A note for integer sequences to be potentially K_{r+1}-graphic*

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Abstract: Let K_{r+1} be the complete graph on $r+1$ vertices and $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers. If π has a realization containing K_{r+1} as a subgraph, then π is said to be potentially K_{r+1}-graphic. A.R. Rao obtained the Erdős-Gallai type criterion for π to be potentially K_{r+1}-graphic. In this paper, we give a simplification of the Erdős-Gallai type criterion. Moreover, the Fulkerson-Hoffman-McAndrew type criterion and the Hässelbarth type criterion for π to be potentially K_{r+1}-graphic are also presented.

Keywords: graph, degree sequence, potentially K_{r+1}-graphic sequence.
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1. Introduction

The set of all sequences $\pi = (d_1, d_2, \ldots, d_n)$ of non-negative, non-increasing integers with $d_1 \leq n-1$ is denoted by NS_n. A sequence $\pi \in NS_n$ is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The following Theorem 1.1 gives three criterions for π to be graphic.

Theorem 1.1 Let $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $\sum_{i=1}^{n} d_i$ is even.

The Erdős-Gallai Criterion. [1] π is graphic if and only if

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} min\{k, d_i\}$$

for all k with $1 \leq k \leq n$.

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The Fulkerson-Hoffman-McAndrew Criterion. [2] π is graphic if and only if

$$\sum_{i=1}^{k} d_i \leq k(n-m-1) + \sum_{i=n-m+1}^{n} d_i$$

for all k and m with $1 \leq k \leq n$, $m \geq 0$ and $k+m \leq n$.

The Hässelbarth Criterion. [3] Denote $f = max\{i | d_i \geq i\}$. Define $(d_1^*, \ldots, d_n^*)$ as follows: For $1 \leq i \leq n$, d_i^* is the ith column sum of the $(0,1)$-matrix, which has for each k the d_k leading terms in row k equal to 1, and the remaining entries are 0. The criterion is

$$\sum_{i=1}^{k} d_i \leq \sum_{i=1}^{k} (d_i^* - 1)$$

for each k with $1 \leq k \leq f$.

A sequence $\pi \in NS_n$ is said to be *potentially K_{r+1}-graphic* if there is a realization of π containing K_{r+1} as a subgraph. The following Theorem 1.2 due to A.R. Rao [7] gives the Erdős-Gallai type criterion for π to be potentially K_{r+1}-graphic.

Theorem 1.2 [7] Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_{r+1} \geq r$ and $\sum_{i=1}^{n} d_i$ is even. Then π is potentially K_{r+1}-graphic if and only if

$$\begin{aligned} \sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \quad \leq \quad & (s+t)(s+t-1) \\ & + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \qquad (1) \\ & + \sum_{i=r+t+2}^{n} min\{s+t, d_i\} \end{aligned}$$

for all s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq n-r-1$.

In [7], A.R. Rao gave a lengthy induction proof of Theorem 1.2 via linear algebraic techniques that remains unpublished, but Kézdy and Lehel [4] have given another proof using network flows or Tutte's f-factor theorem. Recently, Yin [9] obtained a short constructive proof of Theorem 1.2. There are several survey articles on the subject of degree sequences of graphs (see, e.g., Lai and Hu [5], Li and Yin [6], S.B. Rao [8]). For $n \geq r+1$, $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$ and $0 \leq s \leq r+1$, we denote $f(s) = max\{i | d_{r+1+i} \geq s+i\}$. A simplification of Theorem 1.2 is the following Theorem 1.3.

Theorem 1.3 Theorem 1.2 remains valid if condition (1) is assumed only for those s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq f(s)$.

Another purpose of the paper is to give Fulkerson-Hoffman-McAndrew type criterion and Hässelbarth type criterion for π to be potentially K_{r+1}-graphic as follows.

Theorem 1.4 (The Fulkerson-Hoffman-McAndrew Type Criterion) Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_{r+1} \geq r$ and $\sum_{i=1}^{n} d_i$ is even. Then π is potentially K_{r+1}-graphic if and only if

$$\begin{aligned} \sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \quad \leq \quad & (s+t)(n-s_1-t_1-1) \\ & + \sum_{i=r+2-s_1}^{r+1} d_i - r + s + \sum_{i=n-t_1+1}^{n} d_i \end{aligned} \tag{2}$$

for all s, s_1, t and t_1 with $0 \leq s \leq r+1$, $0 \leq t \leq n-r-1$, $s_1 \geq 0$, $t_1 \geq 0$, $s+s_1 \leq r+1$ and $t+t_1 \leq n-r-1$.

We define $(d_1^*, \ldots, d_n^*)$ as follows: For $1 \leq i \leq n$, d_i^* is the ith column sum of the $(0,1)$-matrix, which has for each k the d_{r+1+k} leading terms in row k equal to 1, and the remaining entries are 0.

Theorem 1.5 (The Hässelbarth Type Criterion) Let $n \geq r+1$ and $\pi = (d_1, d_2, \ldots, d_n) \in NS_n$, where $d_{r+1} \geq r$ and $\sum_{i=1}^{n} d_i$ is even. Then π is potentially K_{r+1}-graphic if and only if

$$\begin{aligned} \sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \quad \leq \quad & (s+t-1)s + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \\ & + \sum_{i=1}^{t} (d_i^* - 1) + \sum_{i=t+1}^{t+s} d_i^* \end{aligned} \tag{3}$$

for all s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq f(s)$.

2. Proofs of Theorem 1.3–1.5

Proof of Theorem 1.3. Assume that (1) holds for s and t with $0 \leq s \leq r+1$ and $0 \leq t \leq f(s)$. We only need to check that (1) holds for s and t with $0 \leq s \leq r+1$ and $f(s) < t \leq n-r-1$. In this case, we have that

$$\begin{aligned}
\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &= \sum_{i=1}^{s} d_i + \sum_{i=1}^{f(s)} d_{r+1+i} + \sum_{i=f(s)+1}^{t} d_{r+1+i} \\
&\le (s+f(s))(s+f(s)-1) \\
&\quad + \sum_{i=s+1}^{r+1} min\{s+f(s), d_i - r + s\} \\
&\quad + \sum_{i=r+f(s)+2}^{n} min\{s+f(s), d_i\} + \sum_{i=f(s)+1}^{t} d_{r+1+i} \\
&\le (s+f(s))(s+f(s)-1) \\
&\quad + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \\
&\quad + \sum_{i=r+t+2}^{n} min\{s+t, d_i\} + 2(t-f(s))(s+f(s)) \\
&\le (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \\
&\quad + \sum_{i=r+t+2}^{n} min\{s+t, d_i\}. \ \square
\end{aligned}$$

Proof of Theorem 1.4. (1) $\Rightarrow$ (2). For any s, s_1, t and t_1 with $0 \le s \le r+1$, $0 \le t \le n-r-1$, $s_1 \ge 0$, $t_1 \ge 0$, $s+s_1 \le r+1$ and $t+t_1 \le n-r-1$, it follows that

$$\begin{aligned}
\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &\le (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \\
&\quad + \sum_{i=r+t+2}^{n} min\{s+t, d_i\} \\
&\le (s+t)(s+t-1) + \sum_{i=s+1}^{r+1-s_1} min\{s+t, d_i - r + s\} \\
&\quad + \sum_{i=r+2-s_1}^{r+1} min\{s+t, d_i - r + s\} \\
&\quad + \sum_{i=r+t+2}^{n-t_1} min\{s+t, d_i\} \\
&\quad + \sum_{i=n-t_1+1}^{n} min\{s+t, d_i\} \\
&\le (s+t)(s+t-1) + (s+t)(r+1-s-s_1) \\
&\quad +(s+t)(n-t_1-r-t-1) \\
&\quad + \sum_{i=r+2-s_1}^{r+1} (d_i - r + s) + \sum_{i=n-t_1+1}^{n} d_i \\
&= (s+t)(n-s_1-t_1-1) + \sum_{i=r+2-s_1}^{r+1} (d_i - r + s) \\
&\quad + \sum_{i=n-t_1+1}^{n} d_i.
\end{aligned}$$

(2) $\Rightarrow$ (1). Suppose to the contrary that there exist s and t with $0 \le s \le r+1$ and $0 \le t \le n-r-1$ such that

$$\begin{aligned}\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &> (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} \\ &+ \sum_{i=r+t+2}^{n} min\{s+t, d_i\}.\end{aligned}$$

We consider the following cases.

Case 1. $d_{s+1} - r \ge t$ and $d_{r+t+2} \ge s+t$.

Denote $s' = max\{i | d_i - r \ge t\}$ and $t' = max\{i | d_{r+1+i} \ge s+t\}$. Then we have that

$$\begin{aligned}\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &> (s+t)(s'+t'-1) + \sum_{i=s'+1}^{r+1} (d_i - r + s) \\ &+ \sum_{i=r+t'+2}^{n} d_i.\end{aligned}$$

By $s_1 = r+1-s'$ and $t_1 = n-r-1-t'$, it follows from (2) that

$$\begin{aligned}\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &\le (s+t)(s'+t'-1) + \sum_{i=s'+1}^{r+1} (d_i - r + s) \\ &+ \sum_{i=r+t'+2}^{n} d_i,\end{aligned}$$

a contradiction.

Case 2. $d_{s+1} - r \ge t$ and $d_{r+t+2} < s+t$.

Denote $s' = max\{i | d_i - r \ge t\}$. Then we have that

$$\begin{aligned}\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &> (s+t)(s'+t-1) + \sum_{i=s'+1}^{r+1} (d_i - r + s) \\ &+ \sum_{i=r+t+2}^{n} d_i.\end{aligned}$$

By $s_1 = r+1-s'$ and $t_1 = n-r-1-t$, it follows from (2) that

$$\begin{aligned}\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} &\le (s+t)(s'+t-1) + \sum_{i=s'+1}^{r+1} (d_i - r + s) \\ &+ \sum_{i=r+t+2}^{n} d_i,\end{aligned}$$

a contradiction.

Case 3. $d_{s+1} - r < t$ and $d_{r+t+2} \ge s+t$.

Denote $t' = max\{i|d_{r+1+i} \geq s+t\}$. Then we have that

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} > (s+t)(s+t'-1) + \sum_{i=s+1}^{r+1} (d_i - r + s) + \sum_{i=r+t'+2}^{n} d_i.$$

By $s_1 = r+1-s$ and $t_1 = n-r-1-t'$, it follows from (2) that

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leq (s+t)(s+t'-1) + \sum_{i=s+1}^{r+1} (d_i - r + s) + \sum_{i=r+t'+2}^{n} d_i,$$

a contradiction.

Case 4. $d_{s+1} - r < t$ and $d_{r+t+2} < s+t$.

Then we have that

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} > (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} (d_i - r + s) + \sum_{i=r+t+2}^{n} d_i.$$

By $s_1 = r+1-s$ and $t_1 = n-r-1-t$, it follows from (2) that

$$\sum_{i=1}^{s} d_i + \sum_{i=1}^{t} d_{r+1+i} \leq (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} (d_i - r + s) + \sum_{i=r+t+2}^{n} d_i,$$

a contradiction. □

Proof of Theorem 1.5. By the definition of d_i^* for $1 \leq i \leq n$, we can see that

$$(s+t-1)s + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\} + \sum_{i=1}^{t}(d_i^* - 1) + \sum_{i=t+1}^{t+s} d_i^*$$
$$= (s+t)(s+t-1) + \sum_{i=s+1}^{r+1} min\{s+t, d_i - r + s\}$$
$$+ \sum_{i=r+t+2}^{n} min\{s+t, d_i\}.$$

Therefore, Theorem 1.5 immediately follows from Theorem 1.3. □

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