Graph designs for 6-circle with two pendant edges *

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Abstract. Let λK_v be the complete multigraph of order v and index λ, where any two distinct vertices x and y are joined exactly by λ edges $\{x,y\}$. Let G be a finite simple graph. A G-design of λK_v, denoted by (v,G,λ)-GD, is a pair $(X,\mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of λK_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. There are four graphs which is a 6-circle with two pendant edges, denoted by $G_i, i = 1,2,3,4$. In [9], we have solved the existence problems of $(v,G_i,1)$-GD. In this paper, we obtain the existence spectrum of (v,G_i,λ)-GD for any $\lambda > 1$.

Keywords: G-design, G-holey design, G-incomplete design.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by exactly λ edges $\{x,y\}$. A *t-partite graph* is one whose vertex set can be partitioned into t subsets $X_1, X_2, \cdots, X_t$, such that two ends of each edge lie

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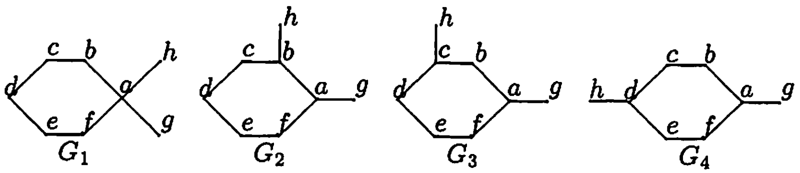
in distinct subsets respectively. Such a partition $(X_1, X_2, \cdots, X_t)$ is called a *t-partition* of the graph. A *complete t-partite graph* with replication λ is a t-partite graph with t-partition $(X_1, X_2, \cdots, X_t)$, in which each vertex of X_i is joined to each vertex of X_j by λ times (where $i \neq j$). Such a graph is denoted by $\lambda K_{n_1,n_2,\cdots,n_t}$ if $|X_i| = n_i$ $(1 \leq i \leq t)$.

Let $(X_1, X_2, \cdots, X_t)$ be the t-partition of $K_{n_1,n_2,\cdots,n_t}$, and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G}=\{X_1, X_2, \cdots, X_t\}$. For any given graph G, if the edges of $\lambda K_{n_1,n_2,\cdots,n_t}$ can be decomposed into subgraphs $\mathcal{A}$, each of which is isomorphic to G (called *block*), then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by G-$HD_\lambda(T)$, where $T = n_1^1 n_2^1 \cdots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \cdots$ denotes i occurrences of 1, r occurrences of 2, etc. A G-$HD_\lambda(1^{v-w}w^1)$ is called an *incomplete G-design*, denoted by G-$ID_\lambda(v; w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$. Obviously, a (v, G, λ)-GD is a G-$HD_\lambda(1^v)$ or a G-$ID_\lambda(v; w)$ with $w = 0$ or 1.

Let G be a finite simple graph. A *G-design* of λK_v, denoted by (v, G, λ)-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. It is well known that if there exists a (v, G, λ)-GD, then

$$\lambda v(v-1) \equiv 0 \pmod{2e(G)}, \text{ and } \lambda(v-1) \equiv 0 \pmod{d},$$

where $e(G)$ denotes the number of edges in G, and d is the greatest common divisor of the degrees of the vertices of G. For the path P_k and the star $K_{1,k}$, the existence problems of their graph designs have been solved (see [1] and [2]). For the graphs which have fewer vertices and fewer edges, the problem of their graph designs has already been researched (see [3]-[8]). There are four graphs which is a 6-circle with two pendant edges, denoted by $G_i, i = 1, 2, 3, 4$. In [9], we have solved the existence problems of $(v, G_i, 1)$-GD. In this paper, we obtain the existence spectrum of (v, G_i, λ)-GD for $\lambda > 1$. The four graphs G_i are drawn as follows.



For convenience, the graphs G_1-G_4 above are denoted by (a, b, c, d, e, f, g, h).

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, m, λ and nonnegative integers w, if there exist G-$HD_\lambda(h^m)$, G-$ID_\lambda(h+w; w)$ and (w, G, λ)-GD (or $(h+w, G, \lambda)$-GD), then there exists $(mh+w, G, \lambda)$-GD.*

Proof. Let $X = (Z_h \times Z_m) \cup W$, where W is a w-set. Suppose there exist

G-$HD_\lambda(h^m) = (Z_h \times Z_m, \mathcal{A})$,

G-$ID_\lambda(h+w; w) = ((Z_h \times \{i\}) \cup W, \mathcal{B}_i)$, $i \in Z_m$ or $i \in Z_m \setminus \{0\}$,

and (w, G, λ)-$GD = (W, \mathcal{C})$ or $(h+w, G, \lambda)$-$GD = ((Z_h \times \{0\}) \cup W, \mathcal{D})$,

then (X, Ω) is a $(mh+w, G, \lambda)$-GD, where

$$\Omega = \mathcal{A} \cup (\bigcup_{i=0}^{m-1} \mathcal{B}_i) \cup \mathcal{C} \quad \text{or} \quad \mathcal{A} \cup (\bigcup_{i=1}^{m-1} \mathcal{B}_i) \cup \mathcal{D}.$$

Note that $|\Omega| = \frac{\lambda\binom{mh+w}{2}}{e(G)}$

$$= \begin{cases} \frac{\lambda\binom{m}{2}h^2}{e(G)} + \frac{\lambda m(\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w}{2}}{e(G)} \\ \frac{\lambda\binom{m}{2}h^2}{e(G)} + \frac{\lambda(m-1)(\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w+h}{2}}{e(G)} \end{cases} = \begin{cases} |\mathcal{A}| + \sum_{i=0}^{m-1} |\mathcal{B}_i| + |\mathcal{C}| \\ |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{D}| \end{cases}. \square$$

The necessary conditions for the existence of (v, G_i, λ)-GD are $\lambda v(v-1) \equiv 0 \pmod{16}$ and $v \geq 8$, that is

$$\begin{cases} v \equiv 0, 1 \pmod{16} & \text{any } \lambda, \\ v \equiv 8, 9 \pmod{16} & \lambda \equiv 0 \pmod{2}, \\ v \equiv 4, 5 \pmod{8} & \lambda \equiv 0 \pmod{4}, \\ v \equiv 2, 3, 6, 7 \pmod{8} & \lambda \equiv 0 \pmod{8}. \end{cases}$$

Lemma 2.2 *Let G be a simple graph. For positive integer m, if there exists a (v, G, λ)-GD, then there exists a $(v, G, m\lambda)$-GD.*

Proof. Let each block in (v, G, λ)-GD repeats m times. □

Lemma 2.3 [9] *There exist $(v, G_i, 1)$-GD if and only if $v \equiv 0, 1 \pmod{16}$ and $v \geq 8$, where $i = 1, 2, 3, 4$.*

Theorem 2.4 *For $v \equiv 0, 1 \pmod{16}$ and any $\lambda \geq 1$, there exist (v, G_i, λ)-GD, where $i = 1, 2, 3, 4$.*

Proof. By Lemma 2.2 and Lemma 2.3. □

Lemma 2.5 *There exist G_i-$HD(8^t)$ for $t \geq 2$, where $i = 1, 2, 3, 4$.*

Proof. In [9], we got G_i-$HD(8^2)$. So there exist G_i-$HD(8^t)$. □

By Theorem 2.1 and the following tables, considering the existence of the needed HD (see Lemma 2.5), we only need to give the constructions of ID and GD for the pointed orders in the Table 2.1.

(Table 2.1) For G_1, G_2, G_3 and G_4

$v \equiv$	λ	HD	ID	GD
2 (mod 8)	8	8^t	(10; 2)	10
3 (mod 8)	8	8^t	(11; 3)	11
4 (mod 8)	4	8^t	(12; 4)	12
5 (mod 8)	4	8^t	(13; 5)	13
6 (mod 8)	8	8^t	(14; 6)	14
7 (mod 8)	8	8^t	(15; 7)	15
8 (mod 8)	2	8^{2t+1}		8
9 (mod 8)	2	8^{2t+1}		9

3 Graph designs

Lemma 3.1 *There exist (w, G_1, λ)-GD for*

(i) $\lambda = 2$ *and* $w = 8, 9$, *(ii)* $\lambda = 4$ *and* $w = 12, 13$,

(iii) $\lambda = 8$ *and* $w = 10, 11, 14, 15$.

Proof. $\underline{\lambda = 2,\ w = 8:}$ $X = Z_7 \cup \{\infty\}$, $(5, 1, 3, \infty, 0, 2, 4, 6)$ mod 7

$\underline{\lambda = 2,\ w = 9:}$ $X = Z_9$, $(0, 1, 2, 4, 8, 3, 6, 7)$ mod 9

$\underline{\lambda = 4,\ w = 12:}$ $X = Z_{11} \cup \{\infty\}$

$(0, \infty, 8, 2, 4, 7, 6, 1)$ $(0, 5, 7, 3, \infty, 1, 8, 4)$ $(0, 1, 2, 4, 6, 3, 5, 7)$ mod 11

$\underline{\lambda = 4,\ w = 13:}$ $X = Z_{13}$

$(0, 1, 2, 3, 4, 9, 6, 10)$ $(0, 2, 4, 6, 11, 5, 9, 7)$ $(0, 3, 6, 8, 12, 7, 9, 10)$ mod 13

$\underline{\lambda = 8,\ w = 10:}$ $X = Z_9 \cup \{\infty\}$

$(0,\infty,1,2,3,4,5,8)$ $(0,\infty,1,3,5,2,8,4)$ $(0,\infty,1,4,7,3,2,6)$
$(0,\infty,1,4,8,7,2,3)$ $(0,1,3,6,2,4,8,5)$ mod 9

$\underline{\lambda = 8,\ w = 11:}$ $X = Z_{11}$

$(0,1,2,3,4,5,6,7)$ $(0,2,4,6,8,10,3,9)$ $(0,3,6,9,1,4,2,10)$
$(0,4,8,1,5,9,6,7)$ $(0,5,10,4,9,3,8,1)$ mod 11

$\underline{\lambda = 8,\ w = 14:}$ $X = Z_{13} \cup \{\infty\}$

$(0,1,2,3,4,5,9,10)$ $(0,2,4,6,8,10,5,9)$ $(0,\infty,1,4,7,10,5,9)$
$(0,\infty,1,5,9,2,10,12)$ $(0,\infty,1,6,11,7,4,12)$ $(0,\infty,1,7,2,8,5,11)$
$(0,6,12,5,11,1,2,4)$ mod 13

$\underline{\lambda = 8,\ w = 15:}$ $X = Z_{15}$

$(0,1,2,3,4,5,6,10)$ $(0,2,4,6,8,10,3,9)$ $(0,3,6,9,12,5,1,14)$
$(0,4,8,12,1,5,2,3)$ $(0,5,10,4,9,2,3,6)$ $(0,6,12,3,10,2,4,11)$
$(0,7,14,6,13,9,1,3)$ mod 15 □

Lemma 3.2 *There exist (w, G_2, λ)-GD for*

(i) $\lambda = 2$ *and* $w = 8, 9$, *(ii)* $\lambda = 4$ *and* $w = 12, 13$,

(iii) $\lambda = 8$ *and* $w = 10, 11, 14, 15$.

Proof. $\underline{\lambda = 2,\ w = 8:}$ $X = Z_7 \cup \{\infty\}$, $(0,1,3,6,5,\infty,2,4)$ mod 7

$\underline{\lambda = 2,\ w = 9:}$ $X = Z_9$, $(2,0,1,3,6,7,5,4)$ mod 9

$\underline{\lambda = 4,\ w = 12:}$ $X = Z_{11} \cup \{\infty\}$

$(7,0,\infty,8,2,4,3,1)$ $(5,7,3,\infty,1,0,9,2)$ $(0,1,2,4,6,3,5,9)$ mod 11

$\underline{\lambda = 4,\ w = 13:}$ $X = Z_{13}$

$(0,1,2,3,4,9,6,11)$ $(0,2,4,6,11,5,9,8)$ $(0,7,12,8,6,3,9,4)$ mod 13

$\underline{\lambda = 8,\ w = 10:}$ $X = Z_9 \cup \{\infty\}$

$(4,0,\infty,1,2,3,5,8)$ $(2,0,\infty,1,3,5,6,4)$ $(3,0,\infty,1,4,7,6,2)$
$(7,0,\infty,1,4,8,5,3)$ $(0,1,3,6,2,4,8,5)$ mod 9

$\underline{\lambda = 8,\ w = 11:}$ $X = Z_{11}$

$(5,0,1,2,3,4,10,7)$ $(0,2,4,6,8,10,9,5)$ $(0,3,6,9,1,4,10,5)$
$(0,4,8,1,5,9,7,10)$ $(0,5,10,4,9,3,1,2)$ mod 11

$\underline{\lambda = 8,\ w = 14:}$ $X = Z_{13} \cup \{\infty\}$

$(0,1,2,3,4,5,9,10)$ $(0,2,4,6,8,10,9,7)$ $(10,0,\infty,1,4,7,9,5)$
$(2,0,\infty,1,5,9,7,4)$ $(7,0,\infty,1,6,11,5,4)$ $(8,0,\infty,1,7,2,9,3)$
$(0,6,12,5,11,1,3,4)$ mod 13

$\underline{\lambda = 8,\ w = 15:}$ $X = Z_{15}$

$(0,1,2,3,4,5,6,7)$ $(0,2,4,6,8,10,3,5)$ $(0,3,6,9,12,5,2,7)$
$(0,4,8,12,1,5,3,10)$ $(0,5,10,4,9,2,1,8)$ $(0,6,12,3,10,2,1,11)$
$(0,7,14,6,13,9,1,3)$ mod 15 □

Lemma 3.3 *There exist* (w, G_3, λ)*-GD for*

(*i*) $\lambda = 2$ *and* $w = 8, 9$, (*ii*) $\lambda = 4$ *and* $w = 12, 13$,

(*iii*) $\lambda = 8$ *and* $w = 10, 11, 14, 15$.

Proof. $\underline{\lambda = 2,\ w = 8:}$ $X = Z_7 \cup \{\infty\}$, $(6,4,5,1,0,\infty,2,3)$ mod 7

$\underline{\lambda = 2,\ w = 9:}$ $X = Z_9$, $(0,1,2,4,8,3,7,5)$ mod 9

$\underline{\lambda = 4,\ w = 12:}$ $X = Z_{11} \cup \{\infty\}$

$(8,\infty,0,7,4,2,1,3)$ $(5,7,3,\infty,1,0,4,10)$ $(0,1,2,4,6,3,5,7)$ mod 11

$\underline{\lambda = 4,\ w = 13:}$ $X = Z_{13}$

$(0,1,2,3,4,9,10,8)$ $(0,2,4,6,11,5,9,10)$ $(0,3,6,8,12,7,10,2)$ mod 13

$\underline{\lambda = 8,\ w = 10:}$ $X = Z_9 \cup \{\infty\}$

$(0,\infty,1,2,3,4,8,5)$ $(0,\infty,1,3,5,2,8,4)$ $(0,\infty,1,4,7,3,5,2)$
$(0,\infty,1,4,8,7,6,3)$ $(0,1,3,6,2,4,7,8)$ mod 9

$\underline{\lambda = 8,\ w = 11:}$ $X = Z_{11}$

$(0,1,2,3,4,5,6,9)$ $(0,2,4,6,8,10,9,7)$ $(0,3,6,9,1,4,2,7)$
$(0,4,8,1,5,9,7,2)$ $(0,5,10,4,9,3,1,7)$ mod 11

$\underline{\lambda = 8,\ w = 14:}$ $X = Z_{13} \cup \{\infty\}$

$(0,1,2,3,4,5,9,11)$ $(0,2,4,6,8,10,5,3)$ $(0,\infty,1,4,7,10,5,2)$
$(0,\infty,1,5,9,2,4,3)$ $(0,\infty,1,6,11,7,5,4)$ $(0,\infty,1,7,2,8,4,3)$
$(0,6,12,5,11,1,3,8)$ mod 13

$\underline{\lambda = 8,\ w = 15:}$ $X = Z_{15}$

$(0,1,2,3,4,5,6,8)$ $(0,2,4,6,8,10,1,7)$ $(0,3,6,9,12,5,1,2)$
$(0,4,8,12,1,5,2,3)$ $(0,5,10,4,9,2,3,1)$ $(0,6,12,3,10,2,4,9)$
$(0,7,14,6,13,9,1,11)$ mod 15 □

Lemma 3.4 *There exist* (w, G_4, λ)*-GD for*

(*i*) $\lambda = 2$ *and* $w = 8, 9$, (*ii*) $\lambda = 4$ *and* $w = 12, 13$,

(*iii*) $\lambda = 8$ *and* $w = 10, 11, 14, 15$.

Proof. $\underline{\lambda = 2,\ w = 8:}$ $X = Z_7 \cup \{\infty\}$, $(2,1,4,5,0,\infty,6,3)$ mod 7

$\underline{\lambda = 2,\ w = 9:}$ $X = Z_9$, $(2,0,1,3,6,7,5,8)$ mod 9

$\underline{\lambda = 4,\ w = 12:}$ $X = Z_{11} \cup \{\infty\}$

$(0,\infty,8,2,4,7,5,10)$ $(0,5,7,3,\infty,1,6,10)$ $(0,1,2,4,6,3,7,5)$ mod 11

$\underline{\lambda = 4,\ w = 13:}$ $X = Z_{13}$

$(0,1,2,3,4,9,7,6)$ $(0,2,4,6,11,5,7,10)$ $(0,3,6,8,12,7,9,5)$ mod 13

$\underline{\lambda = 8,\ w = 10:}$ $X = Z_9 \cup \{\infty\}$

$(0,\infty,1,2,3,4,8,6)$ $(0,\infty,1,3,5,2,8,7)$ $(0,\infty,1,4,7,3,6,2)$
$(0,\infty,1,4,8,7,3,2)$ $(0,1,3,6,2,4,5,7)$ mod 9

$\underline{\lambda=8,\ w=11:}$ $X=Z_{11}$

$(0,1,2,3,4,5,6,7)$ $(0,2,4,6,8,10,9,3)$ $(0,3,6,9,1,4,2,8)$
$(0,4,8,1,5,9,7,6)$ $(0,5,10,4,9,3,1,7)$ mod 11

$\underline{\lambda=8,\ w=14:}$ $X=Z_{13}\cup\{\infty\}$

$(0,1,2,3,4,5,8,7)$ $(0,2,4,6,8,10,9,3)$ $(0,\infty,1,4,7,10,9,8)$
$(0,\infty,1,5,9,2,4,8)$ $(0,\infty,1,6,11,7,5,8)$ $(0,\infty,1,7,2,8,5,6)$
$(0,6,12,5,11,1,2,4)$ mod 13

$\underline{\lambda=8,\ w=15:}$ $X=Z_{15}$

$(0,1,2,3,4,5,9,6)$ $(0,2,4,6,8,10,9,12)$ $(0,3,6,9,12,5,1,10)$
$(0,4,8,12,1,5,2,9)$ $(0,5,10,4,9,2,1,7)$ $(0,6,12,3,10,2,4,8)$
$(0,7,14,6,13,9,3,2)$ mod 15 □

4 Incomplete G_i-designs

Lemma 4.1 *Let G be a simple graph. For positive integers s,t,p and q, if there exists a G-$HD_\lambda(s^1t^1)$, then there exists a G-$HD_\lambda((ps)^1(qt)^1)$, too.*

Proof. Let $|S_i|=s,\ 1\le i\le p,\ |T_j|=t,\ 1\le j\le q$. Suppose there exist

$$G\text{-}HD_\lambda(s^1t^1)=(S_i\cup T_j,\mathcal{A}_{ij}),1\le i\le p,1\le j\le q,$$

then $(X,\mathcal{B})$ is a G-$HD_\lambda((ps)^1(qt)^1)$, where

$$X=(\bigcup_{1\le i\le p}S_i)\bigcup(\bigcup_{1\le j\le q}T_j),\ \mathcal{B}=\bigcup_{1\le i\le p,1\le j\le q}\mathcal{A}_{ij}.$$ □

Corollary 4.2 *If there exists a G-$HD_\lambda(4^2)$, then there exists a G-$HD_\lambda(8^14^1)$.*

Proof. By Lemma 4.1.(Let $p=2,\ q=1\ s=t=4$.) □

Lemma 4.3 *Let G be a simple graph. For positive integers k,w, if there exist G-$HD_\lambda(k^1w^1)$ and (k,G,λ)-GD, then G-$ID_\lambda(k+w;w)$ exists.*

Proof. Let $X=Z_k\cup W$, where W is a w-set. Suppose there exist

$$G\text{-}HD_\lambda(k^1w^1)=(Z_k\cup W,\mathcal{A}),\ (k,G,\lambda)\text{-}GD=(Z_k,\mathcal{B}),$$

then it is easy to know (X,Ω) is a G-$ID_\lambda(k+w;w)$, where $\Omega=\mathcal{A}\cup\mathcal{B}$. □

Lemma 4.4 *There exist G_1-$ID_\lambda(8+w;w)$ for*

(i) $\lambda=4$ *and* $w=4,5$, *(ii)* $\lambda=8$ *and* $w=2,3,6,7$.

Proof. $\underline{\lambda = 4,\ w = 4:}$ There exist $(8, G_1, 4)$-GD (see Lemma 3.1).

By Lemma 4.3, we only need to construct G_1-$HD_4(8^1 4^1)$.

$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = Z_4 \times \{2\}$.

$(0_2, 0_0, 2_2, 0_1, 1_2, 1_1, 3_0, 2_1)$ $(0_2, 0_0, 3_2, 3_0, 1_2, 0_1, 1_0, 1_1)$
$(3_2, 1_0, 2_2, 0_1, 0_2, 1_1, 0_0, 3_1)$ $(0_2, 0_0, 3_2, 1_0, 2_2, 1_1, 3_0, 3_1)$ mod$(4, -)$

$\underline{\lambda = 4,\ w = 5:}$ As Above, we only need to construct G_1-$HD_4(8^1 5^1)$.

$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = (Z_4 \times \{2\}) \cup \{\infty\}$.

$(2_2, 0_0, 1_2, 2_0, \infty, 0_1, 1_1, 2_1)$ $(0_2, 0_0, 3_2, 3_1, \infty, 2_1, 0_1, 1_1)$
$(3_2, 0_0, 1_2, 2_1, \infty, 1_0, 1_1, 0_1)$ $(0_2, 0_0, 2_2, 2_0, \infty, 3_0, 2_1, 3_1)$
$(2_2, 3_1, 0_2, 0_0, 1_2, 3_0, 1_1, 2_1)$ mod$(4, -)$

$\underline{\lambda = 8,\ w = 2:}$ The following $(X, \mathcal{B})$ is a G_1-$ID_2(8 + 2; 2)$.

$X = X_8 \cup \{A, B\}$, the family $\mathcal{B}$ consists of the following blocks.

$(0, B, 1, 7, A, 5, 3, 4)$ $(0, 5, 1, A, 3, 4, 2, 6)$ $(1, 0, A, 7, B, 4, 3, 6)$
$(1, 7, 3, 6, 4, A, 0, 2)$ $(2, 5, 6, 3, B, 0, 7, 1)$ $(2, A, 6, 0, 7, 3, 5, B)$
$(3, B, 2, 7, 4, 5, 0, 1)$ $(4, 1, 6, B, 5, 3, A, 2)$ $(5, B, 6, 2, 3, A, 4, 7)$
$(6, 5, 1, B, 4, 7, A, 2)$ $(7, 6, 4, 2, A, 0, B, 5)$

$\underline{\lambda = 8,\ w = 3:}$ We only need to construct a G_1-$HD_2(8^1 3^1)$.

$X = X_1 \cup X_2$, where $X_1 = (Z_3 \times Z_2) \cup \{A, B\}$, $X_2 = Z_3 \times \{2\}$.

$(1_2, 0_0, 0_2, 1_0, 2_2, 1_1, A, B)$ $(1_2, 2_1, 2_2, 1_1, 0_2, 1_0, A, B)$ mod$(3, -)$

$\underline{\lambda = 8,\ w = 6:}$ We only need to construct a G_1-$HD_2(8^1 6^1)$.

$X = X_1 \cup X_2$, where $X_1 = (Z_6 \times \{0\}) \cup \{A, B\}$, $X_2 = Z_6 \times \{1\}$.

$(1_0, 1_1, 0_0, 3_1, A, 5_1, 0_1, 4_1)$ $(4_0, 4_1, 0_0, 2_1, B, 0_1, 3_1, 5_1)$ mod$(6, -)$

$\underline{\lambda = 8,\ w = 7:}$ We only need to construct a G_1-$HD_2(8^1 7^1)$.

$X = X_1 \cup X_2$, where $X_1 = (Z_7 \times \{0\}) \cup \{\infty\}$, $X_2 = Z_7 \times \{1\}$.

$(3_1, 5_0, 0_1, 0_0, 1_1, 1_0, \infty, 2_0)$ $(0_1, 3_0, 6_1, 0_0, 4_1, 1_0, \infty, 2_0)$ mod$(7, -)$ □

Lemma 4.5 *There exist* G_2-$ID_\lambda(8 + w; w)$ *for*

(i) $\lambda = 4$ *and* $w = 4, 5$, *(ii)* $\lambda = 8$ *and* $w = 2, 3, 6, 7$.

Proof. $\underline{\lambda = 4,\ w = 4:}$ There exist a $(8, G_2, 4)$-GD (see Lemma 3.2).

By Lemma 4.3, we only need to construct G_2-$HD_4(8^1 4^1)$.

By Corollary 4.2, we only need to construct G_2-$HD_4(4^2)$. $X = Z_4 \times Z_2$.

$(0_1, 0_0, 1_1, 2_0, 3_1, 1_0, 3_0, 2_1)$ $(1_0, 0_1, 0_0, 3_1, 3_0, 1_1, 2_1, 2_0)$ mod$(4, -)$

$\lambda = 4,\ w = 5$: We only need to construct G_2-$HD_4(8^1 5^1)$.

$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = (Z_4 \times \{2\}) \cup \{\infty\}$.

$(0_2, 0_1, 1_2, 0_0, \infty, 2_1, 1_0, 3_2)$ $(2_2, 1_0, 1_2, 0_1, \infty, 0_0, 2_1, 3_2)$
$(1_2, 0_1, 3_2, 0_0, \infty, 2_1, 1_0, 2_2)$ $(0_2, 2_0, 1_2, 1_1, \infty, 0_0, 2_1, 3_2)$
$(1_2, 0_1, 2_2, 0_0, 0_2, 1_0, 1_1, 3_2)$ mod$(4, -)$

$\lambda = 8,\ w = 2$: The following $(X, \mathcal{B})$ is a G_2-$ID_2(8+2; 2)$.

$X = X_8 \cup \{A, B\}$. The family $\mathcal{B}$ is as follows:

$(0,3,B,6,4,5,1,2)$ $(0,4,7,2,6,B,A,1)$ $(0,7,1,A,2,B,5,6)$
$(3,6,A,7,5,2,0,1)$ $(6,1,5,B,3,A,0,4)$ $(4,3,7,0,1,B,2,5)$
$(1,A,5,6,7,3,2,4)$ $(2,1,5,6,3,4,0,7)$ $(4,5,2,B,7,A,0,3)$
$(4,6,0,A,5,B,7,2)$ $(7,2,A,3,1,B,5,0)$

$\lambda = 8,\ w = 3$: The following $(X, \mathcal{B})$ is a G_2-$ID_2(8+3; 3)$.

$X = X_8 \cup \{A, B, C\}$. The family $\mathcal{B}$ is as follows:

$(A,3,0,7,1,5,2,6)$ $(A,2,4,B,5,1,3,6)$ $(2,1,4,3,7,5,C,B)$
$(4,6,1,C,2,0,7,A)$ $(B,4,6,0,5,3,2,A)$ $(B,7,A,1,4,5,6,C)$
$(2,7,4,0,3,1,5,B)$ $(7,6,0,C,5,A,2,1)$ $(C,4,A,0,B,6,7,5)$
$(C,0,1,3,2,4,6,B)$ $(3,2,6,A,0,7,C,B)$ $(5,0,1,C,3,6,7,2)$
$(5,3,B,1,7,6,C,4)$

$\lambda = 8,\ w = 6$: We only need to construct a G_2-$HD_2(8^1 6^1)$.

$X = X_1 \cup X_2$, where $X_1 = (Z_6 \times \{0\}) \cup \{A, B\}$, $X_2 = Z_6 \times \{1\}$.

$(1_0, 5_1, 0_0, 0_1, A, 2_1, 4_1, 2_0)$ $(3_0, 4_1, 0_0, 2_1, B, 5_1, 3_1, 5_0)$ mod$(6, -)$

$\lambda = 8,\ w = 7$: We only need to construct a G_2-$HD_2(8^1 7^1)$.

$X = X_1 \cup X_2$, where $X_1 = (Z_7 \times \{0\} \cup \{\infty\}$, $X_2 = Z_7 \times \{1\}$.

$(3_1, 5_0, 0_1, 0_0, 1_1, 1_0, \infty, 6_1)$ $(0_1, 3_0, 6_1, 0_0, 4_1, 1_0, \infty, 1_1)$ mod$(7, -)$ □

Lemma 4.6 *There exist G_3-$ID_\lambda(8+w; w)$ for*
(*i*) $\lambda = 4$ *and* $w = 4, 5$, (*ii*) $\lambda = 8$ *and* $w = 2, 3, 6, 7$.

Proof. $\lambda = 4,\ w = 4$: There exist $(8, G_3, 4)$-GD (see Lemma 3.3).

By Lemma 4.3, we only need to construct G_3-$HD_4(8^1 4^1)$.

$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = Z_4 \times \{2\}$.

$(0_2, 0_0, 2_2, 0_1, 1_2, 1_1, 2_1, 3_0)$ $(0_2, 0_0, 3_2, 3_0, 1_2, 0_1, 1_1, 2_0)$
$(3_2, 1_0, 2_2, 0_1, 0_2, 1_1, 3_1, 3_0)$ $(0_2, 0_0, 3_2, 1_0, 2_2, 1_1, 3_1, 2_0)$ mod$(4, -)$

$\underline{\lambda = 4,\ w = 5:}$ We only need to construct G_3-$HD_4(8^1 5^1)$.
$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = (Z_4 \times \{2\}) \cup \{\infty\}$.

$$\begin{array}{ll} (0_2, 0_1, 1_2, 1_0, \infty, 2_1, 2_0, 3_1) & (2_2, 1_0, 1_2, 0_1, \infty, 0_0, 2_1, 2_0) \\ (1_2, 0_1, 3_2, 2_0, \infty, 2_1, 0_0, 3_1) & (0_2, 2_0, 1_2, 1_1, \infty, 1_0, 2_1, 0_0) \\ (1_2, 0_1, 2_2, 0_0, 0_2, 1_0, 2_1, 3_1) & \text{mod } (4, -) \end{array}$$

$\underline{\lambda = 8,\ w = 2:}$ The following $(X, \mathcal{B})$ is a G_3-$ID_2(8+2; 2)$.
$X = X_8 \cup \{A, B\}$. The family $\mathcal{B}$ consists of the following blocks.

$$\begin{array}{lll} (0,6,1,B,7,4,5,2) & (0,7,2,B,3,A,1,6) & (0,6,3,7,4,B,2,1) \\ (0,3,1,A,5,B,4,7) & (2,A,1,5,3,B,0,4) & (1,7,5,A,6,B,0,2) \\ (6,A,3,2,5,4,1,0) & (2,3,7,5,B,4,1,6) & (5,6,7,A,4,3,1,0) \\ (6,3,4,A,7,B,5,2) & (4,6,2,A,0,5,1,7) & \end{array}$$

$\underline{\lambda = 8,\ w = 3:}$ We only need to construct a G_3-$HD_2(8^1 3^1)$.
$X = X_1 \cup X_2$, where $X_1 = (Z_3 \times Z_2) \cup \{A, B\}$, $X_2 = Z_3 \times \{2\}$.

$$(1_2, 0_0, 0_2, 1_0, 2_2, 1_1, A, B) \quad (1_2, 2_1, 2_2, 1_1, 0_2, 1_0, A, B) \quad \text{mod } (3, -)$$

$\underline{\lambda = 8,\ w = 6:}$ We only need to construct a G_3-$HD_2(8^1 6^1)$.
$X = X_1 \cup X_2$, where $X_1 = (Z_6 \times \{0\} \cup \{A, B\}$, $X_2 = Z_6 \times \{1\}$.

$$(1_0, 2_1, 0_0, 1_1, A, 4_1, 0_1, 3_1) \quad (1_0, 5_1, 0_0, 0_1, B, 3_1, 1_1, 4_1) \quad \text{mod } (6, -)$$

$\underline{\lambda = 8,\ w = 7:}$ We only need to construct a G_3-$HD_2(8^1 7^1)$.
$X = X_1 \cup X_2$, where $X_1 = (Z_7 \times \{0\}) \cup \{\infty\}$, $X_2 = Z_7 \times \{1\}$.

$$(3_1, 5_0, 0_1, 0_0, 1_1, 1_0, 2_0, \infty) \quad (0_1, 3_0, 6_1, 0_0, 4_1, 1_0, 2_0, \infty,) \text{ mod } (7, -) \ \square$$

Lemma 4.7 *There exist* G_4-$ID_\lambda(8 + w; w)$ *for*
(*i*) $\lambda = 4$ *and* $w = 4, 5$, (*ii*) $\lambda = 8$ *and* $w = 2, 3, 6, 7$.

Proof. $\underline{\lambda = 4,\ w = 4:}$ There exist a $(8, G_4, 4)$-GD (see Lemma 3.4).
By Lemma 4.3, we only need to construct G_4-$HD_4(8^1 4^1)$.
By Corollary 4.2, we only need to construct G_4-$HD_4(4^2)$. $X = Z_4 \times Z_2$.

$$(0_0, 2_1, 1_0, 0_1, 2_0, 3_1, 1_1, 3_0) \quad (3_1, 0_0, 0_1, 2_0, 1_1, 1_0, 3_0, 2_1) \quad \text{mod}(4, -)$$

$\underline{\lambda = 4,\ w = 5:}$ We only need to construct G_4-$HD_4(8^1 5^1)$.
$X = X_1 \cup X_2$, where $X_1 = Z_4 \times Z_2$, $X_2 = (Z_4 \times \{2\}) \cup \{\infty\}$.

$(0_2,0_1,1_2,0_0,\infty,2_1,3_0,2_2)$ $(2_2,1_0,1_2,0_1,\infty,0_0,2_1,3_2)$
$(1_2,0_1,3_2,0_0,\infty,2_1,1_1,0_2)$ $(0_2,2_0,1_2,1_1,\infty,3_0,2_1,3_2)$
$(1_2,0_1,2_2,0_0,0_2,1_0,2_1,3_2)$ mod$(4,-)$

<u>$\lambda=8,\ w=2$:</u> The following $(X,\mathcal{B})$ is a G_4-$ID_2(8+2;2)$.
$X=X_8\cup\{A,B\}$. The family $\mathcal{B}$ is as follows:

$(0,4,A,1,6,B,3,2)$ $(0,5,3,1,6,B,2,4)$ $(0,3,6,2,7,A,1,B)$
$(0,7,1,3,4,5,6,B)$ $(1,5,A,3,7,B,2,6)$ $(5,4,7,1,B,3,A,0)$
$(2,6,A,7,4,3,0,5)$ $(2,A,0,6,5,B,3,4)$ $(2,4,6,7,3,A,5,0)$
$(5,1,A,4,B,7,2,0)$ $(6,5,B,4,2,7,A,1)$

<u>$\lambda=8,\ w=3$:</u> The following $(X,\mathcal{B})$ is a G_4-$ID_2(8+3;3)$.
$X=X_8\cup\{A,B,C\}$. The family $\mathcal{B}$ is as follows:

$(A,3,0,7,1,2,4,B)$ $(C,4,5,A,7,1,3,6)$ $(4,6,0,3,5,C,B,7)$
$(4,6,1,A,2,0,3,5)$ $(B,6,A,0,7,2,1,5)$ $(B,7,6,1,2,5,3,C)$
$(2,7,4,1,3,6,C,0)$ $(7,6,0,C,2,4,5,3)$ $(C,5,4,0,B,6,7,2)$
$(1,0,B,3,2,4,5,7)$ $(3,5,0,C,6,2,4,7)$ $(5,B,4,A,3,6,1,0)$
$(5,7,A,1,B,2,6,3)$

<u>$\lambda=8,\ w=6$:</u> We only need to construct a G_4-$HD_2(8^16^1)$.
Let $X=X_1\cup X_2$, where $X_1=(Z_6\times\{0\})\cup\{A,B\}$, $X_2=Z_6\times\{1\}$.

$(0_0,0_1,1_0,4_1,A,1_1,3_1,2_0)$ $(0_0,4_1,5_0,5_1,B,1_1,2_1,1_0)$ mod $(6,-)$

<u>$\lambda=8,\ w=7$:</u> We only need to construct a G_4-$HD_2(8^17^1)$.
$X=X_1\cup X_2$, where $X_1=(Z_7\times\{0\})\cup\{\infty\}$, $X_2=Z_7\times\{1\}$.

$(0_1,0_0,1_1,1_0,3_1,5_0,\infty,6_1)$ $(6_1,0_0,4_1,1_0,0_1,3_0,\infty,2_1)$ mod $(7,-)$ □

5 Results

Theorem 5.1 *There exist (v,G_1,λ)-GD if and only if $\lambda v(v-1)\equiv 0 \pmod{16}$ and $v\geq 8$.*

Proof. By Theorem 2.1, Theorem 2.4, Lemma 3.1 and Lemma 4.4. □

Theorem 5.2 *There exist (v,G_2,λ)-GD if and only if $\lambda v(v-1)\equiv 0 \pmod{16}$ and $v\geq 8$.*

Proof. By Theorem 2.1, Theorem 2.4, Lemma 3.2 and Lemma 4.5. □

Theorem 5.3 *There exist (v,G_3,λ)-GD if and only if $\lambda v(v-1)\equiv 0 \pmod{16}$ and $v\geq 8$.*

Proof. By Theorem 2.1, Theorem 2.4, Lemma 3.3 and Lemma 4.6. □

Theorem 5.4 *There exist* (v, G_4, λ)*-GD if and only if* $\lambda v(v-1) \equiv 0 \pmod{16}$ *and* $v \geq 8$.

Proof. By Theorem 2.1, Theorem 2.4, Lemma 3.4 and Lemma 4.7. □

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