The Complexity of Computing Signed (Total) Domatic Numbers of Graphs

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Abstract

Let $G = (V, E)$ be a graph. A function $f : V \to \{-1, 1\}$ is called a signed dominating function on G if $\sum_{u \in N_G[v]} f(u) \geq 1$ for each $v \in V$, where $N_G[v]$ is the closed neighborhood of v. A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on G is called a signed dominating family (of functions) on G if $\sum_{i=1}^{d} f_i(v) \leq 1$ for each $v \in V$. The signed domatic number of G is the maximum number of functions in a signed dominating family on G. The signed total domatic number is defined similarly, by replacing the closed neighborhood $N_G[v]$ with the open neighborhood $N_G(v)$ in the definition. In this paper, we prove that the problems of computing the signed domatic number and the signed total domatic number of a given graph are both NP-hard, even if the graph has bounded maximum degree. To the best of our knowledge, they are the first NP-hardness results for these two variants of the domatic number.

1 Introduction

In this paper we generally follow the notation and terminology of [3]. Let $G = (V, E)$ be a (simple and undirected) graph. For each $v \in V$, $N_G(v) = \{u \mid \{u, v\} \in E\}$ is the *open neighborhood* of v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighborbood* of v. Let $d_G(v) = |N_G(v)|$ denote the *degree* of v, $\delta(G) = \min_{v \in V} d_G(v)$ be the minimum degree of any vertex in G, and $\Delta(G) = \max_{v \in V} d_G(v)$ be the maximum degree of a vertex in G. A *k-coloring* of G is a mapping $c : V \to \{1, 2, \ldots, k\}$, and the coloring is said to be *legal* if $c(u) \neq c(v)$ whenever $\{u, v\} \in E$. We also say G is *k-colorable* if G has a legal k-coloring.

A *signed dominating function* on G, originally defined in [4], is a mapping $f : V \to \{-1, 1\}$ satisfying that $\sum_{u \in N_G[v]} f(u) \geq 1$ for every $v \in V$.

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A set $\{f_1, f_2, \ldots, f_d\}$ of signed dominating functions on G is called a *signed dominating family* (of functions) on G if $\sum_{i=1}^{d} f_i(v) \leq 1$ for all $v \in V$. The *signed domatic number*, denoted by $d_S(G)$, is the maximum number of functions in any signed dominating family on G. This concept was introduced in [15] and has been further studied in, e.g., [9, 10, 11, 12, 13].

A *signed total dominating function* on G, introduced by [16], is a mapping $f : V \to \{-1, 1\}$ satisfying that $\sum_{u \in N_G(v)} f(u) \geq 1$ for every $v \in V$. A set $\{f_1, f_2, \ldots, f_d\}$ of signed total dominating functions on G is called a *signed total dominating family* (of functions) on G if $\sum_{i=1}^{d} f_i(v) \leq 1$ for all $v \in V$. The *signed total domatic number*, denoted by $d_t^S(G)$, is the maximum number of functions in any signed dominating family on G. The study of this parameter was initiated by [7] and continued in, e.g., [6, 8, 14].

Previous research on the signed domatic number and signed total domatic number of graphs mainly focuses on the graph-theoretic perspectives, such as establishing their upper and lower bounds in general or special classes of graphs. However, unlike signed domination [4, 16] or the original concept of domatic number [5] which have been extensively studied from algorithmic and complexity points of view, these two parameters have not been explored from such aspects before, which motivates our study.

In this paper, we initiate the study of the algorithmic complexity of the natural optimization problems associated with the signed domatic number and signed total domatic number. More specifically, we prove that the problems of computing these two parameters of a given graph are both NP-hard, even if the graph has bounded maximum degree. The proofs are by reductions from two variants of the graph coloring problem, of which one is known and another is new to the best of our knowledge. The reductions require some carefully designed gadgets, and we believe that the techniques used in the reductions are of their own interests and may be useful in future applications.

2 Complexity of Computing the Signed Domatic Number

In this section we show the hardness of computing the signed domatic number of a graph. Our main result is as follows.

Theorem 1. *Given a graph G of maximum degree 8, deciding whether $d_S(G) \geq 3$ is* NP*-complete.*

Proof. The problem is obviously in NP. We now present a polynomial time reduction from the following NP-complete problem [2]: Given a (planar) graph of maximum degree 4, decide whether it has a legal 3-coloring. Let

$G = (V, E)$ be an instance of the latter problem. We assume without loss of generality that G has no isolated vertex. Now construct another graph $H = (V', E')$ as follows. Let $V' = X \cup Y \cup Z$, where $X = \{x_v \mid v \in V\}, Y = \{y_e \mid e \in E\}$, and $Z = \{z_{v,e,j} \mid v \in e \in E, 1 \le j \le 6\}$. That is, we have one vertex x_v for each vertex v of G, one vertex y_e for every edge e of G, and for each pair (v, e) such that v is incident to e in G, we have six vertices $\{z_{v,e,j} \mid 1 \le j \le 6\}$ associated with it. The set of edges of H is defined as: $E' = \{\{x_v, y_e\} \mid v \in e \in E\} \cup \{\{x_v, z_{v,e,1}\} \mid v \in e \in E\} \cup \{\{z_{v,e,j}, z_{v,e,j'}\} \mid v \in e \in E, 1 \le j < j' \le 6\}$. Thus, the subgraph of H induced on $X \cup Y$ is precisely the incidence graph of G, and the six vertices in $\{z_{v,e,j} \mid 1 \le j \le 6\}$ form a clique for every pair (v, e) with $v \in e \in E$. Moreover, the only edges between $X \cup Y$ and Z are $\{\{x_v, z_{v,e,1}\} \mid v \in e \in E\}$. It is easy to see that $\Delta(H) \le \max\{6, 2\Delta(G)\} \le 8$ and $\delta(H) = 2$.

We claim that G is 3-colorable if and only if $d_S(H) \ge 3$, which will finish the reduction. For the "if" part, assume $d_S(H) \ge 3$ and let $\{f_1, f_2, f_3\}$ be a signed dominating family on H. (Note that in this case we actually have $d_S(H) = 3$, since by the result of [15], $d_S(H) \le \delta(H) + 1 = 3$.) For convenience, given $f : V' \to \{-1, 1\}$, we say $v' \in V'$ is *signed dominated* in f if $\sum_{x \in N_H[v']} f(x) \ge 1$. Let v be any vertex in V, and let $e = \{v, u\} \in E$ for some $u \in V$ (recall that G has no isolated vertex). By our construction of H, y_e is only adjacent to x_u and x_v in H. Since $\{f_1, f_2, f_3\}$ is a signed dominating family on H, we have $\sum_{i=1}^{3} f_i(x_v) \le 1$, which implies that at least one of $f_1(x_v), f_2(x_v), f_3(x_v)$ is -1. Now suppose at least two of them are -1, and w.l.o.g., let $f_1(x_v) = f_2(x_v) = -1$. Then we must have $f_1(y_e) = f_1(x_u) = f_2(y_e) = f_2(x_u) = 1$, otherwise y_e is not signed dominated in f_1 and f_2. As $\sum_{i=1}^{3} f_i(y_e) \le 1$, we have $f_3(y_e) = -1$, indicating that $f_3(x_u) = f_3(x_v) = 1$. However, this would give that $\sum_{i=1}^{3} f_i(x_u) = 3 > 1$, violating the property of a signed dominating family. Hence, it holds that exactly one of $f_1(x_v), f_2(x_v), f_3(x_v)$ is -1. Now define a 3-coloring of G as follows. For each $v \in V$, find the unique index $i \in \{1, 2, 3\}$ for which $f_i(x_v) = -1$, and assign v with color i. We show that this is a legal coloring of G. For any edge $e = \{u, v\} \in E$, suppose u and v are assigned with colors j and j' respectively. If $j = j'$, then $f_j(x_u) = f_j(x_v) = -1$, implying that $f_j(y_e) + f_j(x_u) + f_j(x_v) \le -1$. This contradicts with the fact that f_j is a signed dominating function on H. Therefore $j \ne j'$, which proves that the coloring of G obtained in this way is indeed a legal 3-coloring. Hence, G is 3-colorable.

Now comes the "only if" direction, which necessitates a careful design of signed dominating functions using the $\{z_{v,e,j}\}$ gadgets. Suppose G is 3-colorable, and c is a legal 3-coloring of G in which vertex v receives the color $c(v) \in \{1, 2, 3\}$. We now define three functions f_1, f_2, f_3 on V', and

	$z_{v,e,1}$	$z_{v,e,2}$	$z_{v,e,3}$	$z_{v,e,4}$	$z_{v,e,5}$	$z_{v,e,6}$
f_{i_1}	1	1	1	1	−1	−1
f_{i_2}	1	1	−1	−1	1	1
f_{i_3}	−1	−1	1	1	1	1

Table 1: Function values of $z_{v,e,j}$.

prove that they form a signed dominating family on H. For every $v \in V$, let

$$f_i(x_v) = \begin{cases} -1, & \text{if } i = c(v) \\ 1, & \text{otherwise.} \end{cases}$$

For each $e = \{u, v\} \in E$, let

$$f_i(y_e) = \begin{cases} 1, & \text{if } -1 \in \{f_i(x_u), f_i(x_v)\}, \text{ or equivalently, } i \in \{c(u), c(v)\} \\ -1, & \text{otherwise.} \end{cases}$$

For every v, e such that $v \in e \in E$, let

$$f_i(z_{v,e,1}) = \begin{cases} 1, & \text{if } f_i(x_v) = -1, \text{ or equivalently, } i = c(v) \\ -f_i(y_e), & \text{otherwise.} \end{cases}$$

Let $V'' = X \cup Y \cup \{z_{v,e,1} \mid v \in e \in E\}$ be the collection of vertices of H whose function values have already been given. Before assigning function values to the remaining vertices, we first verify that $\sum_{i=1}^{3} f_i(r) = 1$ for every $r \in V''$, that is, among $f_1(r), f_2(r)$ and $f_3(r)$ there are exactly two 1's and one -1. This is clear for all $x_v \in X$. For $y_e \in Y$ where $e = \{u, v\}$, since c is a legal 3-coloring, we have $c(u) \neq c(v)$, and hence $f_i(y_e)$ will be 1 twice (when i is the color of u or v) and be -1 exactly once. Now consider a vertex $z_{v,e,1}$. By our definition, $f_{c(v)}(x_v) = -1$, $f_{c(v)}(y_e) = 1$ and $f_{c(v)}(z_{v,e,1}) = 1$. Assuming $\{1, 2, 3\} \backslash \{c(v)\} = \{i, i'\}$, we have $\{f_i(y_e), f_{i'}(y_e)\} = \{1, -1\}$, and thus $\{f_i(z_{v,e,1}), f_{i'}(z_{v,e,1})\} = \{-f_i(y_e), -f_{i'}(y_e)\} = \{-1, 1\}$. Therefore, the condition $\sum_{i=1}^{3} f_i(r) = 1$ holds for all vertices $r \in V''$.

We proceed to define the functions on $V' \backslash V'' = \{z_{v,e,j} \mid v \in e \in E, 2 \leq j \leq 6\}$. We will do this separately for each pair (v, e) with $v \in e \in E$; so in the following we assume (v, e) is fixed. From previous analysis, we know that there is a permutation (i_1, i_2, i_3) of $(1, 2, 3)$ such that $f_{i_1}(z_{v,e,1}) = f_{i_2}(z_{v,e,1}) = 1$ and $f_{i_3}(z_{v,e,1}) = -1$. (Of course such permutation is not unique, but we only need to choose an arbitrary one.) Now we define the functions on $z_{v,e,j}$ for $2 \leq j \leq 6$ according to Table 1. (The table is viewed in the obvious way; for example, $f_{i_2}(z_{v,e,4})$ is at the crossing of the 3rd row and the 5th column.) Doing this for all pairs (v, e) completes the definition of the three functions.

It is clear that $\sum_{i=1}^{3} f_i(r) = 1$ is fulfilled for all $r \in V'$. Thus, it only remains to show that for any $i \in \{1, 2, 3\}$, f_i is a signed dominating function on H. We will consider all the vertices of H and prove that they are signed dominated in each of the three functions. There are four cases to be examined.

1. y_e with $e = \{u, v\} \in E$. We have $N_H[y_e] = \{y_e, x_u, x_v\}$. By our definition, there are exactly two 1's and one -1 among $f_i(y_e), f_i(x_u), f_i(x_v)$, for each $i \in \{1, 2, 3\}$. Thus, y_e is signed dominated in every function.

2. x_v with $v \in V$. We have $N_H[x_v] = \{x_v\} \cup \{y_e, z_{v,e,1} \mid v \in e \in E\}$. When $i = c(v)$, we have $f_i(x_v) = -1$ and $f_i(r) = 1$ for any $r \in N_H(x_v)$, implying that $\sum_{r \in N_H[x_v]} f_i(x_v) \geq 1$ (note that $|N_H(x_v)| \geq 2$ since G has no isolated vertex). When $i \neq c(v)$, we have $f_i(x_v) = 1$ and $f_i(z_{v,e,1}) = -f_i(y_e)$, which means $\sum_{r \in N_H[x_v]} f_i(x_v) = 1$. Hence, x_v is signed dominated in all three functions.

3. $z_{v,e,j}$ with $v \in e \in E$ and $2 \leq j \leq 6$. We have $N_H[z_{v,e,j}] = \{z_{v,e,k} \mid 1 \leq k \leq 6\}$. From Table 1 we find that $\sum_{r \in N_H[z_{v,e,j}]} f_i(r) = \sum_{k=1}^{6} f_i(z_{v,e,k}) = 2$ for any $i \in \{1, 2, 3\}$. Thus $z_{v,e,j}$ is signed dominated in the three functions.

4. $z_{v,e,1}$ with $v \in e \in E$. We have $N_H[z_{v,e,j}] = \{x_v\} \cup \{z_{v,e,k} \mid 1 \leq k \leq 6\}$. Thus, $\sum_{r \in N_H[z_{v,e,1}]} f_i(r) = f_i(x_v) + \sum_{k=1}^{6} f_i(z_{v,e,k}) \geq -1 + 2 = 1$, for any $i \in \{1, 2, 3\}$. Therefore, $z_{v,e,1}$ is signed dominated in all three functions.

By the above case analysis, we have shown that $\{f_1, f_2, f_3\}$ is indeed a signed dominating family on H, and thus $d_S(H) \geq 3$. This completes the proof of the "only if" part of the reduction, and hence concludes the whole proof. □

The following corollary follows immediately from Theorem 1.

Corollary 1. *It is* NP*-hard to compute the signed domatic number of a given graph of maximum degree 8.*

3 Complexity of Computing the Signed Total Domatic Number

This section is devoted to the hardness proof of computing the signed total domatic number of a graph, as indicated in the following theorem.

Theorem 2. *Given a graph G of maximum degree 11, deciding whether $d_t^S(G) \geq 3$ is* NP*-complete.*

To prove Theorem 2, we design a reduction from a hypergraph coloring problem. A *hypergraph* G is a pair (V, E), where V is the set of vertices of G, and $E \subseteq 2^V \setminus \{\emptyset\}$ is the set of edges of G. The *degree* of a vertex $v \in V$ (in G) is $d_G(v) := |\{e \mid v \in e \in E\}|$, i.e., the number of edges of G that contains v. A hypergraph is called k*-uniform* if every edge of it contains exactly k vertices. (Thus, a graph can be defined as a 2-uniform hypergraph.) A *strong* k*-coloring* of a hypergraph $G = (V, E)$ [1] is a mapping $c : V \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $u \neq v$ and $\{u, v\} \subseteq e$ for some $e \in E$; that is, vertices that are contained in a common edge of G have pairwise distinct colors. The *3-uniform hypergraph strong 3-coloring problem* (3HS3C for short) is to decide whether a given 3-uniform hypergraph has a strong 3-coloring. As far as we are aware, the complexity of this problem has not been investigated before.

Lemma 1. *The 3HS3C problem is* NP*-complete even on hypergraphs of maximum degree 4.*

Proof. The problem is clearly in NP. We give a polynomial time reduction from the problem of deciding whether a given graph of maximum degree 4 is 3-colorable (which is also used in the proof of Theorem 1) to it. Let $G = (V, E)$ be a graph of maximum degree 4. Construct a 3-uniform hypergraph $G' = (V', E')$ by letting $V' = \{x_v \mid v \in V\} \cup \{x_e \mid e \in E\}$ and $E' = \{\{x_u, x_v, x_e\} \mid e = \{u, v\} \in E\}$. It is easy to see that G' has maximum degree 4. If G' has a strong 3-coloring, then this coloring naturally induces a legal 3-coloring of G (by identifying v and x_v and neglecting the x_e's). Conversely, assume G has a legal 3-coloring c where v has the color $c(v)$. Define a 3-coloring c' of G' as follows. For any $v \in V$, let $c'(x_v) = c(v)$. For any $e = (u, v) \in E$, let $c'(x_e)$ be the (unique) color different from $c(u)$ and $c(v)$. It is easy to see that c' is a strong 3-coloring of G'. Therefore, G is 3-colorable if and only if G' has a strong 3-coloring, completing the proof of Lemma 1. □

We now prove Theorem 2.

Proof of Theorem 2. Clearly the problem is in NP, so it suffices to perform a polynomial time reduction from the 3HS3C problem to it. Let $G = (V, E)$ be a 3-uniform hypergraph of maximum degree 4, which is an instance of the 3HS3C problem. We modify G in the following way: for each $v \in V$ such that $d_G(v)$ is even, add two new vertices v', v'' and a new edge $\{v, v', v''\}$, which makes the degrees of v, v' and v'' all become odd. It is easy to see that the new graph has maximum degree at most 5, and that G has a strong 3-coloring if and only if the new hypergraph does. Thus, we can assume w.l.o.g. that $d_G(v)$ is odd for every $v \in V$ and $\Delta(G) \leq 5$.

We create a graph $H = (V', E')$ as follows. Let $V' = X \cup Y \cup Z$, where $X = \{x_v \mid v \in V\}, Y = \{y_e \mid e \in E\}$, and $Z = \{z_{v,i,j} \mid v \in V, 1 \leq i \leq$

$d_G(v)+1, 1 \le j \le 9\}$. Let $E' = \{\{x_v, y_e\} \mid v \in e \in E\} \cup \{\{x_v, z_{v,i,1}\} \mid v \in V, 1 \le i \le d_G(v)+1\} \cup \{\{z_{v,i,j}, z_{v,i,j'}\} \mid v \in V, 1 \le i \le d_G(v)+1, 1 \le j < j' \le 9\}$. It is easy to verify that $\Delta(H) \le \max\{9, 2\Delta(G)+1\} \le 11$ and $\delta(H) = 3$.

We shall prove that G has a strong 3-coloring if and only if $d_t^S(H) \ge 3$, which will complete the reduction. For the "if" direction, suppose $d_t^S(H) \ge 3$. Since $d_t^S(H) \le \delta(H) = 3$ [7], we actually have $d_t^S(H) = 3$. Let $\{f_1, f_2, f_3\}$ be a signed total dominating family on H. Let v be an arbitrary vertex in V, and $e = \{v, u, w\} \in E$ be an edge containing v. Since $\sum_{i=1}^{3} f_i(x_v) \le 1$, at least one of $f_1(x_v), f_2(x_v)$ and $f_3(x_v)$ is -1. Assume that there are two -1's among them, say, $f_1(x_v) = f_2(x_v) = -1$. For convenience, given $f : V' \to \{-1, 1\}$, we say $v' \in V'$ is *signed total dominated* in f if $\sum_{x \in N_H(v')} f(x) \ge 1$. As $N_H(y_e) = \{x_v, x_u, x_w\}$ and y_e is signed total dominated in f_1 and f_2, we have $f_1(x_u) = f_1(x_w) = f_2(x_u) = f_2(x_w) = 1$, from which it follows that $f_3(x_u) = f_3(x_w) = -1$. However, this implies $\sum_{v' \in N_H(y_e)} f_3(v') \le -1$, a contradiction. Thus, exactly one of $f_1(x_v), f_2(x_v)$ and $f_3(x_v)$ is -1. Now define a coloring of G by giving vertex v the unique color j such that $f_j(x_v) = -1$. For every $e = \{u, v, w\} \in E$, its vertices must receive all three colors. Otherwise, w.l.o.g. assume u and v both have color j; that is, $f_j(x_u) = f_j(x_v) = -1$. Then y_e is not signed total dominated in f_j, a contradiction. Thus, G has a strong 3-coloring.

Next we consider the "only if" direction, and assume that G has a strong 3-coloring c in which each vertex $v \in V$ is colored with $c(v) \in \{1, 2, 3\}$. We now define three functions f_1, f_2, f_3 on V', and prove that they form a signed total dominating family on H. For every $v \in V$, let

$$f_i(x_v) = \begin{cases} -1, & \text{if } i = c(v) \\ 1, & \text{otherwise.} \end{cases}$$

For each $e \in E$, let

$$f_i(y_e) = \begin{cases} -1, & \text{if } i = 1 \\ 1, & \text{otherwise.} \end{cases}$$

For every vertex $z_{v,i,j}$, if $1 \le i \le (d_G(v)+1)/2$, then define the functions on it according to Table 2; if $(d_G(v)+1)/2 < i \le d_G(v)+1$, then define the functions on it according to Table 3. (Recall that $d_G(v)$ is odd, and hence $(d_G(v)+1)/2$ is an integer.)

We now show that $\{f_1, f_2, f_3\}$ is a signed dominating function on H. Clearly, $\sum_{i=1}^{3} f_i(r) = 1$ for all $r \in V'$. So we only need to prove that f_i is a signed dominating function on H for every $i \in \{1, 2, 3\}$. We will consider all the vertices of H and prove that they are signed total dominated in each of the three functions. There are three cases to investigate.

	$z_{v,i,1}$	$z_{v,i,2}$	$z_{v,i,3}$	$z_{v,i,4}$	$z_{v,i,5}$	$z_{v,i,6}$	$z_{v,i,7}$	$z_{v,i,8}$	$z_{v,i,9}$
f_1	1	1	1	1	1	1	−1	−1	−1
f_2	1	1	1	−1	−1	−1	1	1	1
f_3	−1	−1	−1	1	1	1	1	1	1

Table 2: Function values of $z_{v,i,j}$ when $1 \le i \le (d_G(v)+1)/2$

	$z_{v,i,1}$	$z_{v,i,2}$	$z_{v,i,3}$	$z_{v,i,4}$	$z_{v,i,5}$	$z_{v,i,6}$	$z_{v,i,7}$	$z_{v,i,8}$	$z_{v,i,9}$
f_1	1	1	1	−1	−1	−1	1	1	1
f_2	−1	−1	−1	1	1	1	1	1	1
f_3	1	1	1	1	1	1	−1	−1	−1

Table 3: Function values of $z_{v,i,j}$ when $(d_G(v)+1)/2 < i \le d_G(v)+1$

1. y_e with $e = \{u,v,w\} \in E$. We have $N_H(y_e) = \{x_u, x_v, x_w\}$. Since c is a legal 3-coloring of G, for every $i \in \{1,2,3\}$, exactly one of $f_i(x_u), f_i(x_v)$ and $f_i(x_w)$ is -1. Hence y_e is signed total dominated in every function.

2. x_v with $v \in V$. We have $N_H(x_v) = \{y_e \mid v \in e \in E\} \cup \{z_{v,i,1} \mid 1 \le i \le d_G(v)+1\}$. By our definition, $f_1(y_e) = -1$ for all e and $f_1(z_{v,j,1}) = 1$ for all v, j. Therefore, $\sum_{r \in N_H(x_v)} f_1(r) = (d_G(v)+1) - d_G(v) = 1$. When $i = 2$ or 3, by our construction, exactly one half of $f_i(z_{v,j,1})$'s, $1 \le j \le d_G(v)+1$, are -1, and another half are all 1. (For example, by Tables 2 and 3, $f_2(z_{v,j,1}) = 1$ if $j \le (d_G(v)+1)/2$, and $f_2(z_{v,j,1}) = -1$ otherwise.) Also, $f_i(y_e) = 1$ for all e. Thus $\sum_{r \in N_H(x_v)} f_i(r) \ge 1$. We have shown that x_v is signed total dominated in all three functions.

3. $z_{v,j,k}$ with $v \in V$, $1 \le j \le d_G(v)+1$ and $1 \le k \le 9$. We have $N_H(z_{v,j,k}) = \{z_{v,j,k'} \mid 1 \le k' \le 9, k' \ne k\}$ if $k \ne 1$, and $N_H(z_{v,j,k}) = \{x_v\} \cup \{z_{v,j,k'} \mid 1 \le k' \le 9, k' \ne k\}$ if $k = 1$. From Tables 2 and 3, we find that for any function f_i, exactly three values among $\{f_i(z_{v,j,k'}) \mid 1 \le k' \le 9\}$ are -1. Thus, in any function f_i, at least 5 neighbors of $z_{v,j,k}$ have function value 1, and at most 4 neighbors of it have function value -1. This shows that $z_{v,j,k}$ is signed total dominated in all three functions.

By the above case analysis, we have shown that $\{f_1, f_2, f_3\}$ is indeed a signed total dominating family on H, and thus $d_t^S(H) \ge 3$, finishing the "only if" part of the reduction. Therefore, G has a strong 3-coloring if and only if $d_t^S(H) \ge 3$. This completes the proof of Theorem 2. □

The following corollary is straightforward.

Corollary 2. *It is* NP*-hard to compute the signed total domatic number of a given graph of maximum degree 11.*

4 Concluding Remarks

In this paper, we proved that it is NP-hard to compute the signed domatic number and the signed total domatic number of a given graph, even if the graph is of bounded maximum degree. On the other hand, these two parameters of a graph of maximum degree 2 (which is the union of disjoint paths and cycles) can be easily decided [15, 7]. What happens if the graph has maximum degree 3? This motivates us to pose the following open question for future research.

Question 1. *Is it* NP*-hard to compute the signed (total) domatic number of a graph of maximum degree 3?*

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