A result on fractional ID-k-factor-critical graphs *†

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Abstract

Let G be a simple graph of order n, and let k be a positive integer. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) if $G-I$ has a fractional k-factor for every independent set I of G. In this paper, we obtain a sufficient condition for a graph G to be fractional ID-k-factor-critical. Furthermore, it is shown that the result in this paper is best possible in some sense.

Keywords: graph, degree condition, independent set, fractional k-factor, fractional ID-k-factor-critical graph.

AMS(2010) Mathematics Subject Classification: 05C70

1 Introduction

Many physical structures can conveniently be modelled by networks. Examples include a communication network with the nodes and links modelling cities and communication channels, respectively, or a railroad network with nodes and links representing railroad stations and railways between two stations, respectively. Factors and factorizations in networks are very useful

*Manuscript Received Date: Oct. 19, 2011

†This research was supported by Natural Science Foundation of the Higher Education Institutions of Jiangsu Province (10KJB110003) and Jiangsu University of Science and Technology (2010SL101J, 2009SL154J), and was sponsored by Qing Lan Project of Jiangsu Province.

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in combinatorial design, network design, circuit layout, and so on. It is well known that a network can be represented by a graph. Vertices and edges of the graph correspond to nodes and links between the nodes, respectively. Henceforth we use the term *graph* instead of *network*.

We investigate the fractional factor problem in graphs, which can be considered as a relaxations of the well-known cardinality matching problem. The fractional factor problem has wide-range applications in areas such as network design, scheduling and combinatorial polyhedra. For instance, in a communication network if we allow several large data packets to be sent to various destinations through several channels, the efficiency of the network will be improved if we allow the large data packets to be partitioned into small parcels. The feasible assignment of data packets can be seen as a fractional flow problem and it becomes a fractional matching problem when the destinations and sources of a network are disjoint (i.e., the underlying graph is bipartite).

The graphs considered in this paper will be finite undirected graphs without loops or multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For each $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the neighborhood of x in G, and we write $N_G[x]$ for $N_G(x) \cup \{x\}$. For $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. If $G[S]$ has no edges, then we call S independent. We use $\delta(G)$ to denote the minimum degree of G. If G_1 and G_2 are disjoint graphs, the join and the union are denoted by $G_1 \vee G_2$ and $G_1 \cup G_2$, respectively. The other terminologies and notations not given here can be found in [1].

Let k be an integer with $k \geq 1$. Then a k-factor of G is a spanning subgraph F of G such that $d_F(x) = k$ for each $x \in V(G)$. A graph G is factor-critical [2] if $G - v$ has a 1-factor for each $v \in V(G)$. In [3], the concept of the factor-critical graph was generalized to the ID-factor-critical graph. We say that G is independent-set-deletable factor-critical (shortly, ID-factor-critical) if for every independent set I of G which has the same parity with $|V(G)|$, $G - I$ has a perfect matching. It is obvious that every ID-factor-critical graph with odd number of vertices is factor-critical.

Let $h : E(G) \to [0,1]$ be a function. If $\sum_{e \ni x} h(e) = k$ holds for any $x \in V(G)$, then we call $G[F_h]$ a fractional k-factor of G with indicator function h where $F_h = \{e \in E(G) : h(e) > 0\}$ and $e \ni x$ denotes the edges adjacent to x in G. A graph G is fractional independent-set-deletable k-factor-critical (in short, fractional ID-k-factor-critical) if $G - I$ has a fractional k-factor for every independent set I of G. If $k = 1$, then a fractional ID-k-factor-critical graph is called a fractional ID-factor-critical graph.

Many authors have investigated factors [4–10] and fractional factors [11-15]. Chang, Liu and Zhu [16] obtained a minimum degree condition for a graph to be a fractional ID-k-factor-critical graph.

Theorem 1 [16] *Let k be a positive integer and G be a graph of order n with $n \geq 6k-8$. If $\delta(G) \geq \frac{2n}{3}$, then G is fractional ID-k-factor-critical.*

In this paper, we proceed to investigate fractional ID-k-factor-critical graphs, and obtain a new degree condition for fractional ID-k-factor-critical graphs. Our main result is the following theorem, which is an improvement of Theorem 1.

Theorem 2 *Let $k \geq 1$ be an integer, and let G be a graph of order n with $n \geq 6k-2$, and $\delta(G) \geq \frac{n}{3}+k$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{2n}{3}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-k-factor-critical.

If $k=1$ in Theorem 2, then we obtain the following corollary.

Corollary 1 *Let G be a graph of order n with $n \geq 4$, and $\delta(G) \geq \frac{n}{3}+1$. If G satisfies*

$$\max\{d_G(x), d_G(y)\} \geq \frac{2n}{3}$$

for each pair of nonadjacent vertices x, y of G, then G is fractional ID-factor-critical.

2 The Proof of Theorem 2

The next result by Liu and Zhang [17] is the main tool for the proof of Theorem 2.

Lemma 2.1 [17] *Let G be a graph. Then G has a fractional k-factor if and only if for every subset S of $V(G)$,*

$$\delta_G(S,T) = k|S| + d_{G-S}(T) - k|T| \geq 0,$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq k-1\}$.

Proof of Theorem 2. Let X be an independent set of G and $H = G - X$. For the proof of Theorem 2, it is sufficient to show that H has a fractional k-factor. We will prove it, by contradiction. Suppose that H has no fractional k-factor. Then by Lemma 2.1, there exists some subset $S \subseteq V(H)$ such that

$$\delta_H(S,T) = k|S| + d_{H-S}(T) - k|T| \leq -1, \tag{1}$$

where $T = \{x : x \in V(H) \setminus S, d_{H-S}(x) \leq k-1\}$.

Using (1) and $H = G - X$, we obtain

$$\delta_H(S,T) = k|S| + d_{G-X-S}(T) - k|T| \leq -1, \tag{2}$$

where $T = \{x : x \in V(G) \setminus (X \cup S), d_{G-X-S}(x) \leq k-1\}$. Firstly, we prove the following claim.

Claim 1. $|X| \leq \frac{n}{3}$.

Proof. The inequality obviously holds for $|X| = 1$. In the following we may assume $|X| \geq 2$. According to the conditions of Theorem 2, there exists $x \in X$ such that $d_G(x) \geq \frac{2n}{3}$. Thus, we obtain

$$|X| + \frac{2n}{3} \leq |X| + d_G(x) \leq n,$$

that is,

$$|X| \leq \frac{n}{3}.$$

This completes the proof of Claim 1.

If $T = \emptyset$, then by (2), $-1 \geq \delta_H(S,T) = k|S| \geq 0$, a contradiction. Hence, $T \neq \emptyset$. Define

$$h_1 = \min\{d_{G-X-S}(x) : x \in T\},$$

and choose $x_1 \in T$ such that $d_{G-X-S}(x_1) = h_1$. According to the definition of T, we have $0 \leq h_1 \leq k-1$.

Case 1. $T = N_T[x_1]$.

In this case, we obtain

$$|T| = |N_T[x_1]| \leq d_{G-X-S}(x_1) + 1 = h_1 + 1. \tag{3}$$

From (3) and $0 \leq h_1 \leq k-1$, we get

$$|T| \leq h_1 + 1 \leq k. \tag{4}$$

In view of $\delta(G) \geq \frac{n}{3} + k$, Claim 1 and $\delta(G) \leq d_G(x_1) \leq d_{G-X-S}(x_1) + |X| + |S| = h_1 + |X| + |S|$, we have

$$|S| \geq \delta(G) - |X| - h_1 \geq \frac{n}{3} + k - |X| - h_1 \geq k - h_1.$$

Combining this with (4) and $0 \leq h_1 \leq k-1$, we obtain

$$\begin{aligned}\delta_H(S,T) &= k|S| + d_{G-X-S}(T) - k|T| \\ &\geq k|S| + h_1|T| - k|T| = k|S| - (k-h_1)|T| \\ &\geq k(k-h_1) - (k-h_1)|T| = (k-h_1)(k-|T|) \geq 0,\end{aligned}$$

which contradicts (2).

Case 2. $T \setminus N_T[x_1] \neq \emptyset$.

Define

$$h_2 = \min\{d_{G-X-S}(x) : x \in T \setminus N_T[x_1]\},$$

and choose $x_2 \in T \setminus N_T[x_1]$ such that $d_{G-X-S}(x_2) = h_2$. Thus, we get $0 \leq h_1 \leq h_2 \leq k-1$ by the definition of T and $d_G(x_i) \leq d_{G-X-S}(x_i) + |X| + |S| = h_i + |X| + |S|$ for $i = 1, 2$.

Since $T \setminus N_T[x_1] \neq \emptyset$, we have

$$h_2 + |X| + |S| \geq \frac{2n}{3}. \tag{5}$$

Otherwise we obtain $h_1 + |X| + |S| \leq h_2 + |X| + |S| < \frac{2n}{3}$, and this implies $d_G(x_1) < \frac{2n}{3}$ and $d_G(x_2) < \frac{2n}{3}$. Since $x_1x_2 \notin E(G)$, that would contradict the assumption of Theorem 2.

Note that $|N_T[x_1]| \leq d_{G-X-S}(x_1) + 1 = h_1 + 1$. According to (2), (5), $|X| + |S| + |T| \leq n$, $0 \leq h_1 \leq h_2 \leq k-1$ and $n \geq 6k-2$, we obtain

$$\begin{aligned}-1 &\geq \delta_H(S,T) = k|S| + d_{G-X-S}(T) - k|T| \\ &\geq k|S| + h_1|N_T[x_1]| + h_2(|T| - |N_T[x_1]|) - k|T| \\ &= k|S| + (h_1 - h_2)|N_T[x_1]| + (h_2 - k)|T| \\ &\geq k|S| + (h_1 - h_2)(h_1 + 1) + (h_2 - k)(n - |X| - |S|) \\ &= (2k - h_2)|S| + (h_1 - h_2)(h_1 + 1) - (k - h_2)n + (k - h_2)|X| \\ &\geq (2k - h_2)(\frac{2n}{3} - h_2 - |X|) + (h_1 - h_2)(h_1 + 1) \\ &\quad -(k - h_2)n + (k - h_2)|X| \\ &= (\frac{h_2 - 1}{2} - h_1)^2 + \frac{3}{4}h_2^2 + (\frac{n}{3} - 2k - \frac{1}{2})h_2 - \frac{1}{4} + (\frac{n}{3} - |X|)k \\ &\geq \frac{3}{4}h_2^2 + (\frac{n}{3} - 2k - \frac{1}{2})h_2 - \frac{1}{4}\end{aligned}$$

$$\begin{aligned}&\geq \frac{3}{4}h_2^2+(\frac{6k-2}{3}-2k-\frac{1}{2})h_2-\frac{1}{4}\\&\geq \frac{3}{4}h_2^2-\frac{7}{6}h_2-\frac{1}{4},\end{aligned}$$

that is,

$$-1\geq \frac{3}{4}h_2^2-\frac{7}{6}h_2-\frac{1}{4}. \tag{6}$$

Let $f(h_2)=\frac{3}{4}h_2^2-\frac{7}{6}h_2-\frac{1}{4}$. Since h_2 is a nonnegative integer with $0\leq h_2\leq k-1$, the function $f(h_2)$ attains its minimum value at $h_2=1$. From (6), we have

$$-1\geq f(h_2)\geq f(1)=\frac{3}{4}-\frac{7}{6}-\frac{1}{4}=-\frac{2}{3}>-1.$$

It is a contradiction.

From the contradictions obtained in all cases we can conclude that H has a fractional k-factor and hence G is fractional ID-k-factor-critical.

Remark 1. Let us show that the condition $\max\{d_G(x),d_G(y)\}\geq \frac{2n}{3}$ in Theorem 2 is sharp. To see this, we construct a graph $G=((kt)K_1\vee(kt)K_1)\vee(kt+1)K_1$, where t is sufficiently large positive integer. Obviously, $|V(G)|=n=3kt+1$ and $\delta(G)=2kt>\frac{n}{3}+k$ and

$$\frac{2n}{3}>\max\{d_G(x),d_G(y)\}=2kt=\frac{2n-2}{3}>\frac{2n}{3}-1$$

for each pair of nonadjacent vertices x,y of $(kt+1)K_1\subset G$. Let $X=(kt)K_1$. Obviously, X is an independent set of G. Set $H=G-X=(kt)K_1\vee(kt+1)K_1$, $S=(kt)K_1$ and $T=(kt+1)K_1$. Then $|S|=kt$, $|T|=kt+1$ and $d_{H-S}(T)=0$. Thus we have

$$\begin{aligned}\delta_H(S,T)&= k|S|+d_{H-S}(T)-k|T|\\&= k^2t-k(kt+1)=-k<0.\end{aligned}$$

By Lemma 2.1, H has no fractional k-factor. Hence, G is not fractional ID-k-factor-critical. In the sense above, the result in Theorem 2 is best possible.

Remark 2. Let us show that the condition $\delta(G)\geq\frac{n}{3}+k$ in Theorem 2 cannot be replaced by $\delta(G)\geq\frac{n}{3}+k-1$. Consider a graph G constructed from ktK_1, $(kt-1)K_1$, $\frac{kt}{2}K_2$ and K_1 as follows: let $\{x_1,x_2,\cdots,x_{k-1}\}\subset(kt-1)K_1$ and $K_1=\{u\}$, where t is sufficiently large positive integer and kt is even. Set $V(G)=V(ktK_1\cup(kt-1)K_1\cup\frac{kt}{2}K_2\cup\{u\})$ and $E(G)=E(ktK_1\vee(kt-1)K_1\vee\frac{kt}{2}K_2)\cup E(ktK_1\vee\{u\})\cup\{ux_i:i=1,2,\cdots,k-1\}$.

It is easily seen that $\max\{d_G(x), d_G(y)\} \geq \frac{2n}{3}$ for each pair of nonadjacent vertices x, y of G, $n = 3kt \geq 6k - 2$ and $\delta(G) = \frac{n}{3} + k - 1$. Let $X = ktK_1$. Obviously, X is an independent set of G. Set $H = G - X$, then $\delta(H) = d_H(u) = k - 1$. Obviously, H has no fractional k-factor, that is, G is not fractional ID-k-factor-critical. In the sense above, the condition $\delta(G) \geq \frac{n}{3} + k$ in Theorem 2 is sharp.

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