP_3 \cup P_2$. Then $\gamma_2'(G) = \chi_{F_2}'(G) = a$.

Case 2. $a < b$. For $a = 2$, let $G = K_{1,b}$. Then $\gamma_2'(G) = 2$ and $\chi_{F_2}'(G) = b$. For $a = 3$, let G be a double of size $b + 1$. Then $\gamma_2'(G) = 3$ and $\chi_{F_2}'(G) = b$. We now assume $a \ge 4$. Let G be the graph described in Case 1 such that $\gamma_2'(G) = \chi_{F_2}'(G) = a$. Let $v \in V(G)$ be the vertex of G such that $\deg v = a$ if a is even and $\deg v = a + 1$ if a is odd. Let H be the graph obtained from the graph G in Case 1 by adding $b - a$ new vertices and joining each of these new vertices to the vertex v. Then $\gamma_2'(H) = a$ and $\chi_{F_2}'(H) = b$. ■

5.3 The Red-Red-Blue Color Frame F_3 of P_4

Consider the red-red-blue color frame F_3 of P_4 in Figure 1 in which the only blue edge is the root edge of F. Since every F_3-coloring is also an F_0-coloring, $\chi_{F_0}'(G) \le \chi_{F_3}'(G)$ for every connected graph G of size at least 3.

A *total edge dominating set* in a connected graph G is a subset S of $E(G)$ such that every edge of G is adjacent to an edge of S. Thus a total edge dominating set contains no independent edges. If G is a nonempty graph containing no component K_2, then $E(G)$ is a total edge dominating set and so every connected graph of order at least 3 has a total edge dominating set. The *total edge domination number* $\gamma_t'(G)$ is the minimum size of a total edge dominating set. A total edge dominating set of size $\gamma_t'(G)$ is a *minimum total edge dominating set* of G.

Theorem 5.4 *If G is a connected graph of size at least 3, then*

$$\gamma_t'(G) \leq \chi_{F_3}'(G).$$

Proof. First, we make an observation. If G has a minimum F_3-coloring c such that $E_{c,r}$ contains no independent edges, then each red edge is adjacent to a red edge and each blue edge is adjacent to a red edge, which implies that $E_{c,r}$ is a total edge dominating set of G and so $\gamma_t'(G) \leq |E_{c,r}| = \chi_{F_3}'(G)$. Thus, it suffices to show that such a minimum F_3-coloring exists. Among all minimum F_3-colorings of G, let c be one such that the number of independent edges in $E_{c,r}$ is minimum. If $E_{c,r}$ has no independent edges, then c has the desired property. Thus, we may assume that $E_{c,r}$ contains an independent edge f. Thus f does not belong to any copy of F_3 in this coloring c. Since G is connected and f is not adjacent to any red edge, f is adjacent to an blue edge e. Because c is an F_3-coloring, e belongs to a copy of F_3. Suppose that $f = uv$, $e = vw$ and e belongs to the copy (v, w, x, y) of F_3 where then wx and xy are red edges.

Now the coloring c' obtained from c by interchanging the colors of f and e is a minimum F_3-coloring the number of whose independent edges is smaller than that of c. This contradicts the defining property of c. ∎

For each positive integer ℓ,

$$\gamma_t'(P_{4\ell}) = \chi_{F_3}'(P_{4\ell}) = 2\ell \text{ and } \gamma_t'(P_{4\ell+2}) = \chi_{F_3}'(P_{4\ell+2}) = 2\ell + 1.$$

Therefore, for each positive integer k, there is a connected graph G such that $\gamma_t'(G) = \chi_{F_3}'(G) = k$. On the other hand, $\chi_{F_3}'(G) - \gamma_t'(G)$ can be arbitrarily large. In fact, more can be said. Two end-vertices of a graph are said to be *similar* if they are adjacent to a same vertex.

Proposition 5.5 *For each positive integer k, there is a connected graph G_k such that*

$$\chi_{F_3}'(G_k) - \gamma_t'(G_k) = k.$$

Proof. We recursively construct a sequence $G_1, G_2, \ldots$ of graphs as follows. Let G_1 be the graph shown in Figure 6. The graph G_1 has two pairs of similar end-vertices, namely $\{u_1, u_2\}$ and $\{w_1, w_2\}$. The graph G_2 is obtained from G_1 and another copy of G_1 by identifying a pair of similar end-vertices in each graph (see Figure 6 where the solid vertices are identified vertices). Thus G_2 has two pairs of similar end-vertices. For each $k \geq 3$, the graph G_k is obtained from G_{k-1} and a copy of G_1 by identifying a pair of similar end-vertices in each graph. The graph G_3 is also shown in Figure 6.

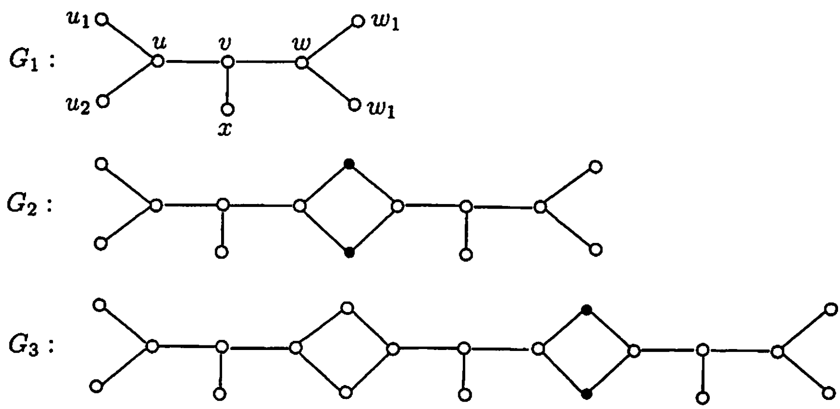

Figure 6: The graphs G_1 and G_2

For each integer $k \geq 1$, let S_k be the set of bridges that are not pendant edges in G_k, where then the subgraph $G_k[S_k]$ induced by S_k is kP_3, and let X_k be the set of pendant edges each of which is incident to the center vertex of some component P_3 in $G_k[S_k]$. For example, $S_1 = \{uv, vw\}$ and $X_1 = \{vx\}$. Thus $S_k \cap X_k = \emptyset$, $|X_k| = k$ and the subgraph $G_k[S_k \cup X_k]$ induced by $S_k \cup X_k$ is $kK_{1,3}$. Since S_k is a minimum total edge dominating set of G_k and the red-blue coloring c with $E_{c,r} = S_k \cup X_k$ is a minimum F_3-coloring of G_k, it follows that

$$\chi'_{F_3}(G_k) = |S_k \cup X_k| = |S_k| + |X_k| = \gamma'_t(G_k) + k$$

for $k \geq 1$. ∎

5.4 The Two Red-Blue-Blue Color Frames F_4 and F_5 of P_4

The two red-blue-blue color frames F_4 and F_5 of P_4 are shown in Figure 1 in which one of the two blue edges is the root edge of the color frame. Again, we indicate the root edge in each of F_4 and F_5 by a bold edge. It appears that $\chi'_{F_4}(G)$ is not related to any known edge domination parameters.

We consider the F_5-colorings of a connected graph of size at least 3. A set $S \subseteq E(G)$ is a *restrained edge dominating set* if every edge not in S is adjacent to an edge in S and to an edge in $E(G) - S$. Every graph has a restrained edge dominating set since $E(G)$ is such a set. The *restrained edge domination number* $\gamma'_r(G)$ is the minimum size of a restrained edge dominating set of G. A restrained edge dominating set of size $\gamma'_r(G)$ is a *minimum restrained edge dominating set* of G. If c is an F_5-coloring of G, then every blue edge is adjacent to a red edge and a blue edge. Thus the set $E_{c,r}$ of red edges is a restrained edge dominating set of G and so $\gamma'_r(G) \leq |E_{c,r}|$. Therefore,

$$\gamma'_r(G) \leq \chi'_{F_5}(G) \text{ for every connected graph } G \text{ of size at least 3.} \qquad (3)$$

Proposition 5.6 *Let a and b be positive integers with $a \leq b$. Then there is a connected graph G of size at least 3 such that $\gamma'_r(G) = a$ and $\chi'_{F_5}(G) = b$ if and only if $(a,b) \neq (1,1)$.*

Proof. Assume, to the contrary, that there is a connected graph G of size at least 3 such that $\gamma'_r(G) = \chi'_{F_5}(G) = 1$. Let $S = \{e\}$ be a minimum restrained edge dominating set of G. Then each edge f distinct from e in G must be adjacent to e and another edge f' that is also adjacent to e. Since (1) f and f' are adjacent and (2) f and f' are both adjacent to e, it follows that $\{e, f, f'\}$ forms a triangle. This implies that $G - e$ is the complete bipartite graph $K_{2,k}$ for some positive integer k and the edge e joins the only two vertices in one of the partite sets of G. If $k = 1$, then $G = K_3$ and $\chi'_{F_5}(G) = 3$, a contradiction. Thus $k \geq 2$ and the graph G is shown in Figure 7. Let c be a minimum F_5-coloring of G where then $|E_{c,r}| = 1$. First assume that $E_{c,r} = \{e\}$. If F is a copy of F_5 in G, then e is the middle edge of F and so no blue edge belongs to a copy of F_5. Thus $E_{c,r} = \{f\}$ where $f \neq e$. However then, the edge f' adjacent to both f and e does not belong to a copy of F_5 in G, a contradiction.

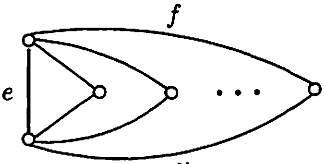

Figure 7: A graph G with $\gamma'_r(G) = 1$

For the converse, let a and b of positive integers with $a \leq b$ such that $(a,b) \neq (1,1)$. First, assume that $a = b \geq 2$. Let $G = S(K_{1,a})$ be the subdivision of $K_{1,a}$; that is, G is obtained from $K_{1,a}$ by subdividing each edge exactly once. Let S be the set of pendant edges of G. Then S is a minimum restrained edge dominating set of G and the red-blue coloring c with $E_{c,r} = S$ is a minimum F_5-coloring of G. Thus $\gamma'_r(G) = \chi'_{F_5}(G) = a$. Next, assume that $a < b$. Let G be the graph obtained from $S(K_{1,a})$ by adding $b - a$ pendant edges at a vertex of degree 2 in $S(K_{1,a})$. Then $\gamma'_r(G) = a$ and $\chi'_{F_5}(G) = b$. ■

There is no connected graph G of size 3 or more such that $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 1$. On the other hand, since $\chi'_{F_4}(P_{3\ell+2}) = \chi'_{F_5}(P_{3\ell+2}) = \ell + 1$ for each $\ell \geq 1$, there is a connected graph G such that $\chi'_{F_4}(G) = \chi'_{F_5}(G) = k$ for each integer $k \geq 2$.

Proposition 5.7 *For each positive integer k, there is a connected graph G_k such that*

$$\chi'_{F_5}(G_k) - \chi'_{F_4}(G_k) = k.$$

Proof. For each $k \geq 1$, let $G_k = S(K_{1,k+4})$ be the subdivision of $K_{1,k+4}$ and let S be the set of pendant edges of G_k. Since (1) every F_5-coloring of G_k must assign red to each edge in S and (2) the red-blue coloring c of G_k such $E_{c,r} = S$ is an F_5-coloring, $\chi'_{F_5}(G_k) = |S| = k + 4$.

Next, we show that $\chi'_{F_4}(G_k) = 4$. Let

$$V(G_k) = \{u, u_1, u_2, \ldots, u_{k+4}, v_1, v_2, \ldots, v_{k+4}\}$$

where u is the central vertex of the subgraph $K_{1,k+4}$ in G_k, u is adjacent to u_i for $1 \leq i \leq k+4$ and u_i is adjacent to v_i for $1 \leq i \leq k+4$. The red-blue coloring c_0 with $E_{c_0,r} = \{uu_1, u_1v_1, u_2v_2, u_3v_3\}$ is an F_4-coloring of G and so $\chi'_{F_4}(G_k) \leq 4$. Assume, to the contrary, that $\chi'_{F_4}(G_k) \leq 3$. Let c' be a minimum F_4-coloring of G_k where then $|E_{c'r}| \leq 3$. Note that c' must assign blue to at least two edges in $\{uu_i : 1 \leq i \leq k+4\}$, say uu_1 and uu_2 are blue. Since each of uu_1 and uu_2 belongs to a copy of F_4, it follows that c' must assign red to at least two pendant edges in G_k, say u_sv_s and u_tv_t are red, where then $1 \leq s \neq t \leq k+4$ and uu_s and uu_t are blue. Let $j \in \{1, 2, \ldots, k+4\} - \{s, t\}$. Either u_jv_j is red or u_jv_j is blue. If u_jv_j is blue, then some edge uu_p must be red where $p \notin \{1, 2, j, s, t\}$. Hence $|E_{c'r}| = 3$ and either

$$\text{(i) } E_{c'r} = \{u_sv_s, u_tv_t, u_jv_j\} \text{ or (ii) } E_{c'r} = \{u_sv_s, u_tv_t, uu_p\}.$$

If (i) occurs, then each blue edge u_iv_i where $i \in \{1, 2, \ldots, k+4\} - \{s, t, j\}$ does not belong to a copy of F_4; while if (ii) occurs, then the blue edge u_pv_p does not belong to a copy of F_4. In any case, a contradiction is produced and so $\chi'_{F_4}(G_k) = 4$. Therefore, $\chi'_{F_5}(G_k) - \chi'_{F_4}(G_k) = k$. ■

Although there are graphs G for which $\chi'_{F_4}(G) > \chi'_{F_5}(G)$ (see the graph G in Figure 8), it is not known whether there is a connected graph G_k such that $\chi'_{F_4}(G_k) - \chi'_{F_5}(G_k) = k$ for *every* positive integer k.

For the five color frames $F_1, F_2, \ldots, F_5$ of P_4, a 2-element set $\{i, j\}$, where $i, j \in \{1, 2, \ldots, 5\}$, is called a *realizable* set if there exist a graph G such that $\chi'_{F_i}(G) < \chi'_{F_j}(G)$ and a graph H such that $\chi'_{F_j}(H) < \chi'_{F_i}(H)$. We will see that every 2-element subset of $\{1, 2, \ldots, 5\}$ is realizable with one possible exception. In order to show this, we first present three examples. In each of the following figures, the solid lines are red edges and the dashed lines are blue edges.

- For the graph G of Figure 8, $\chi'_{F_1}(G) = 1$, $\chi'_{F_i}(G) = 3$ for $i = 2, 3, 4$ and $\chi'_{F_5}(G) = 2$. For $1 \le i \le 5$, minimum F_i-colorings of G are also shown in Figure 8.

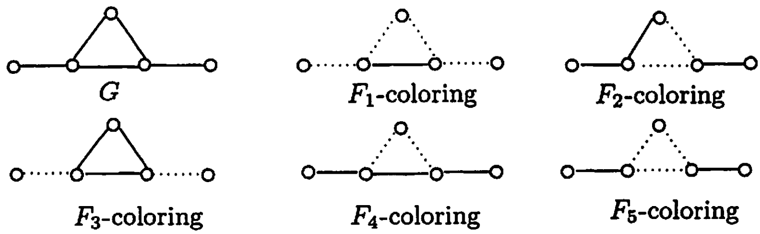

Figure 8: Minimum F_i-colorings of a graph G for $1 \le i \le 5$

- For the graph G of Figure 9, $\chi'_{F_1}(G) = 3$, $\chi'_{F_2}(G) = 5$, $\chi'_{F_3}(G) = 2$ and $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 4$. Minimum F_i-colorings of G are shown in Figure 9 for $1 \le i \le 5$.

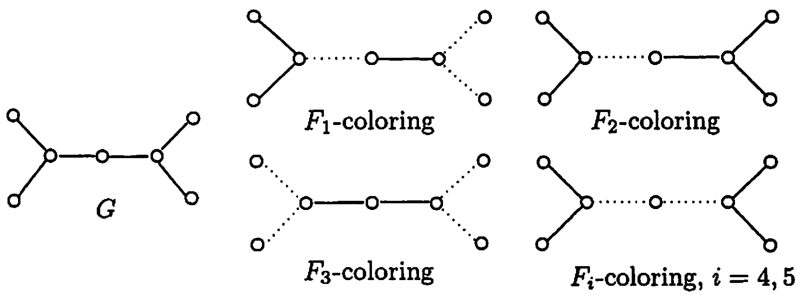

Figure 9: Minimum F_i-colorings of a graph G for $1 \le i \le 5$

- For the graph G of Figure 10, $\chi'_{F_1}(G) = 1$, $\chi'_{F_2}(G) = 3$, $\chi'_{F_3}(G) = 2$ and $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 4$. Minimum F_i-colorings of G are shown in Figure 9 for $1 \le i \le 5$.

We are now prepared to present the following.

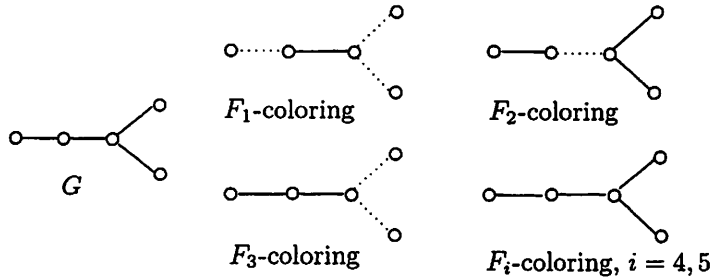

Figure 10: Minimum F_i-colorings of a graph G for $1 \le i \le 5$

Theorem 5.8 *Every 2-element subset of* $\{1, 2, \ldots, 5\}$ *is realizable except possibly* $\{1, 2\}$.

Proof. For each subset $\{i, j\} \subseteq \{1, 2, \ldots, 5\}$ and $\{i, j\} \neq \{1, 2\}$, where $i < j$, we construct two graphs $G_{i,j}$ and $H_{i,j}$ such that $\chi'_{F_i}(G_{i,j}) < \chi'_{F_j}(G_{i,j})$ and $\chi'_{F_j}(H_{i,j}) < \chi'_{F_i}(H_{i,j})$. Recall that $\chi'_{F_1}(Q_3) = 3$, $\chi'_{F_2}(Q_3) = \chi'_{F_3}(Q_3) = 4$, $\chi'_{F_4}(Q_3) = 2$ and $\chi'_{F_5}(Q_3) = 3$.

- For the set $\{1, 3\}$, let $G_{1,3} = P_4$ and $H_{1,3}$ be the graph in Figure 9. Then $1 = \chi'_{F_1}(P_4) < \chi'_{F_3}(P_4) = 2$ and $2 = \chi'_{F_3}(H_{1,3}) < \chi'_{F_1}(H_{1,3}) = 3$.
- For the set $\{1, 4\}$, let $G_{1,4}$ be the graph in Figure 8 and $H_{1,4} = Q_3$. Then $1 = \chi'_{F_1}(G_{1,4}) < \chi'_{F_4}(G_{1,4}) = 3$ and $2 = \chi'_{F_4}(Q_3) < \chi'_{F_1}(Q_3) = 3$
- For the set $\{1, 5\}$, let $G_{1,5}$ be the graph in Figure 8 and $H_{1,4} = P_4 + 2K_1$ (the join of P_4 and $2K_1$) shown in Figure 11. Then $1 = \chi'_{F_1}(G_{1,5}) < \chi'_{F_5}(G_{1,5}) = 2$ and $2 = \chi'_{F_5}(H_{1,5}) < \chi'_{F_1}(H_{1,5}) = 3$. (If $G = P_4 + kK_1$, then $\chi'_{F_1}(G) = 2$ and $\chi'_{F_5}(G) = k + 1$ and so $\chi'_{F_1}(G) - \chi'_{F_5}(G)$ can be arbitrarily large.)

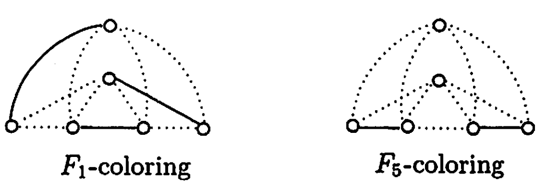

Figure 11: A graph G with $\chi'_{F_1}(G) = 3$ and $\chi'_{F_5}(G) = 2$

- For the set $\{2, 3\}$, let $G_{2,3} = C_6$ and $H_{2,3} = P_5$. Then $3 = \chi'_{F_2}(C_6) < \chi'_{F_3}(C_6) = 4$ and $2 = \chi'_{F_3}(P_5) < \chi'_{F_2}(P_5) = 3$.

- For the set $\{2,4\}$, let $G_{2,4}$ be the graph of Figure 10 and $H_{2,4}$ be the graph of Figure 12. Then $3 = \chi'_{F_2}(G_{2,4}) < \chi'_{F_4}(G_{2,4}) = 4$ and $3 = \chi'_{F_4}(H_{2,4}) < \chi'_{F_2}(H_{2,4}) = 4$.

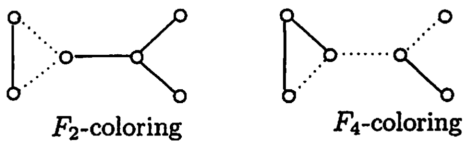

Figure 12: A graph G with $3 = \chi'_{F_4}(G) < \chi'_{F_2}(G) = 4$

- For the set $\{2,5\}$, let $G_{2,5}$ be the graph of Figure 10 and $H_{2,5} = Q_3$. Then $3 = \chi'_{F_2}(G_{2,5}) < \chi'_{F_5}(G_{2,5}) = 4$ and $3 = \chi'_{F_5}(Q_3) < \chi'_{F_2}(Q_3) = 4$.

- For the set $\{3,4\}$, let $G_{3,4}$ be the graph shown in Figure 13 and let $H_{3,4} = Q_3$. Then $2 = \chi'_{F_3}(G_{3,4}) < \chi'_{F_4}(G_{3,4}) = 3$ and $2 = \chi'_{F_4}(Q_3) < \chi'_{F_3}(Q_3) = 4$.

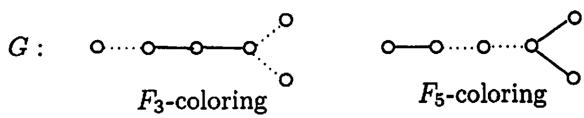

Figure 13: A graph G with $\chi'_{F_3}(G) = 2$ and $\chi'_{F_4}(G) = \chi'_{F_5}(G) = 3$

- For the set $\{3,5\}$, let $G_{3,5}$ be the graph G shown in Figure 13 and let $H_{3,5} = Q_3$. Then $2 = \chi'_{F_3}(G_{3,5}) < \chi'_{F_5}(G_{3,5}) = 3$ and $3 = \chi'_{F_5}(Q_3) < \chi'_{F_3}(Q_3) = 4$.

- For the set $\{4,5\}$, let $G_{4,5} = Q_3$ and let $H_{4,5}$ be the graph G shown in Figure 8. Then $2 = \chi'_{F_4}(Q_3) < \chi'_{F_5}(Q_3) = 3$ and $2 = \chi'_{F_5}(H_{4,5}) < \chi'_{F_4}(H_{4,5}) = 3$. ∎

For every connected graph G that we have encountered, $\chi'_{F_1}(G) \le \chi'_{F_2}(G)$. Furthermore, since $\chi'_{F_0}(G) = \gamma'(G) \le \gamma'_2(G) \le \chi'_{F_2}(G)$, it follows by Theorem 5.1 that if $\delta(G) \ge 3$ or if G is triangle-free and $\delta(G) \ge 2$, then $\chi'_{F_1}(G) \le \chi'_{F_2}(G)$. This leads us to the following conjecture.

Conjecture 5.9 *For every connected graph G of size at least 3,*

$$\chi'_{F_1}(G) \le \chi'_{F_2}(G).$$

We saw that if G is a disconnected graph with components G_1, G_2, ..., G_k where $k \geq 2$, then $\chi'_F(G) = \chi'_F(G_1) + \chi'_F(G_2) + \cdots + \chi'_F(G_k)$. Thus there are graphs G consisting of two components such that the numbers $\chi'_{F_i}(G)$, $i = 1, 2, 3, 4, 5$, are distinct. Whether there is a connected graph with this property is not known.

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