Signed $\{k\}$-domatic numbers of graphs

[1]S.M. Sheikholeslami and [2]L. Volkmann

[1]Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
s.m.sheikholeslami@azaruniv.edu

[2]Lehrstuhl II für Mathematik
RWTH Aachen University
52056 Aachen, Germany
volkm@math2.rwth-aachen.de

Abstract

Let k be a positive integer, and let G be a simple graph with vertex set $V(G)$. A function $f : V(G) \longrightarrow \{\pm1, \pm2, \ldots, \pm k\}$ is called a *signed $\{k\}$-dominating function* if $\sum_{u \in N[v]} f(u) \geq k$ for each vertex $v \in V(G)$. The signed $\{1\}$-dominating function is the same as the ordinary signed domination. A set $\{f_1, f_2, \ldots, f_d\}$ of signed $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed $\{k\}$-dominating family* (of functions) on G. The maximum number of functions in a signed $\{k\}$-dominating family on G is the *signed $\{k\}$-domatic number* of G, denoted by $d_{\{k\}S}(G)$. Note that $d_{\{1\}S}(G)$ is the classical signed domatic number $d_S(G)$. In this paper we initiate the study of signed $\{k\}$-domatic numbers in graphs, and we present some sharp upper bounds for $d_{\{k\}S}(G)$. In addition, we determine $d_{\{k\}S}(G)$ for several classes of graphs. Some of our results are extensions of known properties of the signed domatic number.

Keywords: signed $\{k\}$-domatic number, signed $\{k\}$-dominating function, signed $\{k\}$-domination number, signed dominating function, signed domination number
MSC 2000: 05C69

1 Introduction

In this paper, G is a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. For a vertex $v \in V(G)$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* $N[v]$ is the set $N(v) \cup \{v\}$.

The *open neighborhood* $N(S)$ of a set $S \subseteq V$ is the set $\bigcup_{v\in S} N(v)$, and the *closed neighborhood* $N[S]$ of S is the set $N(S) \cup S$. The minimum degree and maximum degree of a vertex of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We write K_n for the *complete graph* of order n and C_n for a *cycle* of length n. Consult [10] for the notation and terminology which are not defined here.

For a real-valued function $f : V(G) \longrightarrow \mathbb{R}$, the weight of f is $w(f) = \sum_{v\in V} f(v)$. For $S \subseteq V$, we define $f(S) = \sum_{v\in S} f(v)$. So $w(f) = f(V)$. Let $k \geq 1$ be an integer. A *signed* $\{k\}$*-dominating function* (S$\{k\}$D function) is a function $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ satisfying $\sum_{u\in N[v]} f(u) \geq k$ for every $v \in V(G)$. The minimum of the values of $\sum_{v\in V(G)} f(v)$ taken over all signed $\{k\}$-dominating functions f is called the *signed* $\{k\}$*-domination number* and is denoted by $\gamma_{\{k\}S}(G)$. Since the function assigning $+k$ to every vertex of G is a S$\{k\}$D function, called the function ϵ, of weight nk, $\gamma_{\{k\}S}(G) \leq nk$ for every graph G of order n. Hence $\gamma_{\{k\}S}(G) = nk$ if and only if ϵ is the unique S$\{k\}$D function of G. In the special case when $k = 1$, $\gamma_{\{k\}S}(G)$ is the signed domination number $\gamma_S(G)$ investigated in [2] and has been studied by several authors (see, for example, [1, 3]).

Observation 1. If G is the complete graph of order n, then $\gamma_{\{k\}S}(G) = k$.

A set $\{f_1, f_2, \ldots, f_d\}$ of distinct signed $\{k\}$-dominating functions on G with the property that $\sum_{i=1}^{d} f_i(v) \leq k$ for each $v \in V(G)$, is called a *signed* $\{k\}$*-dominating family* on G. The maximum number of functions in a signed $\{k\}$-dominating family on G is the *signed* $\{k\}$*-domatic number* of G, denoted by $d_{\{k\}S}(G)$. The signed $\{k\}$-domatic number is well-defined and $d_{\{k\}S}(G) \geq 1$ for all graphs G since the set consisting of any one S$\{k\}$D function, for instance the function ϵ, forms a S$\{k\}$D family of G. A $d_{\{k\}S}$*-family* of a graph G is a S$\{k\}$D family containing $d_{\{k\}S}(G)$ S$\{k\}$D functions. The signed $\{1\}$-domatic number $d_{\{1\}S}(G)$ is the usual signed domatic number $d_S(G)$ which was introduced by Volkmann and Zelinka in [8] and has been studied by several authors (see for example [4, 5, 6]).

We first study basic properties and sharp upper bounds for the signed $\{k\}$-domatic number of a graph. Some of them generalize the results obtained for the signed domatic number.

In this paper we make use of the following results.

Proposition A. [2] Let G be a graph of order n. Then $\gamma_S(G) = n$ if and only if every nonisolated vertex of G is either an endvertex or adjacent to an endvertex.

Observation 2. Let G be a graph of order n and k a positive integer. Then $\gamma_{kS}(G) = nk$ if and only if G is empty graph or every nonisolated vertex of G is either an endvertex or adjacent to an endvertex when $k = 1$.

Proof. If G is empty graph or every nonisolated vertex of G is either an endvertex or adjacent to an endvertex when $k = 1$, then obviously $\gamma_{kS}(G) = nk$.

Conversely, let $\gamma_{ks}(G) = nk$. If $k = 1$, then the result follows from Proposition A. Assume now that $k \geq 2$ and suppose to the contrary that G is not empty. Then there exists an edge $uv \in E(G)$ and the function $f : V(G) \to \{\pm1, \pm2, \ldots, \pm k\}$ defined by $f(u) = 1, f(v) = k-1$ and $f(x) = k$ for $x \in V(G) - \{u, v\}$ is a signed $\{k\}$-dominating function of weight $(n-1)k$ which is a contradiction. This completes the proof. □

Proposition B. [7] If G is a graph of order n, then

$$\gamma_S(G) + d_S(G) \leq n + 1.$$

Equality $\gamma_S(G) + d_S(G) = n + 1$ occurs if and only if $G = K_n$ with n odd or every nonisolated vertex of G is either an endvertex or adjacent to an endvertex.

Proposition C. [9] For any integer $n \geq 1$, we have

$$\gamma_S(K_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{otherwise.} \end{cases} \tag{1}$$

Proposition D. [8] If $G = K_n$ is the complete graph of order n, then

$$d_S(K_n) = \begin{cases} n & \text{if } n \text{ is odd,} \\ p & \text{if } n = 2p \text{ and } p \text{ is odd} \\ p-1 & \text{if } n = 2p \text{ and } p \text{ is even.} \end{cases} \tag{2}$$

2 Basic properties of the signed $\{k\}$-domatic number

In this section we present basic properties of $d_{\{k\}S}(G)$ and sharp bounds on the signed $\{k\}$-domatic number of a graph.

Theorem 3. If G is a graph of order n, then

$$1 \leq d_{\{k\}S}(G) \leq \delta(G) + 1.$$

Moreover if $d_{\{k\}S}(G) = \delta(G) + 1$, then for each function of any $d_{\{k\}S}$-family $\{f_1, f_2, \cdots, f_d\}$ and for all vertices v of degree $\delta(G)$, $\sum_{u \in N[v]} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(u) = k$ for every $u \in N[v]$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a S$\{k\}$D family of G such that $d = d_{\{k\}S}(G)$ and let v be a vertex of minimum degree $\delta(G)$. Then $|N[v]| = \delta(G) + 1$ and

$$\begin{aligned} 1 \leq d &= \sum_{i=1}^{d} 1 \\ &\leq \sum_{i=1}^{d} \frac{1}{k} \sum_{u \in N[v]} f_i(u) \\ &= \sum_{u \in N[v]} \frac{1}{k} \sum_{i=1}^{d} f_i(u) \\ &\leq \sum_{u \in N[v]} 1 \\ &= \delta(G) + 1. \end{aligned}$$

If $d_{\{k\}S}(G) = \delta(G) + 1$, then the two inequalities occurring in the proof become equalities, which gives the two properties given in the statement. □

The special case $k = 1$ in Theorem 3 can be found in [8]. The next corollaries are consequences of Theorem 3.

Corollary 4. If T is a tree of order $n \geq 2$, then $d_{\{k\}S}(T) = 1$ when $k = 1$, $1 \leq d_{\{k\}S}(T) \leq 2$ when $k = 2$ and $d_{\{k\}S}(T) = 2$ when $k \geq 3$.

Proof. If $k = 1$, then since every signed dominating set assigns 1 to an endvertex, we deduce that $d_{\{k\}S}(T) = 1$.

If $k = 2$, then Theorem 3 implies immediately that $1 \leq d_{\{k\}S}(T) \leq 2$

Let $k \geq 3$. For a fixed vertex $v \in V(T)$, let $V_i = \{u \in V(T) \mid d_T(u, v) = i\}$ for $i = 0, 1, \ldots, h$, where h is the eccentricity of v. Define $f : V(T) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ by $f(u) = k - 1$ for $u \in V_i$ when i is even and $f(u) = 1$ otherwise. Also define $g : V(T) \to \{1, 2, \ldots, k\}$ by $g(u) = 1$ for $u \in V_i$ when i is even and $g(u) = k - 1$ otherwise. Obviously, $\{f, g\}$ is a signed S$\{k\}$D family on T and the result follows from Theorem 3. □

Corollary 5. If T is a tree of order $n \geq 2$ such that every vertex is an endvertex or adjacent to an endvertex, then $d_{\{2\}S}(T) = 1$.

Proof. Suppose to the contrary that $d_{\{2\}S}(T) = 2$, and let $\{f_1, f_2\}$ be a S$\{k\}$D family on T. Let v be an arbitrary endvertex and u its neighbor. Then it follows from Theorem 3 that $f_1(v) + f_1(u) = 2$ and $f_2(v) + f_2(u) = 2$ and thus $f_1(v) = 1$ and $f_2(v) = 1$ and so $f_1(u) = 1$ and $f_2(u) = 1$. Since every vertex of T is an endvertex or adjacent to an endvertex, we obtain the contradiction $f_1 \equiv f_2 \equiv 1$. □

Let T' be the tree consisting of the vertex set

$$V(T') = \{x_1, x_2, x_3, x_4, x_5, x_6, v_1, v_2, v_3, w\}$$

such that w is adjacent to v_1, v_2 and v_3 and x_1 and x_2 are adjacent to v_1, x_3 and x_4 are adjacent to v_2 as well as x_5 and x_6 are adjacent to v_3. Define $f_i : V(T') \to \{\pm 1, \pm 2\}$ for $i \in \{1, 2\}$ by $f_1(x) = 1$ for each $x \in V(T')$ and $f_2(w) = -1$ and $f_2(x) = 1$ for each $x \in V(T') - \{w\}$. Clearly, $\{f_1, f_2\}$ is a signed S$\{k\}$D family on T' and hence if follows from Theorem 3 that $d_{\{2\}S}(T') = 2$.

This example and Corollary 5 demonstrate that if T is a tree, then $d_{\{2\}S}(T) = 1$ and $d_{\{2\}S}(T) = 2$ are possible.

Problem 1. Characterize all trees T with the property that $d_{\{2\}S}(T) = 2$.

Corollary 6. If P is a path of order $n \geq 2$, then $d_{\{2\}S}(P) = 1$.

Proof. Let $P = x_1 x_2 \ldots x_n$ a path of order n. Suppose to the contrary that $d_{\{2\}S}(P) = 2$, and let $\{f_1, f_2\}$ be a S$\{k\}$D family on P. We have seen in the proof of Corollary 5 that $f_1(x_1) = f_1(x_2) = f_2(x_1) = f_2(x_2) = 1$. Since $f_i(x_1) + f_i(x_2) + f_i(x_3) \geq 2$, it follows that $f_i(x_3) \geq 1$ for $i = 1, 2$. As $f_1(x_3) + f_2(x_3) \leq 2$, we conclude that $f_1(x_3) = f_2(x_3) = 1$. If we continue this process, we finally arrive at $f_1 \equiv f_2 \equiv 1$, a contradiction. □

If C_n is the cycle of order n, then it was shown in [8] that $d_s(C_n) = 3$ if $n \equiv 0 \pmod 3$ and $d_s(C_n) = 1$ otherwise.

Corollary 7. For positive integers $k \geq 2$ and $n \geq 3$,

$$d_{\{k\}S}(C_n) = \begin{cases} 3 & \text{if} \quad n \equiv 0 \pmod 3, \\ 2 & \text{if} \quad n \not\equiv 0 \pmod 3 \text{ and } k \geq 3, \\ 1 & \text{if} \quad n \not\equiv 0 \pmod 3 \text{ and } k = 2. \end{cases}$$

Proof. Let $C_n = (v_1 v_2 \dots v_n)$. By Theorem 3, $d_{\{k\}S}(C_n) \leq 3$. Assume first that $n \equiv 0 \pmod 3$. Define $f_i : V(T) \to \{\pm 1, \pm 2, \dots, \pm k\}$ for $i \in \{1, 2, 3\}$ by

$$f_1(v_{3j-2}) = k, \ \ f_1(v_{3j-1}) = k \ \text{ and } \ f_1(v_{3j}) = -k,$$

$$f_2(v_{3j-2}) = -k, \ \ f_2(v_{3j-1}) = k \ \text{ and } \ f_2(v_{3j}) = k,$$

$$f_3(v_{3j-2}) = k, \ \ f_3(v_{3j-1}) = -k \ \text{ and } \ f_3(v_{3j}) = k$$

for $1 \leq j \leq n/3$. Obviously, $\{f_1, f_2, f_3\}$ is a S$\{k\}$D family on C_n and therefore $d_{\{k\}S}(G) = 3$ in that case.

Now let $n \not\equiv 0 \pmod 3$. We show that $d_{\{k\}S}(C_n) \leq 2$. Suppose to the contrary that $d_{\{k\}S}(C_n) = 3$. Let $\{f_1, f_2, f_3\}$ be a S$\{k\}$D family of C_n. It follows from Theorem 3 that for all vertices v, $\sum_{u \in N[v]} f_i(u) = k$ and $\sum_{i=1}^{d} f_i(v) = k$. We claim that $f_i(v) > 0$ for every $i \in \{1, 2, 3\}$ and each $v \in V(G)$. Suppose to the contrary that $f_i(v) < 0$ for some $i \in \{1, 2, 3\}$ and some $v \in V(G)$. We may assume $f_1(v_1) < 0$. Since $\sum_{u \in N[v]} f_1(u) = k$ for all vertices v, it is easy to verify that $f_1(v_1) = f_1(v_4) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor + 1})$, $f_1(v_2) = f_1(v_5) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor + 2})$ and $f_1(v_3) = f_1(v_6) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor})$. If $n \equiv 2 \pmod 3$, then it follows from $2f_1(v_1) + f_1(v_n) = \sum_{u \in N[v_n]} f_1(u) = k$ that $f_1(v_n) > k$ which is a contradiction. If $n \equiv 1 \pmod 3$, then we obtain $f_1(v_1) = f_1(v_n)$ which leads to the contradiction $\sum_{u \in N[v_n]} f_1(u) < 0$. Thus $f_i(v) > 0$ for every $i \in \{1, 2, 3\}$ and $v \in V(G)$.

Since the f_is are distinct, we may assume that $f_1(v_i) > f_2(v_i) \geq f_3(v_i)$ for some i, say $i = 1$. It follows from $\sum_{i=1}^{d} f_i(v_1) = k$ that $f_1(v_1) \geq k/3$. As above we have $f_1(v_1) = f_1(v_4) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor + 1})$, $f_1(v_2) = f_1(v_5) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor + 2})$ and $f_1(v_3) = f_1(v_6) = \dots = f_1(v_{3\lfloor \frac{n}{3} \rfloor})$. If $n \equiv 2 \pmod 3$ (the case $n \equiv 1 \pmod 3$ is similar) then from $\sum_{u \in N[v_1]} f_1(u) = k$ and $\sum_{u \in N[v_n]} f_1(u) = k$, we deduce that $2f_1(v_1) + f_1(v_n) = f_1(v_1) + 2f_1(v_n) = k$ which implies that $f_1(v_1) = f_1(v_n) = k/3$. It follows that $f_1(v) = k/3$ for each $v \in V(C_n)$. Since $f_2(v_1) < k/3$ and $f_2(v_1) + f_2(v_2) + f_2(v_n) = k$, we may assume, without loss of generality, that $f_2(v_2) > k/3$. An argument similar to that described above implies that $f_2(v) = k/3$ for each $v \in V(C_n)$, a contradiction. Thus

$$d_{\{k\}S}(C_n) \leq 2. \qquad (3)$$

If $k \geq 3$, then the method in Corollary 4 shows that $d_{\{k\}S}(C_n) \geq 2$ and hence $d_{\{k\}S}(C_n) = 2$.

Now let $k = 2$. By (3), $d_{\{k\}S}(C_n) \leq 2$. We show that $d_{\{k\}S}(C_n) \leq 1$. Suppose to the contrary that $d_{\{k\}S}(C_n) = 2$. Let $\{f_1, f_2\}$ be a S$\{k\}$D family of C_n.

Fact 1. $f_i(v_j) \in \{-1, 1, 2\}$ for each $i = 1, 2$ and each $1 \leq j \leq n$.
Suppose to the contrary that $f_i(v_j) = -2$ for some i and j. We may assume, without loss of generality, that $f_1(v_1) = -2$. Since $\sum_{u\in N[v_1]} f_1(u) \geq 2$ and $\sum_{u\in N[v_2]} f_1(u) \geq 2$, we obtain $f_1(v_2) = f_1(v_n) = 2$ and $f_1(v_3) = 2$, respectively. Since $f_1(v_2) + f_2(v_2) \leq 2$ and $f_1(v_3) + f_2(v_3) \leq 2$, we deduce that $f_2(v_2) < 0$ and $f_2(v_3) < 0$. This implies that $\sum_{u\in N[v_2]} f_2(u) \leq 0$ which is a contradiction. Thus $f_i(v_j) \neq -2$ for each i and each j.

Fact 2. For each i, there is no $1 \leq j \leq n$ such that $f_i(v_j) = f_i(v_{j+1}) = 2$, where the sum is taken module n.
Suppose to the contrary that $f_i(v_j) = f_i(v_{j+1}) = 2$ for some i and j. We may assume, without loss of generality, that $f_1(v_1) = f_1(v_2) = 2$. Since $f_1(v_1) + f_2(v_1) \leq 2$ and $f_1(v_2) + f_2(v_2) \leq 2$ we deduce that $f_2(v_1) < 0$ and $f_2(v_2) < 0$. It follows that $\sum_{u\in N[v_1]} f_2(u) \leq 0$ which is a contradiction.

Fact 3. For each i, there is some $1 \leq j \leq n$ such that $f_i(v_j) = 2$.
Suppose to the contrary that $f_i(v_j) < 2$ for some i and each j. We may assume $i = 1$. Since $\sum_{u\in N[v_j]} f_1(u) \geq 2$, we deduce that $f_1(v_j) = 1$ for each j. On the other hand, $f_1(v_j) + f_2(v_j) \leq 2$ implies that $f_2(v_j) < 2$ for each j. Since $\sum_{u\in N[v_j]} f_2(u) \geq 2$, we deduce that $f_2(v_j) = 1$ for each j. Thus $f_1 = f_2$ which is a contradiction.

By Fact 3, we may assume, without loss of generality, that $f_1(v_1) = 2$. Since $f_1(v_1) + f_2(v_1) \leq 2$, we obtain $f_2(v_1) = -1$ by Fact 1. It follows from $\sum_{u\in N[v_1]} f_2(u) \geq 2$ that $f_2(v_2) = 2$ or $f_2(v_n) = 2$. Suppose that $f_2(v_2) = 2$. This implies that $f_1(v_2) = -1$. Since $\sum_{u\in N[v_2]} f_1(u) \geq 2$ and $\sum_{u\in N[v_2]} f_2(u) \geq 2$, we must have $f_1(v_3) \geq 1$ and $f_2(v_3) \geq 1$. It follows from $f_1(v_3) + f_2(v_3) \leq 2$ that $f_1(v_3) = f_2(v_3) = 1$. Since $\sum_{u\in N[v_3]} f_1(u) \geq 2$, we obtain $f_1(v_4) = 2$. If we continue this process we finally arrive at $f_1(v_1) = f_1(v_4) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor+1}) = 2$, $f_1(v_2) = f_1(v_5) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor+2}) = -1$, $f_1(v_3) = f_1(v_6) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor}) = 1$, $f_2(v_1) = f_1(v_4) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor+1}) = -1$, $f_1(v_2) = f_1(v_5) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor+2}) = 2$ and $f_1(v_3) = f_1(v_6) = \ldots = f_1(v_{3\lfloor \frac{n}{3} \rfloor}) = 1$. If $n \equiv 1 \pmod 3$, then we obtain $f_2(v_2) = f_2(v_n) = -1$ which implies that $\sum_{u\in N[v_1]} f_2(u) \leq 0$, a contradiction. If $n \equiv 2 \pmod 3$, then we obtain $f_1(v_1) = f_1(v_n) = 2$ which contradicts Fact 2, and the proof is complete. □

Theorem 8. If $k \geq 2$ and $n \geq 3$ are integers, then $d_{\{k\}S}(K_n) = n$.

Proof. Assume that $\{x_1, x_2, \ldots, x_n\}$ is the vertex set of the complete graph K_n.

First let $n = 2p + 1$ be odd. Define the signed $\{k\}$-dominating functions $f_1, f_2, \ldots, f_n$ by

$$f_i(x_i) = f_i(x_{i+1}) = \ldots = f_i(x_{i+p}) = k$$

and $f_i(x_j) = -k$ otherwise for $i = 1, 2, \ldots, n$, where all numbers are taken modulo n. It is easy to see that $\sum_{v\in V(K_n)} f_i(v) = k$ for $1 \leq i \leq n$ and

$\sum_{i=1}^{n} f_i(v) = k$ for each $v \in V(K_n)$. Hence $\{f_1, f_2, \ldots, f_n\}$ is a S$\{k\}$D family on K_n and therefore $d_{\{k\}S}(K_n) \geq n$. In view of Theorem 3, we see that $d_{\{k\}S}(K_n) \leq n$, and thus $d_{\{k\}S}(K_n) = n$.

Second let $n = 2p \geq 4$ be even. Define the signed $\{k\}$-dominating functions $f_1, f_2, \ldots, f_n$ by $f_i(x_i) = k$, $f_i(x_{i+1}) = 2$, $f_i(x_{i+2}) = f_i(x_{i+3}) = -1$, $f_i(x_{i+2j}) = 1$ and $f_i(x_{i+2j+1}) = -1$ for $i = 1, 2, \ldots, n$ and $2 \leq j \leq p-1$, where the indices are taken modulo n. It is easy to see that $\sum_{v \in V(K_n)} f_i(v) = k$ for $1 \leq i \leq n$ and $\sum_{i=1}^{n} f_i(v) = k$ for each $v \in V(K_n)$. Hence $\{f_1, f_2, \ldots, f_n\}$ is a S$\{k\}$D family on K_n and therefore $d_{\{k\}S}(K_n) \geq n$. In view of Theorem 3, we see that $d_{\{k\}S}(K_n) \leq n$, and thus $d_{\{k\}S}(K_n) = n$. □

If $k = 1$, then Proposition D shows Theorem 8 is only valid in the case that n is odd. If $n = 2$, then it follows from Corollary 4 that Theorem 8 is also valid for $k \geq 3$. Now Proposition D, Theorem 3, Corollaries 4 and 5 and Theorem 8 imply the next result immediately.

Corollary 9. If k is a positive integer and G a graph of order n, then

$$d_{\{k\}S}(G) \leq n,$$

with equality if and only if $k = 1$ and G is isomorphic to the complete graph K_n and n is odd or $k = 2$ and G is isomorphic to the complete graph K_n and $n \neq 2$ or $k \geq 3$ and G is isomorphic to the complete graph K_n.

As a further application of Theorem 3, we will prove the following Nordhaus-Gaddum type result.

Proposition 10. Let G be a graph of order n, minimum degree $\delta(G)$, maximum degree $\Delta(G)$, and let $\overline{G}$ be its complementary graph. Then

$$d_{\{k\}S}(G) + d_{\{k\}S}(\overline{G}) \leq n + \delta(G) - \Delta(G) + 1 \leq n + 1. \tag{4}$$

The equality $d_{\{k\}S}(G) + d_{\{k\}S}(\overline{G}) = n + 1$ implies that G is a regular graph.

Proof. Since $\delta(\overline{G}) = n - \Delta(G) - 1$, it follows from Theorem 3 that

$$d_{\{k\}S}(G) + d_{\{k\}S}(\overline{G}) \leq (\delta(G) + 1) + (n - \Delta(G)) \leq n + 1.$$

If $d_{\{k\}S}(G) + d_{\{k\}S}(\overline{G}) = n + 1$, then $\delta(G) = \Delta(G)$ and G is regular. □

If $k = 1$ and n is odd or $k \geq 2$ and $n \geq 4$, then Proposition D or Theorem 8 impliy that $d_{\{k\}S}(K_n) = n$ and consequently

$$d_{\{k\}S}(K_n) + d_{\{k\}S}(\overline{K_n}) = n + 1.$$

This example demonstrates that Proposition 10 is sharp.

Theorem 11. Let G be a graph of order n with signed $\{k\}$-domination number $\gamma_{\{k\}S}(G)$ and signed $\{k\}$-domatic number $d_{\{k\}S}(G)$. Then

$$\gamma_{\{k\}S}(G) \cdot d_{\{k\}S}(G) \leq nk.$$

Moreover, if $\gamma_{\{k\}S}(G) \cdot d_{\{k\}S}(G) = n$, then for each $d_{\{k\}S}$-family $\{f_1, f_2, \cdots, f_d\}$ on G, each function f_i is a $\gamma_{\{k\}S}$-function and $\sum_{i=1}^{d} f_i(v) = k$ for all $v \in V$.

Proof. Let $\{f_1, f_2, \ldots, f_d\}$ be a S$\{k\}$D family on G such that $d = d_{\{k\}S}(G)$ and let $v \in V$. Then

$$\begin{aligned} d \cdot \gamma_{\{k\}S}(G) &= \textstyle\sum_{i=1}^{d} \gamma_{\{k\}S}(G) \\ &\leq \textstyle\sum_{i=1}^{d} \sum_{v \in V} f_i(v) \\ &= \textstyle\sum_{v \in V} \sum_{i=1}^{d} f_i(v) \\ &\leq \textstyle\sum_{v \in V} k \\ &= nk. \end{aligned}$$

If $\gamma_{\{k\}S}(G) \cdot d_{\{k\}S}(G) = nk$, then the two inequalities occurring in the proof become equalities. Hence for the $d_{\{k\}S}$-family $\{f_1, f_2, \cdots, f_d\}$ on G and for each i, $\sum_{v \in V} f_i(v) = \gamma_{\{k\}S}(G)$, thus each function f_i is a $\gamma_{\{k\}S}$-function, and $\sum_{i=1}^{d} f_i(v) = k$ for all v. □

The upper bound on the product $\gamma_{\{k\}S}(G) \cdot d_{\{k\}S}(G)$ leads to a bound on the sum.

Theorem 12. If $k \geq 1$ is an integer and G a graph of order n, then

$$\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq nk + 1$$

with equality if and only if G is isomorphic to the empty graph or $k = 1$ and G is isomorphic to K_n and n is odd or $k = 1$ and every nonisolated vertex of G is either an endvertex or adjacent to an endvertex.

Proof. Applying Theorem 11, we obtain

$$\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq \frac{kn}{d_{\{k\}S}(G)} + d_{\{k\}S}(G). \tag{5}$$

Theorem 3 implies that $1 \leq d_{\{k\}S}(G) \leq n$. Using these inequalities, and the fact that the function $g(x) = x + (kn)/x$ is decreasing for $1 \leq x \leq \sqrt{kn}$ and increasing for $\sqrt{kn} \leq x \leq n$, we deduce the desired bound as follows

$$\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq \max\left\{kn + 1, \frac{kn}{n} + n\right\} = nk + 1.$$

If G is isomorphic to the empty graph, then $\gamma_{\{k\}S}(G) = kn$ and $d_{\{k\}S}(G) = 1$ and thus $\gamma_{\{k\}S}(S) + d_{\{k\}S}(G) = nk + 1$. If $k = 1$ and $G = K_n$ where n is odd or $k = 1$ and every nonisolated vertex of G is either an endvertex or adjacent to an endvertex, then $\gamma_S(G) + d_S(G) = n + 1$ by Proposition B.

Conversely, assume that G is not the empty graph, $G \neq K_n$ when $k = 1$ and n odd, and that not every nonisolated vertex of G is either an endvertex or adjacent to an endvertex when $k = 1$. If $k = 1$, then it follows from Proposition B that $\gamma_S(G) + d_S(G) \leq n$. Thus we assume that $k \geq 2$. Since G is not the empty graph, there exists an edge $vw \in E(G)$. Now define $f : V(G) \to \{\pm 1, \pm 2, \ldots, \pm k\}$ by $f(w) = k-1, f(v) = 1$ and $f(x) = k$ for $x \in V(D) \setminus \{v, w\}$. Then f is a signed $\{k\}$-dominating function on G and hence $\gamma_{\{k\}S}(G) \leq k(n-1)$.

If $d_{\{k\}S}(G) = 1$, then $\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq k(n-1) + 1 \leq kn$.

Assume next that $d_{\{k\}S}(G) \geq 2$. Using these facts and inequality (5), we obtain

$$\begin{aligned} \gamma_{\{k\}S}(G) + d_{\{k\}S}(G) &\leq \frac{kn}{d_{\{k\}S}(G)} + d_{\{k\}S}(G) \\ &\leq \max\left\{\frac{kn}{2} + 2, \frac{kn}{n} + n\right\} \\ &= \max\left\{\frac{kn}{2} + 2, k + n\right\} \\ &= \frac{kn}{2} + 2 \leq kn. \end{aligned}$$

This completes the proof. □

Corollary 13. Let G be a graph of order n and $k \geq 1$ an integer. If

$$\min\{\gamma_{\{k\}S}(G), d_{\{k\}S}(G)\} \geq a,$$

with $2 \leq a \leq \sqrt{nk}$, then

$$\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq a + \frac{nk}{a}.$$

Proof. Since $\min\{\gamma_{\{k\}S}(G), d_{\{k\}S}(G)\} \geq a$, it follows from Theorem 11 that $a \leq d_{\{k\}s}(G) \leq \frac{nk}{a}$. According to Theorem 11, we obtain

$$\gamma_{\{k\}S}(G) + d_{\{k\}S}(G) \leq d_{\{k\}S}(G) + \frac{nk}{d_{\{k\}S}(G)}.$$

The bound results from the facts that the function $g(x) = x + (nk)/x$ is decreasing for $1 \leq x \leq \sqrt{nk}$ and increasing for $\sqrt{nk} \leq x \leq nk$. □

References

[1] E.J. Cockayne and C.M. Mynhardt, *On a generalisation of signed dominating functions of a graph*, Ars Combin. **43** (1996), 235–245.

[2] J. Dunbar, S.T. Hedetniemi, M.A. Henning and P.J. Slater, *Signed domination in graphs*, Graph Theory, Combinatorics, and Applications, Vol. 1, Wiley, New York, 1995, 311–322.

[3] O. Favaron, *Signed domination in regular graphs*, Discrete Math. **158** (1996), 287–293.

[4] D. Meierling, L. Volkmann and S. Zitzen, *The signed domatic number of some regular graphs*, Discrete Appl. Math. **157** (2009), 1905–1912.

[5] L. Volkmann, *Signed domatic numbers of the complete bipartite graphs*, Util. Math. **68** (2005), 71–77.

[6] L. Volkmann, *Some remarks on the signed domatic number of graphs with small minimum degree*, Applied Math. Lett. **22** (2009), 1166–1169.

[7] L. Volkmann, *Bounds on the signed domatic number*, Applied Math. Lett. **24** (2011), 196–198.

[8] L. Volkmann and B. Zelinka, *Signed domatic number of a graph*, Discrete Appl. Math. **150** (2005), 261–267.

[9] C.P. Wang, *The signed k-domination numbers in graphs*, Ars Combin. (to appear).

[10] D.B. West, *Introduction to Graph Theory*, Prentice-Hall, Inc, 2000.