DOMINATION IN THE TOTAL GRAPH OF A COMMUTATIVE RING

T. TAMIZH CHELVAM AND T. ASIR
Department of Mathematics
Manonmaniam Sundaranar University
Tirunelveli 627 012, India.
e-mail : *tamche59@gmail.com, asirjacob75@gmail.com*

Abstract

Let R be a commutative ring and $Z(R)$ be its set of all zero-divisors. The *total graph* of R, denoted by $T_\Gamma(R)$, is the undirected graph with vertex set R, and two distinct vertices x and y are adjacent if and only if $x + y \in Z(R)$. In this paper, we obtain a lower bound as well as an upper bound for domination number of $T_\Gamma(R)$. Further we proved that the upper bound for the domination number of $T_\Gamma(R)$ is attained in the case an Artin ring R. Having proved this, we have identified certain classes of rings corresponding to which the domination number of the total graph equals the upper bound. In view of these assertions, we conjecture that the domination number equals to this upper bound. Certain other domination parameters are also obtained for $T_\Gamma(R)$ under the assumption that the conjecture is true.

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1 Introduction

Let R be a commutative ring with 1, $Z(R)$ be its set of zero-divisors and $Reg(R) = R - Z(R)$, set of all regular elements in R. Anderson and Livingston [3] introduced, the *zero-divisor graph* of R, denoted by $\Gamma(R)$, as the (undirected) graph with vertices $Z^*(R) = Z(R) - \{0\}$, the set of nonzero zero-divisors of R, and for distinct $x, y \in Z(R)$, the vertices x and y are adjacent if and only if $xy = 0$. Several authors [13, 14] have extensively studied about this zero-divisor graph. Recently Anderson and Badawi [2] introduced the concept of the total graph corresponding to a commutative ring and the subgraph $Reg_{\Gamma}(R)$ is the subgraph of $T_{\Gamma}(R)$ induced by $Reg(R)$. In recent years, the study on graphs out of algebraic structures has grown in various directions. At the heart is the interplay between the ring theoretic properties of R and the graph theoretic properties of $T_{\Gamma}(R)$, for which one can refer to reader [2, 1, 7, 18, 17, 16, 15]. For basic definitions and properties about commutative rings, we refer to Kaplansky [12].

Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$, $< S >$ denotes the subgraph of G induced by S and for a vertex $v \in V$, $deg(v)$ is the degree of a vertex v, $N(v) = \{u \in V : u \text{ is adjacent to } v\}$ and $N[v] = N(v) \cup \{v\}$. A subset S of V is called a *dominating set* if every vertex in $V - S$ is adjacent to at least one vertex in S. A dominating set S is called a *strong* (or *week) dominating set* if for every vertex $u \in V - S$ there is a vertex $v \in S$ with $deg(v) \geq deg(u)$ (or $deg(v) \leq deg(u)$) and u is adjacent to v. The *domination number* γ of G is defined to be minimum cardinality of a dominating set in G and such a dominating set is called γ-set of G. In a similar way, we define the *strong dominating number* γ_s and the *weak dominating number* γ_w. A graph G is called *excellent* if, for every vertex $v \in V$, there exists a γ-set S containing v. A *domatic partition* of G is a partition of V, into dominating sets in G. The maximum number of classes of a domatic partition of G is called the *domatic number* of G and is denoted by $d(G)$. A graph G is called *domatically full* if $d(G) = \delta(G) + 1$, which is the maximum possible order of a domatic partition of $V(G)$ and $\delta(G)$ is minimum degree of a vertex of G. The *disjoint domination number* $\gamma\gamma(G)$ defined by $\gamma\gamma(G) = min\{|S_1| + |S_2| : S_1, S_2$ are disjoint dominating sets of $G\}$. Similarly, we can define $ii(G)$ and $\gamma i(G)$. For double domination parameters, we refer to reader [8]. The *bondage number* $b(G)$ is the minimum number of edges whose removal increases the domination number. A set of vertices $S \subseteq V$ is said to be *independent* if no two vertices in S are adjacent in G. The *independent number* $\beta_0(G)$, is the maximum cardinality of an independent set in G. A graph G is called *well-covered* if $\beta_0(G) = i(G)$. The *cartesian product* of graphs G_1 and G_2 is the graph $G_1 \square G_2$ whose vertex set is $V(G_1) \times V(G_2)$ and whose edge set is the set of all pairs $(u_1, v_1)(u_2, v_2)$ such that either $u_1u_2 \in E(G_1)$ and $v_1 = v_2$, or $v_1v_2 \in E(G_2)$ and $u_1 = u_2$. For basic definition and results in domination, see Ref. [9] and for any undefined graph-theoretic terminology, see Ref. [6].

Many researcher studied the interplay between the ring theoretic properties of R and the graph theoretic properties of the zero divisor graph $\Gamma(R)$, see Ref. [1, 2, 3, 7, 13, 14]. The concepts of dominating sets and domination numbers are very important concepts in graph theory. Dominating sets are the focus of many books in graph theory, for example Ref. [9, 10]. But not much research has been done about the domination parameters of graphs associated to algebraic structures (groups, rings, modules) in terms of algebraic properties. Tamizh Chelvam and Asir [16, 17] studied about the domination number, some other domination parameters and dominating sets of $T_\Gamma(\mathbb{Z}_n)$. In fact, it is proved that $\gamma(T_\Gamma(\mathbb{Z}_n)) = p$ where p is the smallest prime divisor of n [17, Theorem 2] and also obtained a characterization for all γ-sets in $T_\Gamma(\mathbb{Z}_n)$ [17, Theorem 3]. The purpose of this article is to study about the domination number of the total graph of a commutative ring through ring theoretic properties. More specifically, we continue the investigation begun in [16] regarding domination in the total graph on $\mathbb{Z}_n$ and we prove that the domination number of $T_\Gamma(R)$ depends up on the maximum cardinality of the maximal annihilator ideal of R.

This paper is organized as follows. In section 2, we obtain a lower bound and an upper bound for the domination number of the total graph of a commutative ring. In the main theorem of section 2, we prove that the domination number of the total graph of an Artin ring equals the upper bound. Using this, we conjecture that $\gamma(T_\Gamma(R)) = \mu$ where μ(upper bound) is the number of distinct cosets of an ideal I of R, where I is an ideal with maximum cardinality among all maximal annihilator ideals of R. Under the assumption that the conjecture is true, in section 3, we determine various domination parameters of $T_\Gamma(R)$. In section 4, with the assumption that $Z(R)$ is an ideal of R, we obtain several domination parameters of $T_\Gamma(R)$.

Throughout this paper, R denotes a commutative ring (not necessarily finite) with $1 \neq 0$ and we denote $1+1$ as 2. For any $a \in R$, $Ann(a) = \{x \in R : ax = 0\}$ is the annihilator ideal of a in R. When R is finite, we take I as a maximum annihilator ideal among all maximal annihilator ideals of R. i.e., I is a maximal annihilator ideal such that $|I| = max\{|A| : A \text{ is a maximal annihilator ideal of } R\}$. When R is infinite, we take I as a maximal annihilator ideal of R such that $|R/I|$ is minimum under the assumption that such an ideal exists. Thus in both the cases $|R/I|$ is finite. Let us take $|I| = \lambda$, $|Z(R)| = \alpha$, $|R/Z(R)| = \beta$ and $|R/I| = \mu$. We denote $R_1 \times R_2$ for the direct product of two rings R_1 and R_2. Also for a subset $S \subseteq V(G)$, $< S >$ denotes the subgraph of G induced by S and $G_1 \Box G_2$ denotes the Cartesian product of two graphs G_1 and G_2.

The following observation proved by H.R. Maimani et al. [7] is used frequently and hence given below.

Observation 1.1. *([7, Lemma 1.1]) Let R be a finite commutative ring, $Z(R)$ be its set of all zero-divisors in R. Then the following are true:*

(i) If $2 \in Z(R)$, then $deg(v) = |Z(R)| - 1$ for every $v \in V(T_\Gamma(R))$.

(ii) If $2 \notin Z(R)$, then $deg(v) = |Z(R)| - 1$ for every $v \in Z(R)$ and $deg(v) = |Z(R)|$ for every vertex $v \notin Z(R)$.

(iii) $T_\Gamma(R)$ has no vertex of degree $|R| - 1$.

2 Domination number of $T_\Gamma(R)$

In this section, we obtain a lower bound and an upper bound for the domination number of the total graph of a commutative ring. Having observed bounds for the domination number, we propose a conjecture for equality of the upper bound. As mentioned in the introduction, let I be a maximum annihilator ideal of R. That means, I is a maximal annihilator ideal of R such that $|R/I| = min\{|R/A| : A$ is a maximal annihilator ideal of $R\}$. We begin this section with a theorem, which exhibits a relation between the product of rings and the product of corresponding total graphs. More specifically, the relation is concerning the domination number of the total graph of the direct product of two rings and the domination number of Cartesian product of the total graphs of rings.

Theorem 2.1. *Let R_1 and R_2 be two commutative rings with identity. Then $\gamma(T_\Gamma(R_1 \times R_2)) \leq \gamma(T_\Gamma(R_1)\Box T_\Gamma(R_2))$.*

Proof. Let $G_1 = T_\Gamma(R_1)\Box T_\Gamma(R_2)$ and $G_2 = T_\Gamma(R_1 \times R_2)$. If $(x_1, y_1)(x_2, y_2) \in E(G_1)$, then either $x_1 + x_2 \in Z(R_1)$ or $y_1 + y_2 \in Z(R_2)$. Since $Reg(R_1 \times R_2) = Reg(R_1) \times Reg(R_2)$, $(x_1, y_1) + (x_2, y_2) \in Z(R_1 \times R_2)$ and so $(x_1, y_1)(x_2, y_2) \in E(G_2)$. From this we get that, G_1 is an edge induced spanning subgraph of G_2 and so $\gamma(T_\Gamma(R_1 \times R_2)) \leq \gamma(T_\Gamma(R_1)\Box T_\Gamma(R_2))$. □

For any integral domain R, the maximum degree $\Delta(T_\Gamma(R)) \leq 1$. If R is a finite integral domain, then $\gamma(T_\Gamma(R)) = \frac{|R|-k}{2} + k$ where $k = |\{a \in R : a = -a\}|$. If R is infinite, then there exists no positive integer k such that $\gamma(T_\Gamma(R)) = k$. So hereafter, we assume throughout this section that all rings are commutative which is not an integral domain. In the following theorem, we obtain lower and upper bounds for the domination number of the total graph of a commutative ring.

Lemma 2.2. *Let R be a commutative ring (not necessarily finite) with identity, I be a maximum annihilator ideal of R and $|R/I| = \mu$(finite). Then $2 \leq \gamma(T_\Gamma(R)) \leq \mu$.*

Proof. First assume that R is finite and let $|I| = \lambda$. As noted in Observation 1.1(iii), no vertex in $T_\Gamma(R)$ has degree $|R| - 1$ and so $\gamma(T_\Gamma(R)) \geq 2$. Let H be the spanning subgraph of $T_\Gamma(R)$ in which two distinct vertices $x, y \in R$ are adjacent if $x + y \in I$. By noting that each maximal annihilator ideal I is prime, one can obtain, in the same way as in Theorem 2.2 [2], H is a spanning subgraph of G and further

$$H = \begin{cases} K_\lambda \cup \underbrace{K_\lambda \cup K_\lambda \cup \ldots \cup K_\lambda}_{(\mu-1)\ copies} & \text{if } 2 \in I \\ K_\lambda \cup \underbrace{K_{\lambda,\lambda} \cup K_{\lambda,\lambda} \cup \ldots \cup K_{\lambda,\lambda}}_{(\frac{\mu-1}{2})\ copies} & \text{if } 2 \notin I. \end{cases} \quad (1)$$

Note that each of the connected components of H as described above corresponds to a coset of I in R. Take one element in each of the cosets of I in R and they form a minimum dominating set of H and so $\gamma(H) = \mu$. Since H is a spanning subgraph of $T_\Gamma(R)$, $\gamma(T_\Gamma(R)) \leq \mu$.

If R is infinite. By the assumption that R/I is finite, we have I is infinite. If $2 \in Z(R)$, then for all $x \in R$, $< x + I > \subseteq T_\Gamma(R)$ is an infinite complete graph. On the other hand, if $2 \notin Z(R)$, then for all $x \in R$, $< x + I \cup -x + I > \subseteq T_\Gamma(R)$ is an infinite complete bi-partite graph. Thus, in either of these cases $\gamma(T_\Gamma(R)) \leq \mu$. □

Hereafter, by H, we mean the edge induced spanning subgraph induced by a maximum annihilator ideal I of R. The following example shows that the lower and upper bounds are sharp.

Example 2.3. *(i)* If $R = \mathbb{Z}_3 \times \mathbb{Z}_4$, then $I = \{(0,0), (0,2), (1,0), (1,2), (2,0), (2,2)\}$, $\lambda = 6$, $\mu = 2$ and $\gamma(G) = 2$.
(ii) If $R = \mathbb{Z}_{35}$, then $I = \{0, 5, \ldots, 30\}$, $\lambda = 7$, $\mu = 5$ and $\gamma(G) = \mu$.

Now we find the domination number for the total graph of certain classes of commutative rings. First of all, if $Z(R)$ is an ideal of R, then the maximal annihilator ideal in R is $Z(R)$ and so by Lemma 2.2, one can have the following lemma.

Lemma 2.4. *If R is a commutative rings with identity, $Z(R)$ is an ideal of R and $|R/Z(R)| = \mu$, then $\gamma(T_\Gamma(R)) = \mu$.*

Now we prove the main theorem of this section. If R is an Artin ring with $|R/J|$ is finite for at least one maximal ideal J of R, then the domination number of $T_\Gamma(R)$ is $min\{|R/I| : I$ is an maximal annihilator ideal of $R\}$.

Theorem 2.5. *Let R be an Artin ring, I be a maximum annihilator ideal of R and $|R/I| = \mu$. Then $\gamma(T_\Gamma(R)) = \mu$.*

Proof. Clearly by Lemma 2.2, $\gamma(T_\Gamma(R)) \leq \mu$. Further, since R is an Artin ring, by [4, Theorem 8.7], one can write $R \cong R_1 \times \ldots \times R_m$ where each R_i is an Artin local ring with maximal ideal m_i. Note that $Z(R_i) = m_i$ is the unique maximal annihilator ideal of R_i, and so by Lemma 2.4, $\gamma(T_\Gamma(R_i)) = |R_i/m_i|$ for all i. Let $I_i = R_1 \times \ldots \times R_{i-1} \times m_i \times R_{i+1} \times \ldots \times R_n$ and $\mu_i = |R/I_i|$. It is easy to see that I_i's $(1 \leq i \leq n)$ are the only maximal annihilator ideals of R and so $I = \max_{1\leq i\leq m} I_i$. From this $\mu = \min_{1\leq i\leq n} \gamma(T_\Gamma(R_i))$. Suppose $S = \{(x_{11}, x_{12}, \ldots, x_{1m}), (x_{21}, x_{22}, \ldots, x_{2m}), \ldots, (x_{(\mu-1)1}, x_{(\mu-1)2}, \ldots, x_{(\mu-1)m})\}$ is a dominating set of $T_\Gamma(R)$. Let $S_k = \{x_{1k}, x_{2k}, \ldots, x_{(\mu-1)k}\}$ for $1 \leq k \leq m$. Since $\gamma(T_\Gamma(R_j)) = \mu_j > \mu - 1$, S_j is not a dominating set of $T_\Gamma(R_j)$ and so there exists an element $y_j \in R_j$ such that y_j is not dominated by S_j for each $j = 1, 2, \ldots, m$. Therefore $(y_1, y_2, \ldots, y_m) \in R$ which is not dominated by S, contraction to S is a dominating set of $T_\Gamma(R)$. Hence $\gamma(T_\Gamma(R)) = \mu$. □

Let R be a finite commutative ring. Since every finite ring is an Artin ring, we have the following corollary.

Corollary 2.6. *Let R is a finite commutative ring, I be a maximum annihilator ideal of R and $|R/I| = \mu$. Then $\gamma(T_\Gamma(R)) = \mu$.*

Now using Theorem 2.5, one can find the domination number for the total graph of certain classes of commutative rings. In this fact this exhibits that, there are families of infinite graphs whose domination number is finite.

Corollary 2.7. *(i) [17, Theorem 2] If n is a composite integer, then $\gamma(T_\Gamma(\mathbb{Z}_n)) = p$ where p is the smallest prime divisor of n.*

(ii) For any $n, k \in \mathbb{Z}^+$, $\gamma(T_\Gamma(\frac{\mathbb{Z}_n[x]}{<x^k>})) = \gamma(T_\Gamma(\frac{\mathbb{Z}_n[x,y]}{<x^k, xy, y^k>}) = p$ where p is the smallest prime divisor of n.

(iii) If R_i's are finite integral domains, then $\gamma(T_\Gamma(R_1 \times R_2 \times \ldots \times R_k)) = min\{|R_1|, |R_2|, \ldots, |R_k|\}$.

(iv) If n is a composite positive integer, then $\gamma(T_\Gamma(\mathbb{Z}_n \times \mathbb{Z} \times \ldots \times \mathbb{Z})) = p$ where p is the smallest prime divisor of n.

(v) Let n be a composite positive integer, $k \in \mathbb{Z}^+$ and $\mathbb{F}$ be a field. If $R = \mathbb{Z}_n \times \mathbb{F} \times \ldots \times \mathbb{F}$ or $R = \frac{\mathbb{Z}_n[x]}{<x^k>} \times \mathbb{Z} \times \ldots \times \mathbb{Z}$, then $\gamma(T_\Gamma(R)) = p$ where p is the smallest prime divisor of n.

Proof. (i) Note that the maximum annihilator ideal of $Z(\mathbb{Z}_n)$ is $I = Ann(\frac{n}{p}) = \{0, p, 2p, \ldots, n-p\}$ and so $|R/I| = \mu = p$. Thus by Theorem 2.5, $\gamma(T_\Gamma(\mathbb{Z}_n)) = p$.

(ii) Let $\frac{\mathbb{Z}_n[x]}{<x^k>} = \{a_0 + a_1x + \ldots + a_{k-1}x^{k-1} + <x^k>: a_i \in \mathbb{Z}_n$ for $i = 0, 1, \ldots, k-1\}$ and let us take the maximum annihilator ideal of $\mathbb{Z}_n$ as $J = Ann(\frac{n}{p})$. Now the maximum annihilator ideal of $\frac{\mathbb{Z}_n[x]}{<x^k>}$ is $I = \{a_0 + a_1x + \ldots + a_{k-1}x^{k-1} + <x^k>: a_0 \in J$ and $a_i \in \mathbb{Z}_n$ for $i = 1, \ldots, k-1\}$ and so $|I| = \frac{n^k}{p}$ and $|R/I| = p$. Therefore, by Theorem 2.5, $\gamma(T_\Gamma(\frac{\mathbb{Z}_n[x]}{<x^k>})) = p$. Similar way one can get, $\gamma(T_\Gamma(\frac{\mathbb{Z}_n[x,y]}{<x^k, xy, y^k>}) = p$.

(iii) Assume that $|R_i| = min\{|R_1|, \ldots, |R_k|\}$. The maximum annihilator ideal of $R_1 \times \ldots \times R_k$ is $I = \{(a_1, \ldots, a_{i-1}, 0, a_{i+1} \ldots, a_k) : a_j \in R_j\}$ and so $|R/I| = |R_i|$. Thus $\gamma(T_\Gamma(R_1 \times R_2 \times \ldots \times R_k)) = min\{|R_1|, \ldots, |R_k|\}$.

(*iv*) Let $R = \mathbb{Z}_n \times \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{k-1 \text{ times}}$. Note that $Z(R) = \{(a_1, \ldots, a_k) \in R : a_1 \in Z(\mathbb{Z}_n) \text{ or } a_i = 0 \text{ for some } i = 2, \ldots, k\}$. Let $J = Ann(\frac{n}{p})$ in $\mathbb{Z}_n$. Clearly, the maximum annihilator ideal in R is $I = J \times \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{k-1 \text{ times}}$ and so $|R/I| = |\mathbb{Z}_n/J| = p$.

(*v*) In line with the proof of part (*iv*), one can find $\gamma(T_\Gamma(R)) = p$. □

Having obtained the domination number of the total graph of some classes of rings, now we propose a conjecture for the domination number of the total graph of a commutative ring.

Conjecture 2.8. *Let R be a commutative ring with identity which is not an Artin ring, $Z(R)$ be not an ideal of R and I_i's are maximal annihilator ideals of R. If $|R/I_i| = finite$ for some i, then $\gamma(T_\Gamma(R)) = min\{|R/I_i| : I_i$ is a maximal annihilator ideal of $R\}$, where the minimum is taken over all I_i for which $|R/I_i|$ is finite.*

3 Some domination parameters of $T_\Gamma(R)$

In this section, we find certain domination parameters of $T_\Gamma(R)$ under the assumption that $\gamma(T_\Gamma(R)) = \mu$. As mentioned earlier, I is a maximum annihilator ideal in R, $|I| = \lambda$ and $|R/I| = \mu$. By Lemma 2.2, we have the following.

Lemma 3.1. *Let R be a commutative ring. If $\gamma(T_\Gamma(R)) = \mu$, then the set $S = \{x_1, x_2, \ldots, x_\mu\} \subset V(T_\Gamma(R))$ is a γ-set of G where $x_j \notin x_i + I$ for all $i, j = 1, \ldots, \beta$ and $i \neq j$.*

Corollary 3.2. *Let R be a commutative ring. If $\gamma(T_\Gamma(R)) = \mu$, then*
(i) $\gamma'(T_\Gamma(R)) = \mu$, where $\gamma'(G)$ is the inverse domination number of G.
(ii) $T_\Gamma(R)$ is excellent.
(iii) the domatic number $d(T_\Gamma(R)) = \lambda$.

Proof. Since $\lambda \geq 2$, (*i*) is true. The proof for (*ii*) and (*iii*) are trivial. □

Theorem 3.3. *For a commutative ring R, if $Z(R)$ is not an ideal of R, $R =< Z(R) >$(i.e., R is generated by $Z(R)$) and $\gamma(T_\Gamma(R)) = \mu$, then $\gamma_t(T_\Gamma(R)) = \gamma_c(T_\Gamma(R)) = \mu$.*

Proof. If $Z(R)$ is not an ideal of R and $R =< Z(R) >$, then by Theorem 3.3 [2], $T_\Gamma(R)$ is connected. Let I be a maximum annihilator ideal in R and $x_1 \in I$. Since $T_\Gamma(R)$ is connected, there exists a vertex $x_2 \in a_1 + I$ for some $a_1 \in R - I$ such that x_2 is adjacent to x_1. Again by connectedness of $T_\Gamma(R)$, there exists a coset $a_2 + I$ for some $a_2 \notin I$ as well as $a_2 \notin a_1 + I$

such that at least one element of $a_2 + I$ is adjacent to either a vertex in I or in $a_1 + I$, say I.

If there exists an element $a \in a_i + I$ which is adjacent to some $b \in a_j + I$ with $a \notin a_j + I$, then each vertex in $a_i + I$ is adjacent to at least one vertex in $a_j + I$. For, if $a + b = c$ for some $c \in Z(R)$, then $c \in a_i + a_j + I$. Let $d_1 \in a_i + I$. Take $d_2 \in R$ such that $d_1 + d_2 = c$. From this $d_2 \in a_j + I$ and d_1 is adjacent to d_2. Therefore, each vertex in $a_i + I$ is adjacent to at least one vertex in $a_j + I$.

Thus x_1 is adjacent to some vertex $x_3 \in a_2 + I$. Similarly, we can select coset representatives x_i, for $4 \le i \le \mu$, in distinct cosets of I in R other than I, $a_1 + I$ and $a_2 + I$ such that $< x_1, x_2, \dots, x_\mu > \subseteq T_\Gamma(R)$ is connected. Then, by Lemma 3.1, $\{x_1, x_2 \dots, x_\mu\}$ is a γ_c-set of $T_\Gamma(R)$ and so $\gamma_c(T_\Gamma(R)) = \mu$. Since, for any graph G, we have $\gamma(G) \le \gamma_t(G) \le \gamma_c(G)$, $\gamma_t(T_\Gamma(R)) = \mu$. □

Remark 3.4. Let R be a commutative ring and $G = T_\Gamma(R)$. If R is not an integral domain, then G satisfies $\gamma(G - v) = \gamma(G)$ for all $v \in V(G)$. Having observed this, we obtain the bondage number of the total graph.

Theorem 3.5. *For a finite commutative ring R, if $\gamma(T_\Gamma(R)) = \mu$, then bondage number $b(T_\Gamma(R)) = |Z(R)| - 1$.*

Proof. Let x be a vertex in $T_\Gamma(R)$ of minimum degree. Then $deg(x) = |Z(R)| - 1$. Take some arbitrary $|Z(R)| - 2$ edges incident at x and let y be the remaining vertex adjacent to x. By Lemma 3.1, there exists a γ-set S in $T_\Gamma(R)$ with cardinality μ and containing y. If we remove all the $|Z(R)| - 1$ edges incident at x, then x is an isolated vertex and so we have to add two vertices in S from the coset of I in R containing x. Thus $b(T_\Gamma(R)) \le |Z(R)| - 1$. Also by the structure of H and $T_\Gamma(R)$, removal of any other set with less than $|Z(R)| - 1$ edges cannot increase the value of domination number and so $b(T_\Gamma(R)) = |Z(R)| - 1$. □

4 Domination parameters of $T_\Gamma(R)$ and $\overline{T_\Gamma(R)}$ when $Z(R)$ is an ideal of R

In this section, we assume that $Z(R)$ is an ideal of R and so $I = Z(R)$, $\lambda = \alpha$, $\mu = \beta$ and $\gamma[T_\Gamma(R)] = \beta$. We recall the following structure theorem for further discussion.

Lemma 4.1. *[2, Theorem 2.2] Let R be a finite commutative ring such that $Z(R)$ is an ideal of R, $|Z(R)| = \alpha$ and $|R/Z(R)| = \beta$. Then*

$$T_\Gamma(R) = \begin{cases} K_\alpha \cup \underbrace{K_\alpha \cup K_\alpha \cup \ldots \cup K_\alpha}_{(\beta-1)\ copies} & if\ 2 \in Z(R) \\ K_\alpha \cup \underbrace{K_{\alpha,\alpha} \cup K_{\alpha,\alpha} \cup \ldots \cup K_{\alpha,\alpha}}_{(\frac{\beta-1}{2})\ copies} & if\ 2 \notin Z(R). \end{cases} \quad (2)$$

Lemma 4.2. *Let R be a finite commutative ring such that $Z(R)$ is an ideal of R. Then $\gamma(\overline{T_\Gamma(R)}) = 2$.*

Proof. For $x \in Z(R)$ and $y \in R - Z(R)$, by Lemma 4.1, $S = \{x, y\}$ is a dominating set of $\overline{T_\Gamma(R)}$. Since $\overline{T_\Gamma(R)}$ has no vertex of degree $|R| - 1$, $\gamma(\overline{T_\Gamma(R)}) = 2$. □

Remark 4.3. Let R be a finite commutative ring and $G =< Reg(R) >\subseteq \overline{T_\Gamma(R)}$. If $2 \in Z(R)$ and $\beta = |R/Z(R)| = 2$, then $G = \overline{K_\alpha}$ and so $\gamma(G) = \alpha$. By Lemma 4.1, in all remaining cases of R, we have $\gamma(G) = 2$. Therefore

$$\gamma(< Reg(R) >) = \begin{cases} \alpha & \text{if } 2 \in Z(R) \text{ and } \beta = 2 \\ 2 & \text{otherwise.} \end{cases}$$

The following corollary is immediate from the definition of the inverse domination number.

Corollary 4.4. *(i) Let R be a commutative ring except the one with $2 \in Z(R)$, $\alpha > 2$, $\beta = 2$ and $G =< Reg(R) >$ in $\overline{T_\Gamma(R)}$. Then $\gamma'(G) = 2$.*
(ii) For any commutative ring R, $\gamma'(\overline{T_\Gamma(R)}) = 2$.

Theorem 4.5. *Let R be a commutative ring such that $Z(R)$ is an ideal of R and $G = T_\Gamma(R)$. A set $S = \{x_1, x_2, \ldots, x_\beta\} \subset V(G)$ is a γ-set of G if and only if $x_j \notin x_i + Z(R)$ for all $1 \leq i, j \leq \beta$ and $i \neq j$.*

Proof. If part follows from Lemma 3.1. Conversely, let S be a γ-set of G. Suppose, there exist $j, k \in \{1, \ldots, \beta\}$ such that $x_j \in x_k + Z(R)$. Since $|S| = \beta$, there is a coset $x + Z(R)$ such that $x_i \notin x + Z(R)$ for all $x_i \in S$. Now vertices in $-x + Z(R)$ can not be dominated by S, a contradiction. □

As proved above, one can prove the following:

Corollary 4.6. *Let R be a commutative ring such that $Z(R)$ is an ideal of R and $G = \overline{T_\Gamma(R)}$. A set $S = \{x_1, x_2\} \subset V(G)$ is a γ-set of G if and only if $x_2 \notin x_1 + Z(R)$.*

Corollary 4.7. *Let R be a finite commutative ring with $Z(R)$ is an ideal of R. Then*

(i) $T_\Gamma(R)$ and $\overline{T_\Gamma(R)}$ are excellent.

(ii) $d(T_\Gamma(R)) = \alpha$ and $d(\overline{T_\Gamma(R)}) = \left\lfloor \frac{|R|}{2} \right\rfloor$.

(iii) If $G_1 = \overline{Reg_\Gamma(R)}$, then

$$d(G_1) = \begin{cases} 1 & \text{if } 2 \in Z(R) \text{ and } \beta = 2 \\ \left\lfloor \frac{|Reg(R)|}{2} \right\rfloor & \text{otherwise.} \end{cases}$$

(iv) $T_\Gamma(R)$ is domatically full.

Theorem 4.8. *Let R be a finite commutative ring with $Z(R)$ is an ideal of R and $G = T_\Gamma(R)$. Then G and $\overline{G}$ are well-covered.*

Proof. If $2 \in Z(R)$, then by Lemma 4.1, $i(G) = \beta$. If $2 \notin Z(R)$, all vertices in one partition of each $K_{\alpha,\alpha}$ together with a vertex of $Z(R)$, form an i-set of G and so $i(G) = (\frac{\beta-1}{2})\alpha + 1$. Similarly $\beta_0(G)$ is same as $i(G)$. Thus

$$i(G) = \beta_0(G) = \begin{cases} \beta & \text{if } 2 \in Z(R) \\ (\frac{\beta-1}{2})\alpha + 1 & \text{otherwise.} \end{cases}$$

Similarly, if $2 \in Z(R)$, then by equation (1), each coset of $R/Z(R)$ is an i-set of $\overline{G}$ and so $i(\overline{G}) = \alpha$. If $2 \notin Z(R)$, the set $\{x, y\}$ where $y \in -x + Z(R)$ is an i-set in $\overline{G}$. Also $\beta_0(G) = i(G)$. Therefore

$$i(\overline{G}) = \beta_0(\overline{G}) = \begin{cases} \alpha & \text{if } 2 \in Z(R) \\ 2 & \text{otherwise.} \end{cases}$$

Hence G and $\overline{G}$ are well-covered. □

Corollary 4.9. *If R is a finite commutative ring such that $Z(R)$ is an ideal of R and $|Z(R)| = \alpha$, then $\omega(T_\Gamma(R)) = \alpha$.*

As proved above, one can prove the following.

Theorem 4.10. *Let R be a finite commutative ring such that $Z(R)$ is an ideal of R, $|Z(R)| = \alpha$, $|R/Z(R)| = \beta$ and $G = T_\Gamma(R)$. Then*

(i) $$\gamma_t(G) = \begin{cases} 2\beta & \text{if } 2 \in Z(R) \\ \beta + 1 & \text{otherwise.} \end{cases}$$

(ii) $\gamma_t(\overline{G}) = 2$.

(iii) $\gamma_c(\overline{G}) = 2$.

(iv) $\gamma_s(G) = \gamma_w(G) = \beta$ and $\gamma_s(\overline{G}) = \gamma_w(\overline{G}) = 2$.

(v) $\gamma_p(G) = \beta$.

(vi) $\gamma_p(\overline{G}) = 2$ if $\beta = 2$.

(vii) If $G_1 = < Reg(R) >$ in $\overline{T_\Gamma(R)}$, $\beta = 2$ and $2 \notin Z(R)$, then $\gamma_p(G_1) = 2$.

Theorem 4.11. *Let R be a finite commutative ring with $Z(R)$ is an ideal of R, $|Z(R)| = \alpha$, $|R/Z(R)| = \beta$ and $G = T_\Gamma(R)$. Then*

(i) $\gamma\gamma(G) = 2\beta$.

(ii) $\gamma i(G) = \begin{cases} 2\beta & if\ 2 \in Z(R) \\ \beta + (\frac{\beta-1}{2})\alpha + 1 & otherwise. \end{cases}$

(iii) $ii(G) = \begin{cases} 2\beta & if\ 2 \in Z(R) \\ 2(\frac{\beta-1}{2})\alpha + 2 & otherwise. \end{cases}$

(iv) $tt(G) = \begin{cases} 4\beta & if\ 2 \in Z(R)\ and\ \alpha \geq 4 \\ 2(\beta + 1) & if\ 2 \notin Z(R) \\ does\ not\ exists & otherwise. \end{cases}$

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