On some Ramsey numbers of C_4 versus $K_{2,n}$

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Abstract

For given graphs H_1, H_2, the Ramsey number $R(H_1, H_2)$ is the smallest positive integer n such that if we arbitrarily color the edges of the complete graph K_n with two colors 1 (red) and 2 (blue), then there is monochromatic copy of H_1 colored with 1 or H_2 colored with 2. We show that if n is even, $q = \lceil\sqrt{n}\rceil$ is odd, and $s = n-(q-1)^2 \leq q/2$, then $R(K_{2,2}, K_{2,n}) \leq n+2q-1$, where $K_{n,m}$ are complete bipartite graphs. The latter bound gives the exact value of $R(K_{2,2}, K_{2,18}) = 27$. Moreover, we show that $R(K_{2,2}, K_{2,14}) = 22$ and $R(K_{2,2}, K_{2,15}) = 24$.

1 Introduction

For given graphs H_1, H_2, the Ramsey number $R(H_1, H_2)$ is the smallest positive integer n such that if we arbitrarily color the edges of the complete graph K_n with two colors 1 (red) and 2 (blue), then there is a subgraph $G_1 \subset K_n$ isomorphic to H_1 and with all its edges red, or a subgraph $G_2 \subset K_n$ isomorphic to H_2 and with all its edges blue. Ramsey numbers for complete bipartite graphs are quite well investigated, see [3, 4, 5, 6, 7, 8]. All known values and bounds can be found in [9]. Harary [3] proved that $R(K_{1,n}, K_{1,m}) = n+m+\epsilon$, where $\epsilon = 1$ when n and m are even and $\epsilon = 0$ otherwise. Harborth and Mengersen [4] studied the properties of numbers $R(K_{2,2}, K_{m,n})$ for $2 \leq m \leq 3$ and $m \leq n$. The case $m = 1$ was studied by Parsons [8] and the cases of $m = 3$ and $3 \leq n \leq 10$ by Lortz [5].

In this paper we are interested in Ramsey numbers for $H_1 = K_{2,2} = C_4$ and $H_2 = K_{2,n}$. Harborth and Mengersen [4] proved the following theorem.

Theorem 1 *[see [4]] For* $n \geq 2$ *let* $q = \lceil\sqrt{n}\rceil$, $s = n - (q-1)^2$ *and* $M = \{2, 5, 37, 3137\}$. *Then*

$$R(C_4, K_{2,n}) \leq \begin{cases} n + 2q - 1 & \textit{for } s = 1 \textit{ and } n \notin M \\ n + 2q & \textit{for } 2 \leq s \leq q - 1 \textit{ or } n \in M \\ n + 2q + 1 & \textit{otherwise} \end{cases}$$

Moreover, if q is a prime power then

$$R(C_4, K_{2,n}) = \begin{cases} n + 2q - 1 & \text{for } s = 1 \text{ and } n \notin \{2, 5, 37\} \\ n + 2q & \text{for } s = q - 1 \geq 2 \text{ or } n \in \{2, 5, 37\} \\ n + 2q + 1 & \text{for } s = q \end{cases}$$

Additionally, $R(C_4, K_{2,n}) = n+2q+1$ if $q+1$ is a prime power and $s = 2q-1$.

Table 1 presents all known values of $R(K_{2,2}, K_{2,n})$ for n up to 21.

n	1	2	3	4	5	6	7
$R(K_{2,2}, K_{2,n})$	4	6	8	9	11	12	14
n	8	9	10	11	12	13	14
$R(K_{2,2}, K_{2,n})$	15	16	17	18	20	22	**22**
n	15	16	17	18	19	20	21
$R(K_{2,2}, K_{2,n})$	**24**	25	26	**27**	28-29	30	32

Table 1: $R(K_{2,2}, K_{2,n})$ for $n \leq 21$

The values for $n = 1$ and $n = 2$ were obtained by Chvátal and Harary in [1] and [2]. The values in bold are obtained in this paper. All other were presented by Harborth and Mengersen [4], who also proved the bounds:

(E1) $22 \leq R(K_{2,2}, K_{2,14}) \leq 23$,

(E2) $22 \leq R(K_{2,2}, K_{2,15}) \leq 24$,

(E3) $27 \leq R(K_{2,2}, K_{2,18}) \leq 28$.

The main result of this paper is an improvement of Theorem 1. Namely, we show that if n is even, q is odd, and $s \leq q/2$, then $R(K_{2,2}, K_{2,n}) \leq n + 2q - 1$. The latter gives the exact value of $R(K_{2,2}, K_{2,18}) = 27$. In Section 3 we describe an algorithm which we used to determine the values of $R(K_{2,2}, K_{2,14}) = 22$ and $R(K_{2,2}, K_{2,15}) = 24$.

In the sequel by $N_b(v)$ we shall denote the blue neighborhood of the vertex v and by $deg_b(v) = |N_b(v)|$ the blue degree of v. Similarly we define N_r and deg_r.

2 Main Theorem

Theorem 2 *For even $n \geq 2$ let $q = \lceil \sqrt{n} \rceil$ and $s = n - (q-1)^2$. If q is odd and $s \leq q/2$, then $R(K_{2,2}, K_{2,n}) \leq n + 2q - 1$.*

In order to prove the theorem we will need the following lemma.

Lemma 3 *Let $q = \lceil \sqrt{n} \rceil$ be odd and $s = n - (q-1)^2 \leq q/2$. Then any 2-coloring of the edges of K_{n+2q-1} contains red $K_{2,2}$ or blue $K_{2,n}$, when there exists a vertex $v \in V$ with*

(1) *$deg_r(v) > q$, or*

(2) *$deg_r(v) < q$*

Proof (1) combine in pairs vertices from $N_r(v)$ (see Fig. 1). Firstly, two vertices v_i, v_j form a pair if the edge $\{v_i, v_j\}$ is red. We may assume that each vertex from $N_r(v)$ can by connected by a red edge with at most one other vertex from $N_r(v)$. Note that, if two vertices $u, w \in N_r(v)$ have a common red neighbor $x \neq v$, the coloring contains red C_4. Other vertices from $N_r(v)$ we combine in pairs arbitrarily. There is $\lfloor deg_r(v)/2 \rfloor \geq (q+1)/2$ pairs. In $N_b(v)$ there is $deg_b(v) = n + 2q - 1 - (deg_r(v) + 1) \leq n + q - 3$ vertices and, again, we may assume that each of them can be connected by a red edge with at most one vertex from $N_r(v)$. That means that there is a pair, without loss of generality say (v_1, v_2) in $N_r(v)$, which is connected by red edges with at most $(n+q-3)/\lfloor deg_r(v)/2 \rfloor$ vertices from $N_b(v)$. Let $B \subset N_b(v)$ be the set of these vertices. Thus $|B| \leq \lfloor (n+q-3)/\lfloor deg_r(v)/2 \rfloor \rfloor \leq \lfloor 2(n+q-3)/(q+1) \rfloor$. The pair (v_1, v_2) with vertices from the set $(N_r(v) \setminus \{v_1, v_2\}) \cup (N_b(v) \setminus B)$ creates blue complete bipartite graph $K_{2,l}$ where $l = |V| - 3 - |B| \geq n + 2q - 4 - \lfloor 2(n+q-3)/(q+1) \rfloor \geq n - \lfloor 2s/(q+1) \rfloor$. The latter inequality follows from the fact that $s = (2-q)(q+1) + n + q - 3$. Since $2s < q + 1$, we have $l \geq n$ and the coloring contains blue $K_{2,n}$.

(2) Let v be a vertex with $0 < deg_r(v) < q$ and $u \in N_r(v)$ (see Fig. 2), then the vertices $\{u, v\}$ together with the vertices from $N_b(v) \cap N_b(u)$ create blue $K_{2,l}$ where $l = n + 2q - 1 - (deg_r(v) + deg_r(u)) \geq n$. If $deg_r(v) = 0$ then v creates blue $K_{2,n}$ with any other u and the set $N_b(v) \cap N_b(u)$. □

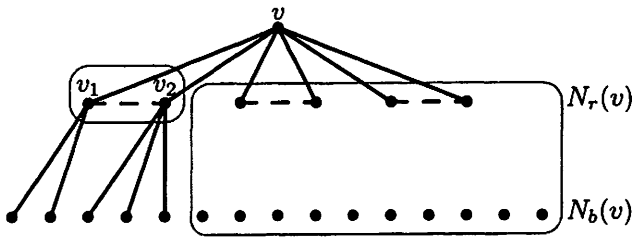

Figure 1: Example for $deg_r(v) = 6$ and $|(N_r(v_1) \cup N_r(v_2)) \cap N_b(v)| = 5$

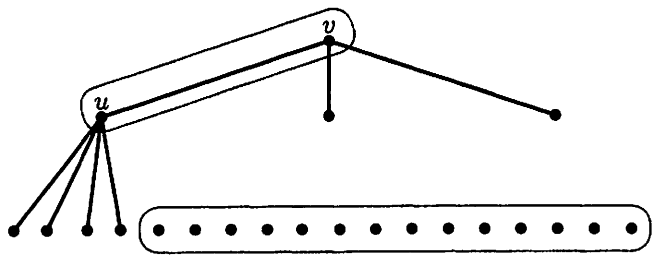

Figure 2: Example for $deg_r(v) = 3$ and $deg_r(u) = 5$

Proof of theorem 2 From Lemma 3, we know that every coloring of K_{n+2q-1} contains a red C_4 or a blue $K_{2,n}$, if there exists a vertex v with $deg_r(v) \neq q$. On the other hand K_{n+2q-1} has odd number of vertices and by the hand shaking rule, there is no coloring with odd number of vertices having odd red degree. This means that every coloring of K_{n+2q-1} contains a red C_4 or blue $K_{2,n}$, so $R(C_4 = K_{2,2}, K_{2,n}) \leq n + 2q - 1$. □

Corollary 3.1 $R(K_{2,2}, K_{2,18}) = 27$.

Proof For $n = 18$, we have $q = 5$ and $s = 2 \leq q/2$, so by Theorem 2, $R(K_{2,2}, K_{2,18}) \leq 27$ and by the bound (E3), we get the final result. □

3 Coloring Algorithm

In this section we describe the algorithm used for the searching of critical colorings of the complete graph K_p, i.e. colorings which contain neither a red $K_{2,2}$ nor a blue $K_{2,n}$. Every coloring of K_p is represented by the adjacency matrix $A = (a_{i,j})_{p \times p}$, where $a_{i,j} \in \{1, 2\}$ is the color of the edge $\{i, j\}$. The coloring is fully described by the values above diagonal and represented as the $\binom{p}{2}$ bit number $rep(G) = b_{1,2}b_{1,3}...b_{1,p}b_{2,3}...b_{2,p}...b_{p-1,p}$, where $b_{i,j} = 1$ if the edge $\{i,j\}$ is red and $b_{i,j} = 0$ otherwise.

Roughly speaking the algorithm checks one by one colorings of K_p which are possible good. In order to speed up the process of checking we skip some colorings which are bad. At each step, if the algorithm finds that a coloring G_{act} is bad, because it contains a red $K_{2,2}$ or a blue $K_{2,n}$, then it may skip many colorings which also contain a red $K_{2,2}$ or a blue $K_{2,n}$. To do this it looks for the most significant bit $b_{i,j}$ satisfying one of the conditions:

- $b_{i,j} = 1$ and the edge $\{i,j\}$ together with some edges represented by more significant bits create a red $K_{2,2}$
- $b_{i,j} = 0$ and the edge $\{i,j\}$ together with some edges represented by more significant bits create a blue $K_{2,n}$
- $b_{i,j} = 1$ and the edge $\{i,j\}$ together with some edges represented by more significant bits create a vertex u with $deg_r(u) > \Delta_r$
- $b_{i,j} = 0$ and the edge $\{i,j\}$ together with some edges represented by more significant bits create a vertex u with $deg_r(u) < \delta_r$.

where Δ_r and δ_r are two parameters of the algorithm (maximum and minimum degree of a vertex in critical coloring). If we find such a bit $b_{i,j}$ we can omit many colorings. Regardless of which value will have less significant bits the coloring will contain red $K_{2,2}$ or blue $K_{2,n}$. All such colorings can be omitted and the coloring considered next should be represented by the smallest number with flipped $b_{i,j}$ bit and greater than $rep(G_{act})$.

4 $R(K_{2,2}, K_{2,14}) = 22$

We use algorithm described in previous section to bound the values of $R(K_{2,2}, K_{2,14})$. The algorithm checks all possible coloring of K_{22}. In order to speed up the process of searching we shall limit down to 6 cases the ways the edges going out of the first 6 vertices (with numbers 1, 2, 3, 4, 5, 6) are colored.

Lemma 4 *The coloring of K_{22} contains a red $K_{2,2}$ or a blue $K_{2,14}$ when it satisfies one of the following conditions:*

(1) *there exists $v \in V$ with $deg_r(v) > 5$*

(2) *there exists $v \in V$ with $deg_r(v) < 4$*

(3) *there exist* $u, v \in V$ *with* $deg_r(v) = deg_r(u) = 4$ *and the edge* $\{u, v\}$ *is red.*

Proof (1) combine in pairs vertices from $N_r(v)$ like in proof of Lemma 4 (see Fig. 1). $N_b(v)$ has $22 - deg_r(v) - 1 \leq 15$ elements and we may assume that each of them can by connected by a red edge with at most one vertex from $N_r(v)$. There are at least 3 pairs. Hence, there is a pair (v_1, v_2) which is connected by red edges with at most 5 elements from $N_b(v)$, so v_1, v_2 and the set $N_b(v_1) \cap N_b(v_2)$ create blue $K_{2,14}$.

(2) First, let us assume that $1 \leq deg_r(v) \leq 3$ and let $u \in N_r(v)$ (see Fig. 1). By (1), we can assume that $deg_r(u) \leq 5$. Then $|N_b(v) \cap N_b(u)| = 22 - deg_r(u) - deg_r(v) \geq 22 - 5 - 3 = 14$ so u, v and $N_b(v) \cap N_b(u)$ create blue $K_{2,14}$. If $deg_r(v) = 0$ then v creates a blue $K_{2,14}$ with any other u and the set $N_b(v) \cap N_b(u)$.

(3) In this case $|N_b(u) \cap N_b(v)| = 22 - deg_r(u) - deg_r(v) = 14$, so u, v with $N_b(u) \cap N_b(v)$ create a blue $K_{2,14}$. □

Observation 4.1 *From Lemma 4 it follows that if we are looking for a coloring of* K_{22} *which contains neither a red* $K_{2,2}$ *nor a blue* $K_{2,14}$ *then it is enough to check colorings satisfying the following properties.*

- *Every vertex* v *has* $deg_r(v) \in \{4, 5\}$
- *There is a vertex* v *with* $deg_r(v) = 5$

Observation 4.2 *Let* v_Δ *be a vertex with* $deg_r(v_\Delta) = 5$. *If a vertex* u *from* $N_r(v_\Delta)$ *has less then 3 red neighbors in* $N_b(v_\Delta)$ *then the coloring of* K_{22} *contains blue* $K_{2,14}$ *formed by* v_Δ, u *and* $N_b(v_\Delta) \setminus N_r(u)$.

By Observation 4.1, we can assume that there is a vertex v_Δ with 5 red neighbors. Let v_Δ have the number 1, and his red neighbors the numbers 2, 3, 4, 5, 6. We can consider three cases: (a) there are no red edges between $N_r(1)$ (see Fig. 3), (b) there is one red edge $(2, 3)$ (see Fig. 4), or there are two separate red edges $(2, 3)$ and $(4, 5)$ (see Fig. 5). Each of the vertices from $N_r(1)$ has 3 or 4 red neighbors in $N_b(1)$. Since $N_b(1)$ has 16 elements we have only two possibilities: either all 5 vertices have 3 neighbors or 4 of them have 3 neighbors and one has 4 neighbors. We may assume that the vertex with 4 neighbors has number 6. Observe that we can exclude the case when a vertex from $N_r(1)$ has four red neighbors in $N_b(1)$ and a red neighbor in $N_r(1)$, because then it could have 6 red neighbors. All these

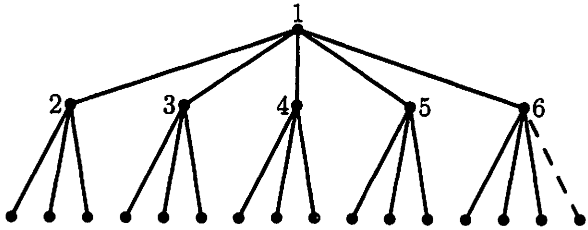

Figure 3: Possibility 1a) when the dashed edge is red and 1b) when dashed edge is blue

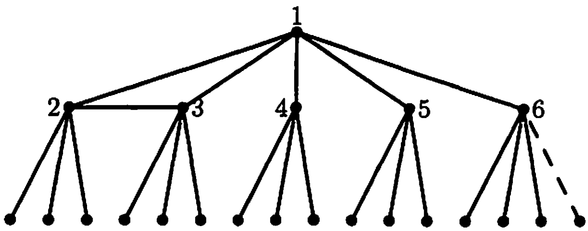

Figure 4: Possibility 2a) when the dashed edge is red and 2b) when dashed edge is blue

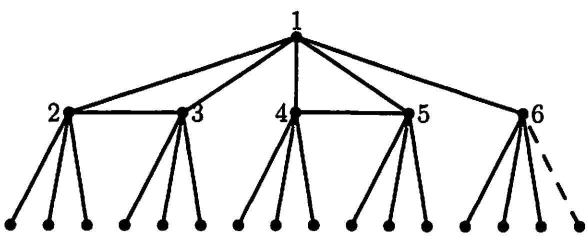

Figure 5: Possibility 3a) when the dashed edge is red and 3b) when dashed edge is blue

means that we can consider only 6 cases of the coloring of the edges going out from the vertices 1, 2, 3, 4, 5, 6. Hence we have only six versions of the first six rows of the matrix. Running algorithm described in section 3 on a computer we checked that all colorings of K_{22} contain either red $K_{2,2}$ or blue $K_{2,14}$. Thus $R(K_{2,2}, K_{2,14}) \leq 22$ which with the bound (E1) gives the theorem.

Theorem 5 $R(K_{2,2}, K_{2,14}) = 22$.

5 $R(K_{2,2}, K_{2,15}) = 24$

Theorem 6 $R(K_{2,2}, K_{2,15}) = 24$.

Proof We used algorithm described in section 3 and find the following coloring of K_{23} that contains neither a red $K_{2,2}$ nor a blue $K_{2,15}$.

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X2222222222222222211111
2X222222222222111122221
22X22222222111222122212
222X2222211221221222122
2222X222121212212221222
22222X22221122122212222
222222X1222221212212222
2222221X112222122222212
22221221X22212122222122
222122212X2122222221222
2221112222X222212222212
22122122212X12222222221
221212221221X2221222222
2211221222222X222122122
21222121122222X22122222
212212122212222X2222221
2121222222221222X212222
21122222222221122X21222
122221122222222212X1222
1222122221222222211X222
12212222122221222222X21
121222212212222222222X2
1122222222212221222212X
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Hence, $R(K_{2,2}, K_{2,15}) > 23$ and by the bound (E2), we get the final result. □

6 Conclusion

In this paper we have studied Ramsey numbers $R(K_{2,2}, K_{2,n})$. The next open problem to attack is the number $R(K_{2,2}, K_{2,19})$. We know that $28 \leq R(K_{2,2}, K_{2,19}) \leq 29$ and one can try to use the computer and look for critical coloring in K_{28}. The number of possible colorings of K_{28} is much greater than the number of possible colorings of K_{23} and we do not know how to narrow the searching space. We only know that we can exclude colorings with vertices of red degree lower than 3 or greater than 6.

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