On (k, λ)-magically total labeling of graphs[1]

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Abstract

If there are integers k and $\lambda \neq 0$ such that a total labeling f of a connected graph $G = (V, E)$ from $V \cup E$ to $\{1, 2, \ldots, |V| + |E|\}$ satisfies $f(x) \neq f(y)$ for distinct $x, y \in V \cup E$ and $f(u) + f(v) = k + \lambda f(uv)$ for each edge $uv \in E$, then f is called a (k, λ)-*magically total labeling* ((k, λ)-*mtl* for short) of G. Several properties of (k, λ)-*mtls* of graphs are shown. The sufficient and necessary connections between (k, λ)-*mtls* and several known labelings (such as graceful, odd-graceful, felicitous and (b, d)-edge antimagic total labelings) are given. Furthermore, every tree is proven to be a subgraph of a tree having super (k, λ)-*mtls*.

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1 Introduction and concepts

The problems of transforming a graph labelling into another one have applications in areas such as bioinformatics, (scale-free, small-world) networks, VLSI, and so on. Labellings including $f(u) + f(v)$ ($uv \in E(G)$) were used to many problems. An example, studied first by Graham and Sloane in [8], is the *harmonious graphs* of modular versions of additive bases problems stemming from error-correcting codes [9]. Various graph labellings can be found in the survey paper [6] in which the author collects more than 1000 articles on graph labellings. The conjectures listed in Conjecture 1 are extensively studied [6].

Conjecture 1. *Trees mentioned in the following have at least three vertices.*

1. (Graceful Tree Conjecture, 1966 [10]) *Every tree is graceful.*
2. (1970 [5]) *Every tree admits an edge-magic total labelling.*
3. (1980 [8]) *Every tree is harmonious.*

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4. (1991 [7]) *Every tree is odd-graceful.*
5. (1998 [4]) *Every tree admits a super edge-magic total labelling.*

It would be interesting to find connections between the conjectures listed in Conjecture 1. For this reason, we show a new labelling in finding equivalent connections among labellings. The graphs under consideration are simple, finite, and loopless, unless otherwise specified. We use standard terminology and notation of graph theory. Graph labellings mentioned here can be found in [6]. The set of vertices that are adjacent to a vertex u of a graph G is denoted as $N(u)$, thus, the *degree* $\deg_G(u)$ of the vertex u is equal to the cardinality $|N(u)|$. The shorthand notation $[m, n]$ stands for an integer set $\{m, m+1, \dots, n\}$ with $n > m \geq 0$. A *k-set* S is a set containing exactly k integers, and $\max(S) = \max\{x : x \in S\}$, $\min(S) = \min\{x : x \in S\}$. A *$(p, q)$-graph* is a graph having p vertices and q edges. For a graph G, a labelling $f : S \rightarrow [m, n]$ is *proper* if $f(x) \neq f(y)$ for distinct $x, y \in S$, where $\emptyset \neq S \subseteq V(G) \cup E(G)$. Write $f(S) = \{f(x) : x \in S\}$. Furthermore, f is *bijection* if $f(S) = [m, n]$.

William et al. [11] defined a θ-edge magic total labelling as: Let G be a (p, q)-graph and θ be a positive integer. A bijection f from $V(G) \cup E(G)$ to $[1, p+q]$ is called a *θ-edge magic total labelling* if for all edges uv, the number $f(u) + f(v) + \theta f(uv)$ is equal to a fixed constant, and furthermore f is a *super θ-edge magic total labelling* if the set of vertex labels is equal to $[1, p]$. Yao et al. ([12], [13]) introduced a magical type of labelling: A (p, q)-graph G has a k-magic coloring $f : V(G) \cup E(G) \rightarrow [1, p+q]$ if $f(u) + f(v) = k + f(uv)$ whenever $uv \in E(G)$, where k is a constant. Motivated from the above labellings, we have

Definition 1. Let G be a connected (p, q)-graph. If there are integers k and $\lambda \neq 0$ such that a proper total labelling f of G from $V(G) \cup E(G)$ to $[1, p+q]$ satisfies $f(u) + f(v) = k + \lambda f(uv)$ for $uv \in E(G)$. Then f is called a *(k, λ)-magically total labelling* (*(k, λ)-mtl* for short) of G, k and λ are called a *magical constant* and a *balanced number*, respectively. Further, f is *super* if $f(V(G)) = [1, p]$, and G is *f-saturated* if f is no longer a (k, λ)-*mtl* of $G + uv$ for any $uv \in E(\overline{G})$, where $\overline{G}$ is the complement of G.

A $(0, 1)$-*mtl* of a (p, q)-graph G is also a *1-sequentially additive labelling* defined by Bange et al.[6] in 1983. A super $(0, 1)$-*mtl* of G is also a $(p + 1, 1)$-edge antimagic vertex labelling of G [6]. A $(k, -1)$-*mtl* of G is just an *edge-magic total labelling* of G defined first by Kotzig and Rosa [5] in 1970, 1966 under the name of *magic valuations*. In 1972, they proposed the ***magical tree problem***: *Whether do all trees have the edge-magic total labellings?* Kotzig and Rosa contributed various labellings for solving problems of graph decompositions, and they proposed some interesting open problems and conjectures, such as the Graceful Tree Conjecture, a longstanding conjecture up to now. Rosa [10] has identified essentially three reasons why a graph G fails to be graceful: (1) G has "too many vertices" and "not enough edges"; (2) G has "too many edges"; and (3) G has "the wrong parity". In fact, many problems of graph labelling have some of these three characteristics.

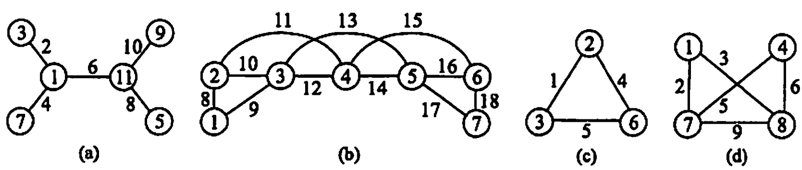

Figure 1: (a) A bi-star has a $(0,2)$-*mtl*; (b) a graph with a super $(-5,1)$-*mtl*; (c) K_3 admits a $(4,1)$-*mtl*; (d) $K_4 - e$ admits a $(6,1)$-*mtl*.

Let G be a connected (p,q)-graph having a proper total labelling $f : V(G) \cup E(G) \to [1, p+q]$, and let m be an integer. Several labellings related with the above labelling f are

1. The *dual labelling* h of f is defined as $h(x) = p+q+1-f(x)$ for $x \in V(G) \cup E(G)$ (see Figure 2(b)).

2. A *total m-float labelling* h of f is defined as $h(x) = m + f(x)$ for $x \in V(G) \cup E(G)$.

3. An *edge* m^+*-float labelling* (respectively, an *edge* m^-*-float labelling*) h of f is defined as $h(u) = f(u)$ for $u \in V(G)$ and $h(uv) = m + f(uv)$ (respectively, $h(uv) = m - f(uv)$) for $uv \in E(G)$.

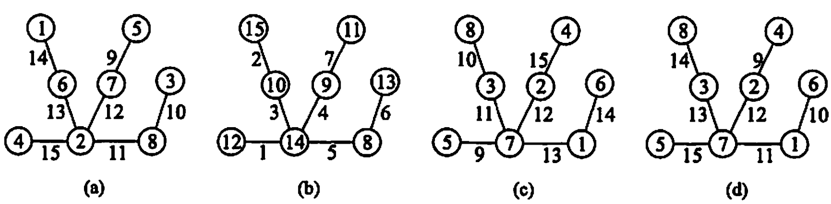

Figure 2: A tree T has: (a) a super $(21,-1)$-*mtl* f; (b) the dual labelling of f is a $(27,-1)$-*mtl*; (c) the partially dual labelling h of f is a super $(21,-1)$-*mtl*; (d) the edge-partially dual labelling of h is a $(-3,1)$-*mtl*.

We will define some labellings that are relevant to (k,λ)-*mtl*s in the following. Let G be a connected (p,q)-graph having a (k,λ)-*mtl* f.

1. Suppose that f is super. (i) The *partially dual labelling* h of f is defined as $h(x) = p+1-f(x)$ for $x \in V(G)$ and $h(xy) = 2p+q+1-f(xy)$ for $xy \in E(G)$ (see Figure 2(c)). (ii) The *edge-partially dual labelling* h of f is defined as $h(x) = f(x)$ for $x \in V(G)$ and $h(xy) = 2p+q+1-f(xy)$ for $xy \in E(G)$ (see Figure 2(d) and Figure 3(d)).

2. Suppose that $f(E(G)) = [1, q]$. The *vertex-partially dual labelling* h of f is defined as $h(x) = 2q + p + 1 - f(x)$ for $x \in V(G)$ and $h(xy) = f(xy)$ for $xy \in E(G)$ (see Figure 3(b)).

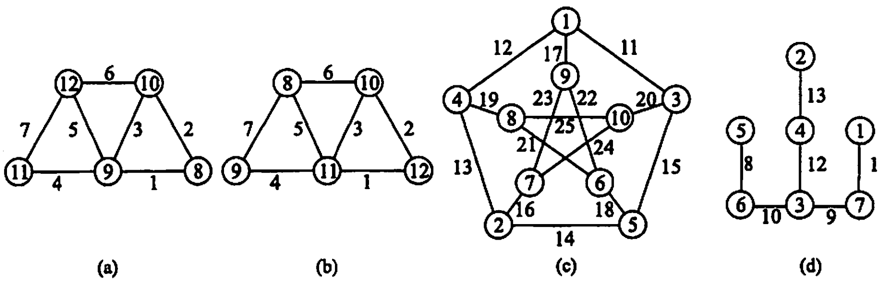

Figure 3: (a) A graph G admits a $(-4, 1)$-*mtl* f; (b) G has a $(16, -1)$-*mtl* that is the vertex-partially dual labelling of f; (c) Petersen graph has a super $(-7, 1)$-*mtl* α; (d) a 2-star has a super $(19, -1)$-*mtl*.

Example 1. Let G be a connected (p, q)-graph having a (k, λ)-*mtl* f. Let $P = u_1 u_2 \cdots u_n$ and $C = P + u_1 u_n$ be a path and a cycle on n vertices in G, respectively. By the definition of a (k, λ)-*mtl*, the path P holds

$$f(u_j) = \frac{1 + (-1)^j}{2} k + (-1)^{j+1} \left[f(u_1) + \lambda \sum_{i=1}^{j-1} (-1)^i f(u_i u_{i+1}) \right], j \in [1, n-1];$$

and the cycle C holds $\frac{1+(-1)^{n+1}}{2}(2f(u_1) - k) = (-1)^n \lambda \sum_{i=1}^{n} (-1)^i f(u_i u_{i+1})$ $(u_{n+1} = u_1 \bmod n)$. If C is an even cycle, the sum $\sum_{i=1}^{n} (-1)^i f(u_i u_{i+1})$ $(u_{n+1} = u_1 \bmod n)$ equals to zero. Conversely, these two properties can be used to define a (k, λ)-*mtl*.

Example 2. Let $n_1(G)$ be the number of vertices of degree one in a graph G. If a tree T admits a (k, λ)-*mtl* f, then T has $2^{n_1(T)}$ (k, λ)-*mtls*. Suppose that an edge $uv \in E(T)$ satisfies $\deg_T(u) = 1$. We define a labelling h of T as: $h(x) = f(x)$ for $x \in V(T) \cup E(T)$ and $x \notin \{u, uv\}$; and $h(u) = f(uv)$, $h(uv) = f(u)$. Clearly, h also is a (k, λ)-*mtl* of T. Thereby, depending on the (k, λ)-*mtl* f, there are $2^{n_1(T)}$ (k, λ)-*mtls* of T.

2 Results on (k, λ)-*mtls*

Theorem 2. *Let G be a connected (p, q)-graph.*

(i) G admits a (k, λ)-mtl f if and only if for two adjacent edges uv and vw of G,

$$f(u) - f(w) = \lambda [f(uv) - f(vw)]. \quad (1)$$

(ii) G admits a (k,λ)-mtl f if and only if the dual labelling h of f is a (k',λ)-mtl, where $k'=(2-\lambda)(p+q+1)-k$.

(iii) G admits a (k,λ)-mtl g with $\lambda \geq 1$ if and only if it admits a $(k',-\lambda)$-mtl h, where $k'=k+\lambda(M+m)$ for $M=\max(f(E(G)))$ and $m=\min(f(E(G)))$.

(iv) G admits a super (k,λ)-mtl f if and only if the partially dual labelling h of f is a super (k',λ)-mtl, where $k'=2(p+1)-k-\lambda(2p+q+1)$.

(v) G admits a super (k,λ)-mtl f if and only if G admits an edge-partially dual labelling h of f which is a super $(k',-\lambda)$-mtl such that $k'=k+\lambda(2p+q+1)$.

(vi) G admits a (k,λ)-mtl f with $f(E(G))=[1,q]$ if and only if G admits a vertex-partially dual labelling h of f which is a $(k',-\lambda)$-mtl, where $k'=4q+2(p+1)-k$.

Proof. *(i)* Let f be a (k,λ)-*mtl* of a connected (p,q)-graph G. For distinct $u,w\in N(v)$ $(v\in V(G))$, we subtract $f(v)+f(w)=k+\lambda f(vw)$ from $f(u)+f(v)=k+\lambda f(uv)$, and then get the equation (1).

Conversely, for an arbitrary vertex $v\in V(G)$ and distinct $u,w\in N(v)$, the equation (1) holds. Hence,

$$f(u)+f(v)-\lambda f(uv)=f(v)+f(w)-\lambda f(vw) \tag{2}$$

Let $k=f(u)+f(v)-\lambda f(uv)$. Notice that $v\in N(u)\cap N(w)$ and G is connected. By (1) and (2), then $f(u)+f(s)-\lambda f(us)=k$ for each vertex $s\in N(u)$, and $f(w)+f(t)-\lambda f(wt)=k$ for each vertex $t\in N(w)$, which means that f really is a (k,λ)-*mtl*.

(ii) By the definition of the dual labelling h of the (k,λ)-*mtl* f, we are able to testify

$$\begin{aligned}h(u)+h(v)&=2(p+q+1)-\big[f(u)+f(v)\big]\\&=2(p+q+1)-\big[k+\lambda f(uv)\big]\\&=2(p+q+1)-\{k+\lambda\big[(p+q+1)-h(uv)\big]\}\\&=\big[(2-\lambda)(p+q+1)-k\big]+\lambda h(uv),\end{aligned}$$

which means that h is a (k',λ)-*mtl* with $k'=(2-\lambda)(p+q+1)-k$. The proof of 'only if' is similar with the above one, since f is also the dual labelling of h.

(iii) We define directly an edge $(M+m)^{-}$-float labelling h of the labelling g by setting $h(u)=g(u)$ for $u\in V(G)$ and $h(uv)=M+m-g(uv)$ for any edge $uv\in E(G)$. Since $\lambda\geq 1$ we have

$$\begin{aligned}&h(u)+h(v)=g(u)+g(v)=k+\lambda g(uv)\\&=k+\lambda\big[M+m-h(uv)\big]=\big[k+\lambda(M+m)\big]+(-\lambda)h(uv)\end{aligned}$$

for each edge $uv \in E(G)$. Let $k' = k + \lambda(M + m)$. Clearly, h is a $(k', -\lambda)$-*mtl* of G. Notice that $g(V(G)) = h(V(G))$, that is, the labellings g and h both have the same labels of vertices. Furthermore, a super (k, λ)-*mtl* h corresponds to a super $(k', -\lambda)$-*mtl* g, and vice versa.

(iv) From the definition of the partially dual labelling h of the super (k, λ)-*mtl* f, we obtain $h(V(G)) = [1, p]$, and moreover

$$\begin{aligned} h(x) + h(y) &= 2(p+1) - [f(x) + f(y)] = 2(p+1) - [k + \lambda f(xy)] \\ &= 2(p+1) - \{k + \lambda[(2p+q+1) - h(xy)]\} = k' + \lambda h(xy), \end{aligned}$$

where $k' = 2(p+1) - k - \lambda(2p+q+1)$. The proof of 'only if' is similar with one above.

The proofs of the assertions (v) and (vi) can be obtained directly by the definitions of the edge-partially and vertex-partially dual labellings. The proof of the theorem is complete. □

Theorem 3. *Let G be a connected (p, q)-graph having a (k, λ)-mtl f.*

(i) *If $k \leq 0$, then $\lambda \geq 1$.*

(ii) *If $|\lambda|$ is even, then G is bipartite and k is odd, and otherwise all labels of vertices of G have the same parity and k is even.*

(iii) $|\lambda|(\Delta(G) - 1) \leq p + q - 1$.

Proof. Let f be a (k, λ)-*mtl* of a connected (p, q)-graph G. (i) Since $f(u)$ and $f(uv)$ are positive, thus, $0 \leq f(u) + f(v) - k = \lambda f(uv)$ when $k \leq 0$, which leads to $\lambda \geq 1$.

(ii) From the assertion (i) of Theorem 2, for any vertex v of G and distinct $u, w \in N(v)$, the difference $f(u) - f(w) = \lambda[f(uv) - f(vw)]$ is even for $f(u) > f(w)$ since $|\lambda|$ is even. So, the labels of any two neighbors of the vertex v have the same parity. Therefore, we can partition $V(G)$ into two subsets, that is, $V(G) = V_e \cup V_o$, where $V_e = \{f(x) = \text{even} : x \in V(G)\}$ and $V_o = \{f(x) = \text{odd} : x \in V(G)\}$. If one of V_e and V_o is empty, thus, all labels of vertices of G have the same parity and k is even. Suppose that $V_e \neq \emptyset$ and $V_o \neq \emptyset$, and G contains an odd cycle C. Then there is an edge $xy \in E(C)$ such that $x, y \in V_e$ (or $x, y \in V_o$), which means $N(x) \cup N(y) \subseteq V_e$. Immediately, V_o is empty since G is connected; a contradiction. Therefore, G is bipartite and k is odd.

(iii) The proof of the assertion (iii) is as the same as one shown in the assertion (iii) of Theorem 4. □

Theorem 4. *Let G be a connected (p, q)-graph having a super (k, λ)-mtl f.*

(i) *$|\lambda|$ is odd.*

(ii) *If $k > 0$, then $\lambda < 0$.*

(iii) *Let Δ be the maximum degree of G, then $|\lambda|(\Delta - 1) \leq p - 1$.*

(*iv*) $k \geq 5 - \lambda(p+q)$ *and* $k \leq (2-\lambda)p - (\lambda+3)$ *for* $p \geq 3$.

(*v*) *If* G *is regular, then* q *is odd and* $k = (p+1) - \frac{1}{2}\lambda(2p+q+1)$.

(*vi*) *If* G *is bipartite,* (X,Y) *is the bipartition of* G *such that* $\max(f(X)) < \min(f(Y))$ *and* $\lambda = 1$. *Then* $k = 1 - |Y|$.

Proof. Notice that $f(V(G)) = [1,p]$ and $f(E(G)) = [p+1, p+q]$.

(*i*) Since f is a super (k,λ)-mtl, this assertion (*i*) is an immediate consequence of the assertion (*ii*) of Theorem 3.

(*ii*) Notice that G is connected, and there exists an edge $uv \in E(G)$ with $f(uv) = p+q$. Thereby, $2p-1-k \geq f(u)+f(v)-k = \lambda f(uv) = \lambda(p+q) \geq \lambda(2p-1)$. Immediately, $\lambda < 0$ follows $k > 0$.

(*iii*) Notice that $\lambda \neq 0$. Since two distinct vertices x and y of $N(u)$ satisfy $f(x) - f(y) = \lambda\big[f(ux) - f(uy)\big]$ by the assertion (*i*) of Theorem 2. Let $f(ux) = \max\{f(uv) : v \in N(u)\}$ and $f(uy) = \min\{f(uv) : v \in N(u)\}$. Thereby, $|N(u)|-1 \leq |f(ux)-f(uy)| = \frac{1}{|\lambda|}|f(x)-f(y)| \leq \frac{1}{|\lambda|}(p-1)$, which implies $|\lambda|(\Delta - 1) \leq p-1$.

(*iv*) Notice that there exists an edge $wz \in E(G)$ such that $f(wz) = p+q$. We have $k = f(w)+f(z)-\lambda f(wz) = f(w)+f(z)-\lambda(p+q) \geq 3-\lambda(p+q)$. By the assertion (*iv*) of Theorem 2 the partially dual labelling h of f, also, is a (k', λ)-*mtl*, where $k' = 2(p+1) - k - \lambda(2p+q+1)$. Notice the bound $3-\lambda(p+q)$ is the smallest one, so it implies that the bound $2(p+q)-\lambda-1$ is the largest one. We discuss the bounds of k again.

If $k = 3 - \lambda(p+q)$ (which means $\lambda \geq 1$ by Theorem 3), then we have an edge $xy \in E(G)$ with $f(xy) = p+q$ such that $f(x)+f(y) = k + \lambda f(xy) = 3 + \lambda[f(xy) - p - q] \leq 3$. For any edge $uv \neq xy$ of G, $7 \leq f(u)+f(v) = k+\lambda f(uv) = 3+\lambda[f(xy)-p-q] \leq 3+(p+q-1)-p-q = 2$; a contradiction. Thereby, we have: (1) $E(G) \setminus \{xy\} = \emptyset$, that is $p = 2$; (2) $k \geq 4 - \lambda(p+q)$. We shall consider the following cases.

To consider the case $k = 4 - \lambda(p+q)$, we take an edge $xy \in E(G)$ with $f(xy) = p+q$, which turns out $f(x)+f(y) = k+\lambda f(xy) = 4+\lambda(f(xy)-p-q) \leq 4$. Then, for any edge $uv \neq xy$ of G, $6 \leq f(u)+f(v) = k+\lambda f(uv) = 4+\lambda(f(uv)-p-q) \leq 4+(p+q-1)-p-q = 3$; a contradiction.

From $k \geq 5-\lambda(p+q)$, $k' = 2(p+1) - k - \lambda(2p+q+1) \leq 2(p+1) - 5 + \lambda(p+q) - \lambda(2p+q+1) = (2-\lambda)p - (\lambda+3)$.

(*v*) Let G be t-regular, so $2q = tp$. The addition of $f(u)+f(v) = k + \lambda f(uv)$ about q edges $uv \in E(G)$ yields $t(1+2+\cdots+p) = qk + \lambda\left[pq + \frac{q(q+1)}{2}\right]$, and furthermore $(p+1)q = t \cdot \frac{p(p+1)}{2} = qk + \lambda\left[pq + \frac{q(q+1)}{2}\right]$, solve $k = (p+1) - \frac{\lambda}{2}(2p+q+1)$, as we have wished. Clearly, q is odd since $|\lambda|$ is odd by the assertion (*i*) of this theorem.

(*vi*) Notice that $\max(f(X)) < \min(f(Y))$. For each edge $xy \in E(G)$ with $x \in X$ and $y \in Y$ we have $k + \lambda f(xy) = f(x)+f(y) \geq |X|+2$, and $k + \lambda f(xy) = f(x)+f(y) \leq |X|+p$. There are $f(uv) = 2p-1$ and

$f(wz) = p+1$ for some certain edges $uv, wz \in E(G)$. Thereby, $k+\lambda f(wz) = k+\lambda(p+1) \geq |X|+2$, and $k+\lambda f(uv) = k+\lambda(2p-1) \leq |X|+p$. Notice that $p = |X|+|Y|$. We have $|X|+2-\lambda(p+1) \leq k \leq |X|+p-\lambda(2p-1)$. Clearly, $k = 1-|Y|$ when $\lambda = 1$. □

Example 3. Suppose that a connected (p,q)-graph G is bipartite and regular. If G admits a super $(k,1)$-*mtl* f with $\max(f(X)) < \min(f(Y))$, where (X,Y) is the partition of $V(G)$, then we have $q+1 = 2|Y|$.

Theorem 5. *If a connected (p,q)-graph G admits a super (k,λ)-mtl f with $\lambda \geq 1$, then $q \leq \lceil (2p-3)/\lambda \rceil$, and the equality holds if and only if G is f-saturated.*

Proof. Let $V(G) = \{u_i : i \in [1,p]\}$ and $f(u_i) < f(u_{i+1})$ for $i \in [1,p-1]$, where f is defined as the statement of the theorem. The neighborhood of each vertex u_i of degree d_i is denoted by $N(u_i) = \{v_{i,1}, v_{i,2}, \dots, v_{i,d_i}\}$ such that $f(v_{i,j}) < f(v_{i,j+1})$ for $j \in [1, d_i-1]$. It is not hard to obtain $f(u_1) + [f(v_{11}) + (j-1)\lambda] = k+\lambda[f(u_1v_{11}) + (j-1)]$, $j \in [1,d_1]$. In other words, $f(v_{1,j}) = f(v_{11}) + (j-1)\lambda$ and $f(u_1v_{1,j}) = f(u_1v_{11}) + (j-1)$ for $j \in [1,d_1]$. In general, $f(u_i) + [f(v_{i,1}) + (j-1)\lambda] = k+\lambda[f(u_iv_{i,1}) + (j-1)]$, $j \in [1,d_i]$, $i \in [1,p]$, which means that

$$f(v_{i,j}) = f(v_{i,1}) + (j-1)\lambda \text{ and } f(u_iv_{i,j}) = f(u_iv_{i,1}) + (j-1), \quad (3)$$

For $j \in [1,d_i]$, $i \in [1,p]$. Since f is super, so $f(u_1) = 1$, we then have

$$1+j = k+\lambda\left[f(u_1v_{11}) + \frac{j-f(v_{11})}{\lambda}\right],\ j \in [2,p]; \quad (4)$$

and from $f(u_p) = p$,

$$m+p = k+\lambda\left[f(u_pv_{p,1}) + \frac{m-f(v_{p,1})}{\lambda}\right],\ m \in [1,p-1]. \quad (5)$$

Clearly, $\frac{j-f(v_{11})}{\lambda} \leq \frac{p-1}{\lambda}$ for $j \in [2,p]$ and $\frac{m-f(v_{p,1})}{\lambda} \leq \frac{(p-1)-1}{\lambda}$ for $m \in [1,p-1]$ since $\lambda \geq 1$. Notice that the inequalities $1+j \leq f(x)+f(y) \leq m+p$ hold for all edges $xy \in E(G)$, and furthermore $f(xy) \neq f(uv)$ for distinct edges $xy, uv \in E(G)$ according to the definition of a (k,λ)-*mtl*. Thereby, the forms (3), (4) and (5) imply $q \leq \lceil (2p-3)/\lambda \rceil$.

The inequality $q < \lceil (2p-3)/\lambda \rceil$ shows that none of edges of G is labeled with some numbers of the form $f(u_1v_{11}) + \frac{1}{\lambda}(j-f(v_{11}))$ for $j \in [s,p]$ or $f(u_pv_{p,1}) + \frac{1}{\lambda}(m-f(v_{p,1}))$ for $m \in [t,p-1]$. Thereby, we can add some edges to G, the resulting graph is denoted by H, and then label the edges of $E(H) \setminus E(G)$ such that the labelled edges of $E(H) \setminus E(G)$ satisfy the form (4) or the form (5), the last labelling is denoted by h. It is evident that h is a super (k,λ)-*mtl* and the graph H is h-saturated. □

Example 4. A graph H_n^* has its vertex set $V(H_n^*) = \{u_1, u_2, \ldots, u_n\}$ and its edge set $E(H_n^*) = \{u_iu_{i+1} : i \in [1, n-1]\} \cup \{u_iu_{i+2} : i \in [1, n-2]\}$ for $n \geq 3$. It is sufficient to define directly a super $(k_n, 1)$-*mtl* f_n of H_n^* in the way that $f_n(u_i) = i$ for $i \in [1, n]$; $f_n(u_iu_{i+1}) = n+2i-1$ for $i \in [1, n-1]$; $f_n(u_iu_{i+2}) = n + 2i$ for $i \in [1, n-2]$; and $f_n(u_{n-1}u_n) = 3(n-1)$. It is not hard to show that $(2-n)$ is a magical constant of the $(k_n, 1)$-*mtl* f_n, i.e., $k_n = 2 - n$. Notice that H_n^* has $(2n-3)$ edges, $k_n = n - (2n-3) - 1$, so H_n^* is f_n-*saturated.* We have $H_{n-1}^* = H_n^* - u_n$, $f_{n-1}(u) = f_n(u)$ and $f_{n-1}(uv) = f_n(uv) - 1$ as well as $k_{n-1} = k_n + 1$. Another characteristic of H_n^* is that it is an *outer planar graph* having each inner face being triangular. The inequality in Theorem 5 is tightened by some H_n^*. The graph H_n^* admits another super $(3n, -1)$-*mtl* g_n defined by $g_n(u) = f_n(u)$ for $u \in V(H_n^*)$ and $g_n(uv) = 3n - 2 - f_n(uv)$ for $uv \in E(H_n^*)$. H_6^* and its saturated labellings are shown in Figure 4(a) and (b).

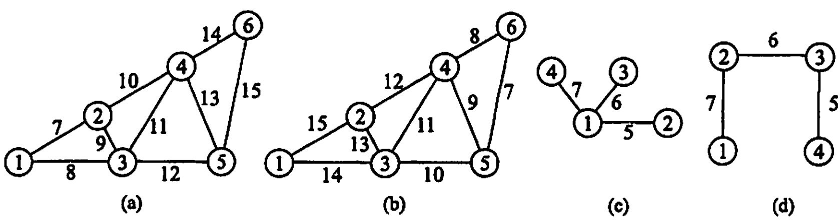

Figure 4: (a) H_6^* admits a super $(-4, 1)$-*mtl*; (b) H_6^* admits a super $(18, -1)$-*mtl*; (c) a star $K_{1,3}$ has a super $(-2, 1)$-*mtl*; (d) a path on 4 vertices has a super $(17, -2)$-*mtl*.

Theorem 6. *Let G be a connected (p, q)-graph possessing a super (k, λ)-mtl f. If $\lambda \leq 2$, there then is a graph $H = G + uw$ obtained by adding a new vertex u to G and joining u with a certain vertex w of G such that H admits a super $(k - \lambda, \lambda)$-mtl.*

Proof. Since $f(uv) \in [p+1, p+q]$ for every edge $uv \in E(G)$, there exists an edge xy such that $f(x) + f(y) = k + \lambda(p+q)$, where $f(xy) = p + q$, and $f(x) + f(y) \leq 2p - 1$. We can select a vertex w from G which holds $f(w) = \lambda + f(x) + f(y) - p - 1 \leq p$, no matter k is positive or negative. Now, joining w with a new vertex u out of G enables us to obtain a new graph $H = G + uw$. Before we define a super (k', λ)-*mtl* of H let us define an edge 1-float labelling h of the super labelling f, thus,

$$h(s)+h(t) = f(s)+f(t) = k+\lambda f(st) = k+\lambda\big[h(st)-1\big] = k-\lambda+\lambda h(st) \quad (6)$$

for each edge $st \in E(G)$. Now we define a desired labelling g of H in the way that $g(x) = h(x)$ when $x \in V(G) \cup E(G)$, $g(uw) = p + q + 2$ and

$g(u) = p + 1$. Thereby,

$$\begin{aligned} g(u) + g(w) &= p + 1 + h(w) = p + 1 + f(w) \\ &= p + 1 + \lambda + f(x) + f(y) - p - 1 = \lambda + f(x) + f(y) \\ &= \lambda + k + \lambda f(xy) = \lambda + k + \lambda(p + q) \\ &= (k - \lambda) + \lambda(p + q + 2) = (k - \lambda) + \lambda g(uw). \end{aligned}$$

For $st \in E(H)\setminus\{uw\}$, the definition of g shows $g(s) + g(t) = k - \lambda + \lambda g(st)$ from (6). Hence, g is a super $(k - \lambda, \lambda)$-*mtl* of H. □

3 Connections between (k, λ)-*mtls* and known labellings

Graceful type of labellings. If a (p, q)-graph G has a labelling $f : V(G) \to [0, q]$ with $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ such that the edge label set $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = [1, q]$, then we say that G is *graceful*, and f is a *graceful labelling* of G. Similarly, a (p, q)-graph G is *odd-graceful* if it admits a mapping $f : V(G) \to [0, 2q - 1]$ with $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ such that the edge label set $f(E(G)) = \{f(uv) = |f(u) - f(v)| : uv \in E(G)\} = \{1, 3, 5, \ldots, 2q - 1\}$, and f is called an *odd-graceful labelling* of G.

Theorem 7. *Let T be a tree on p vertices, and (X, Y) be the bipartition of T. Then T admits a super $(k, -1)$-mtl f with* $\max(f(X)) < \min(f(Y))$ *and* $k = 2p + 1 + |X|$ *if and only if T admits a graceful (odd-graceful) labelling g with* $\max(g(X)) < \min(g(Y))$.

Proof. Let $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$, where $s + t = p$, for the partition (X, Y) of vertex set of a tree T on p vertices. We show the first part of the theorem: T admits a super $(k, -1)$-mtl f with $\max(f(X)) < \min(f(Y))$ and $k = 2p + 1 + |X|$ if and only if T admits a graceful labelling g with $\max(g(X)) < \min(g(Y))$.

To show the proof of 'if' we take a super (k, λ)-*mtl* f of T such that $\max(f(X)) < \min(f(Y))$, where $k = 2p + 1 + s$. Since f is super, without loss of generality, $f(x_i) = i$ for $i \in [1, s]$, $f(y_j) = s + j$ for $j \in [1, t]$. Clearly, $\max(f(X)) < \min(f(Y))$. Notice that $f(E(T)) = [p + 1, 2p - 1]$. To find a desired labelling g of T, we set $g(x_i) = f(x_i) - 1 = i - 1$ for $i \in [1, s]$, $g(y_j) = f(y_{t-j+1}) - 1 = s + t - j + 1 - 1 = p - j$ for $j \in [1, t]$. Hence, $\max(g(X)) < \min(g(Y))$. According to $f(x_i) + f(y_j) = (2p+1+s) - f(x_iy_j)$ for each edge $x_iy_j \in E(T)$ we have $f(x_iy_j) = (2p + 1 + s) - f(x_i) - f(y_j) = 2p - i - j + 1 = p + (p - j) - (i - 1) = p + g(y_j) - g(x_i)$, thus, $g(x_iy_j) = g(y_j) - g(x_i) = f(x_iy_j) - p$, which means $g(E(T)) = [1, p - 1]$. Hence, g is a graceful labelling of T with $\max(g(X)) < \min(g(Y))$.

We present the proof of 'only if'. Suppose that α is a graceful labelling of T with $\max(\alpha(X)) < \min(\alpha(Y))$. By the definition of a graceful labelling, it is reasonable to set $\alpha(x_i) = i - 1$ for $i \in [1, s]$, $\alpha(y_j) = s + j - 1$ for $j \in [1, t]$. It is straightforward to define another labelling β of T as: $\beta(x_i) = \alpha(x_i)+1 = i$ for $i \in [1, s]$; $\beta(y_j) = \alpha(y_{t-j+1})+1 = s+(t-j+1)-1 = p-j+2$ for $j \in [1, t]$; $\beta(x_iy_j) = \alpha(x_iy_j)+p$ for every edge $x_iy_j \in E(T)$. Notice that $\alpha(x_iy_j) = \alpha(y_j)-\alpha(x_i) = s+j-1-i$ for each edge $x_iy_j \in E(T)$. Thereby, $\beta(x_i) + \beta(y_j) + \beta(x_iy_j) = i + (p-j+2) + (s+j-1-i) + p = 2p + s + 1$, that is, T admits β as a super $(2p + s + 1, -1)$-*mtl*.

We, now, consider the second part of the theorem about the odd-graceful labellings of trees.

For the proof of 'if', assume that f is a super $(2p+1+s, \lambda)$-*mtl* of T such that $\max(f(X)) < \min(f(Y))$. By the choice of f, we can set $f(x_i) = i$ for $i \in [1, s]$, $f(y_j) = s + j$ for $j \in [1, t]$. Next we extend the labelling f to another labelling h of T by setting $h(x_i) = 2(f(x_i) - 1) = 2(i - 1)$ for $i \in [1, s]$, $h(y_j) = 2(f(y_{t-j+1})-1)-1 = 2(s+t-j+1-1)-1 = 2(p-j)-1$ for $j \in [1, t]$. For every edge $x_iy_j \in E(T)$, we have $h(x_iy_j) = h(y_j) - h(x_i) = 2(p - i - j) + 1$, and

$$\begin{aligned} 2\big[f(x_iy_j) - p\big] - 1 &= 2\big[2p + 1 + s - f(x_i) - f(y_j) - p\big] - 1 \\ &= 2(p + 1 + s - i - s - j) - 1 = 2(p - i - j) + 1. \end{aligned}$$

Since $f(E(T)) = [p + 1, 2p - 1]$, so $h(E(T)) = \{1, 3, 5, \ldots, 2(p-1) - 1\}$. Evidently, h is an odd-graceful labelling of T with $\max(h(X)) < \min(h(Y))$.

To see the proof of 'only if', we take an odd-graceful labelling φ of T with $\max(\varphi(X)) < \min(\varphi(Y))$. Clearly, all elements of $\varphi(X)$ have the same parity, so do all elements of $\varphi(Y)$. Without loss of generality, we may assume that each element of $\varphi(X)$ is even, and each element of $\varphi(Y)$ is odd. By $\varphi(E(T)) = \{1, 3, 5, \ldots, 2(p-1) - 1\}$ and $\max(\varphi(X)) < \min(\varphi(Y))$, we have $\varphi(x_i) = 2(i - 1)$ for $i \in [1, s]$ and $\varphi(y_j) = 2(s - 1) + 2j - 1$ for $j \in [1, t]$. Thus, $\varphi(x_iy_j) = \varphi(y_j) - \varphi(x_i) = 2(s-1) + 2j - 1 - 2(i-1) = 2(s + j - i) - 1$ for each edge $x_iy_j \in E(T)$. Based on the labelling φ, we can define another labelling ψ of T in the way that $\psi(x_i) = \frac{1}{2}\varphi(x_i) + 1 = i$, $\psi(y_j) = \frac{1}{2}[\varphi(y_{t-j+1})+1]+1 = \frac{1}{2}[2(s-1)+2(t-j+1)-1+1]+1 = p-j+1$, and $\psi(x_iy_j) = \frac{1}{2}[\varphi(x_iy_j)+1]+p = p+s+j-i$ for every edge $x_iy_j \in E(T)$. Thereby, $\psi(x_i) + \psi(y_j) + \psi(x_iy_j) = i+p-j+1+p+s+j-i = 2p+1+s$ is a constant, which means that ψ is a super $(2p + 1 + s, -1)$-*mtl* with $\max(f(X)) < \min(f(Y))$.

The proof of the theorem is complete. □

An explanation of Theorem 7 is shown in Figure 5(a), (b) and (c).

Example 5. Since every caterpillar T admits a graceful labelling g with $\max(g(X)) < \min(g(Y))$, where (X, Y) is the bipartition of $V(T)$, by Theorem 7, T admits a super (k, λ)-*mtl*. We show that T has another supper

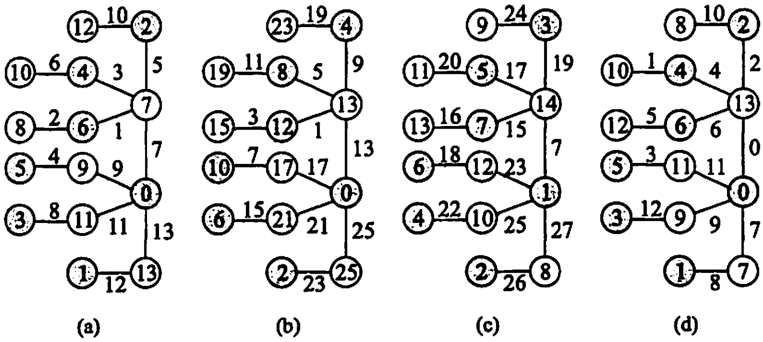

Figure 5: A tree T has: (a) a graceful labelling f with $\max(f(X)) < \min(f(Y))$; (b) an odd-graceful labelling g with $\max(g(X)) < \min(g(Y))$; (c) a super $(36,-1)$-*mtl* h with $\max(h(X)) < \min(h(Y))$; (d) a felicitous labelling obtained from h.

$(k_0, 1)$-*mtl* in the following. Let $\mathcal{L}(T)$ be the set of all leaves of a caterpillar T on p vertices. Hence, the graph $T - \mathcal{L}(T)$ is a path, say $P = u_1u_2\cdots u_m$, $m \geq 1$. Let $S_i = \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_i}\}$ be the set of all leaves adjacent to $u_i \in V(P)$ for $i \in [1, m]$. Clearly, $\mathcal{L}(T) = \bigcup_{i=1}^{m} S_i$. If $m = 1$, T is a star. We have a $(2-p, 1)$-magical labelling h of T as: $h(u_1) = 1$, $h(u_{1,j}) = 1 + j$ and $h(u_1u_{1,j}) = p + j$ for $j \in [1, p-1]$. For $m \geq 2$, we define directly a labelling f of T in the following. Let $\sum_{l=1}^{0} n_{2l} = 0$ and $s = \lfloor (m+1)/2 \rfloor$, we have $f(u_{2t-1}) = t + \sum_{l=1}^{t-1} n_{2l}$ and $f(u_{2t,j}) = f(u_{2t-1}) + j$ with respect to $j \in [1, n_{2t}]$ and $t \in [1, s]$.

If $m = 2s$, let $Q = f(u_{2s,n_{2s}}) = f(u_{2s-1}) + n_{2s}$. Set $f(u_{1,j}) = Q + j$ for $j \in [1, n_1]$, and $f(u_{2t}) = Q + t + \sum_{l=1}^{t} n_{2l-1}$ and $f(u_{2t+1,j}) = f(u_{2t}) + j$ for $j \in [1, n_{2t+1}]$ and $t \in [1, s-1]$. Thereby, $f(u_{2s}) = f(u_m) = p$. Now we go on to the case $m = 2s - 1$. Let $R = f(u_m) = s + \sum_{l=1}^{s-1} n_{2l}$, and set $f(u_{1,j}) = R + j$ for $j \in [1, n_1]$, and $f(u_{2t}) = R + t + \sum_{l=1}^{t} n_{2l-1}$ and $f(u_{2t+1,j}) = f(u_{2t}) + j$ for $j \in [1, n_{2t+1}]$ and $t \in [1, s-1]$. Clearly, $f(u_{2s-1,n_{2s-1}}) = f(u_{m,n_m}) = p$.

The rest labels are assigned to all edges of T. We have $f(u_1u_{1,j}) = p + j$ for $j \in [1, n_1]$ and $f(u_tu_{t+1}) = p + t + \sum_{l=1}^{t} n_l$ and $f(u_tu_{t,j}) = f(u_tu_{t+1}) + j$, where $j \in [1, n_t]$ and $t \in [1, m-1]$. It is not hard to compute the magical constant $k_0 = 1 - p + \lfloor \frac{m+1}{2} \rfloor + \sum_{l=1}^{M} n_{2l}$, where $M = \lceil \frac{1}{2}(m-1) \rceil$. Thereby, f is a supper $(k_0, 1)$-magical labelling. □

Example 6. Acharya [1] proved that every connected graph can be embedded as an induced subgraph of a connected graceful graph. Acharya, Rao, and Arumugam [2] proved: every planar graph can be embedded as an induced subgraph of a planar graceful graph. In [14], it has been shown that a graceful tree is a subgraph of a tree T having a graceful labelling f such that $\max(f(X)) < \min(f(Y))$, where (X, Y) is the bipartition of T. From the above facts and Theorem 7, we conclude that *every tree is a subgraph of a tree having super (k, λ)-mtls.*

Felicitous labelling. A (p, q)-graph G is *felicitous* if it admits a mapping $f : V(G) \to [0, q]$ with $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ such that the edge label set $f(E(G)) = \{f(uv) = f(u) + f(v) \pmod q : uv \in E(G)\} = [0, q-1]$, and we call f a *felicitous labelling* of G. Furthermore, f is *super* if $f(V(G)) = [0, p-1]$.

Theorem 8. *A connected (p,q)-graph admits a super (k, λ)-mtl, where $\lambda = 1$ for $1 - p \leq k < p + q$ and $\lambda = -1$ for $p + q + 2 \leq k$, if and only if it admits a super felicitous labelling.*

Proof. To see the proof of 'if', we may assume that f is a super (k, λ)-*mtl* of a connected (p, q)-graph G with $\lambda = 1$ for $1 - p \leq k < p + q$ and $\lambda = -1$ for $p + q + 2 \leq k$. For the case $\lambda = 1$ for $1 - p \leq k < p + q$, we define a new labelling $h(x) = f(x) - 1$ for $x \in V(G)$. Hence, for edges $xy \in E(G)$, $h(x) + h(y) = f(x) + f(y) - 2 = k + f(xy) - 2 \geq 1 - p + f(xy) - 2 = f(xy) - (p + 1) \geq 0$. Since f is super, we have a set $\{k + f(xy) - 2 : xy \in E(G)\} = \{k + p - 1, k + p, \cdots, k + p - 2 + q\}$. Clearly, $\{k + f(xy) - 2 \pmod q : xy \in E(G)\} = [0, q - 1]$. Consider the case $\lambda = -1$ for $p + q + 2 \leq k$. To the above labelling h, we have $h(x) + h(y) = f(x) + f(y) - 2 = k - f(xy) - 2 \geq p + q + 2 - f(xy) - 2 \geq 0$. Under modulo q, $\{k - f(xy) - 2 \pmod q : xy \in E(G)\} = [0, q - 1]$. Thereby, h is a super felicitous labelling.

We present the proof of 'only if'. For a super felicitous labelling α of G, we have $\{\alpha(x) + \alpha(y) : xy \in E(G)\} = \{a + i : i \in [1, q]\}$ for some positive integer a. If $a \leq p$, we define a desired labelling β in the way that $\beta(x) = \alpha(x) + 1$ for $x \in V(G)$, and $\beta(uv) = \alpha(u) + \alpha(v) + p - a$ for each edge $uv \in E(G)$. Hence, for $xy \in E(G)$, $\beta(x) + \beta(y) = \alpha(x) + \alpha(y) + 2 = \beta(uv) + a - p + 2$. Notice that $a - p + 2$ is a constant. It has shown that β really is a super $(a - p + 2, 1)$-*mtl* of G.

If $a > p$, we can define a labelling $\gamma(x) = \alpha(x) + 1$ for $x \in V(G)$, and $\gamma(uv) = a + p + q + 1 - [\alpha(u) + \alpha(v)]$ for each edge $uv \in E(G)$. Consequently, $\gamma(x) + \gamma(y) = \alpha(x) + \alpha(y) + 2 = a + p + q + 1 - \gamma(uv) + 2$ for $xy \in E(G)$. So, γ is a super $(a + p + q + 3, -1)$-*mtl*. □

Two examples for illustrating Theorem 8 are shown in Figure 5(d) and Figure 6 (b).

Antimagic total labelling. A (b, d)-*edge antimagic total labelling* of a connected (p, q)-graph G is a bijection f: $V(G) \cup E(G) \to [1, p + q]$ such that $\{f(u) + f(v) + f(uv) : uv \in E(G)\} = \{b, b + d, b + 2d, \ldots, b + (q - 1)d\}$ for some certain integers b and d. And f is *super* if $f(V(G)) = [1, p]$.

Theorem 9. *A connected (p, q)-graph G admits a super $(k, -1)$-mtl if and only if it admits a super $(k+1-q, 2)$-edge antimagic total labelling.*

Proof. Let G be a connected (p, q)-graph with $E(G) = \{u_iv_i : i \in [1, q]\}$. We show a constructive proof of 'if'. Let f be a super $(k, -1)$-*mtl* of G such that $f(u_iv_i) = p + i$ for $i \in [1, q]$. Successively, we extend the labelling f to another labelling g of G as: $g(x) = f(x)$ for $x \in V(G)$, and $g(u_iv_i) = f(u_{p+q-i+1}v_{p+q-i+1}) = 2p + q + 1 - f(u_iv_i)$ for $i \in [1, q]$. Therefore, $g(u_i) + g(v_i) + g(u_iv_i) = f(u_i) + f(v_i) + f(u_iv_i) + (2p + q + 1) - 2f(u_iv_i) = k + q + 1 - 2i$ for $i \in [1, q]$, which distributes the set $\{k - q + 1, k - q + 1 + 2, k - q + 1 + 4, \dots, k - q + 1 + 2(q - 1)\}$. Hence, g really is a super $(k - q + 1, 2)$-edge antimagic total labelling.

The proof of 'only if'. Let α be a super $(m - q, 2)$-edge antimagic total labelling of G. By the definition of a (b, d)-edge antimagic total labelling, $\alpha(u_i) + \alpha(v_i) + \alpha(u_iv_i) = m - q + 2(i - 1)$ for $i \in [1, q]$, where $u_iv_i \in E(G) = \{u_iv_i : i \in [1, q]\}$. It is straightforward to define a labelling β of G as: $\beta(x) = \alpha(x)$ for $x \in V(G)$, and $\beta(u_iv_i) = \alpha(u_{q-i+1}v_{q-i+1})$ for $i \in [1, q]$. Therefore, we have $\beta(u_i)+\beta(v_i)+\beta(u_iv_i) = \alpha(u_i)+\alpha(v_i)+\alpha(u_{q-i+1}v_{q-i+1})$ and $\beta(u_{q-i+1}) + \beta(v_{q-i+1}) + \beta(u_{q-i+1}v_{q-i+1}) = \alpha(u_{q-i+1}) + \alpha(v_{q-i+1}) + \alpha(u_iv_i)$, and furthermore

$$\begin{aligned}
&[\beta(u_i) + \beta(v_i) + \beta(u_iv_i)] + [\beta(u_{q-i+1}) + \beta(v_{q-i+1}) + \beta(u_{q-i+1}v_{q-i+1})] \\
&= m - q + 2(i - 1) + m - q + 2(q - i + 1 - 1) \\
&= 2(m - 1)
\end{aligned}$$

for $i \in [1, \lfloor \frac{q+1}{2} \rfloor]$, which means that β is a super $(m - 1, -1)$-*mtl*. □

Figure 6(a) and (c) show two labelled Petersen graphs for illustrating Theorem 9.

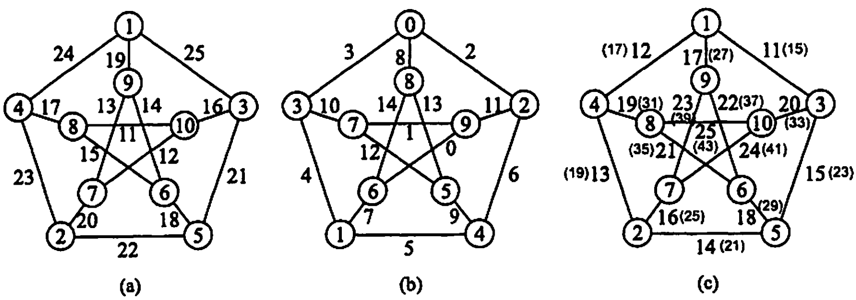

Figure 6: Petersen graph has: (a) a super $(29, -1)$-*mtl*; (b) a felicitous labelling from f; (c) a super $(15, 2)$-edge antimagic total labelling from f.

4 Problems for further works

It has been known that there are many methods for constructing graceful trees that admit such graceful labellings described in Theorem 7 ([3], [6], [14]). Hence, we do not deal with more constructions of trees having super (k, λ)-*mtls*, except Theorem 6. Notice that a $(k, -1)$-*mtl* is just an edge-magic total labelling. Based on Theorems 7, 8 and 9, we propose:

Conjecture 10. *Every tree admits a super (k, λ)-magically total labelling for some integers k and $\lambda \neq 0$.*

Conjecture 11. *Every path on n vertices admits all (k, λ_k)-magically total labellings for each $k = 2, 3, \dots, 2(n-1)$ and some non-zero integers λ_k.*

We have verified Conjecture 11 for all paths on $n \leq 10$ vertices. We show an example that holds Conjecture 11. A path on n vertices is denoted as $P_n = u_1e_1u_2e_2 \cdots u_{n-1}e_{n-1}u_n$. We label each vertex u_i of P_n with ① and each edge e_i of P_n with a number, respectively. For the path P_7, $\lambda_k = 1$ and each integer $k \in [2, 12]$ we have

$k = 2$ ①2③10⑨13⑥12⑧11⑤7④,
$k = 3$ ⑤3①2④10⑨13⑦12⑧11⑥,
$k = 4$ ⑦12⑨13⑧6②1③4⑤11⑩,
$k = 5$ ①5⑨11⑦12⑩13⑧6③2④,
$k = 6$ ③2⑤7⑧12⑩13⑨4①6⑪,
$k = 7$ ③6⑩12⑨4②8⑬7①5⑪,
$k = 8$ ④8⑫7③5⑩11⑨2①6⑬,
$k = 9$ ⑥7⑩9⑧12⑬5①3⑪4②,
$k = 10$ ②5⑬4①3⑫8⑥7⑪10⑨,
$k = 11$ ④5⑫2①3⑬11⑨6⑧7⑩,
$k = 12$ ④3⑪12⑬9⑧2⑥1⑦5⑩.

Problem 12. *Characterize trees having (k, λ)-magically total labellings with one of the following*
(*i*) *even $|\lambda|$.*
(*ii*) *$|\lambda|$ is unique.*
(*iii*) *the labels of vertices have the same parity.*
(*iv*) *$k = a, a+1, a+2, \dots, a+b$ for integers a and $b \geq 1$.*

To consider special kinds of connected graphs having (k, λ)-magically total labellings, we do not discover graphs with large cliques. Furthermore we wish to study the following

Problem 13. *Let $\mathcal{G}_{mtl}(G)$ be the set of all subgraphs having (k,λ)-magically total labellings in a graph G, and let $K(G)$ be the number of vertices of a largest clique of G. Find a graph $G^* \in \mathcal{G}_{mtl}(K_n)$ such that G^* holds one of the following*

(1) $|E(G^*)| \geq |E(H)|$ *for every* $H \in \mathcal{G}_{mtl}(K_n)$.

(2) $K(G^*) \geq K(H)$ *for every* $H \in \mathcal{G}_{mtl}(K_n)$.

(3) $\chi(G^*) \geq \chi(H)$ *for every* $H \in \mathcal{G}_{mtl}(K_n)$, *where* $\chi(G)$ *is the chromatic number of* G.

It has been noticed that one of two graphs shown in Figure 4 (c) and (d) having $(-2,1)$-magically total labelling and $(17,-2)$-magically total labelling is not isomorphic to the rest. It would be interesting to consider the following problem.

Problem 14. *Let G_i be a connected (p_i,q_i)-graph having a super (k_i,λ_i)-magically total labelling f_i for $i=1,2$, we have $V_i=\{f_i(u): u\in V(G_i)\}$ and $E_i=\{f_i(uv): uv\in E(G_i)\}$ for $i=1,2$. If $V_1=V_2$ and $E_1=E_2$, show conditions for $G_1\cong G_2$.*

Now, we rewrite the problem of (k,λ)-magically total labellings on graphs in the manner of integer sets based on the reasons: (1) it may be interesting to the readers who are not familia with knowledge of graph theory; (2) it may be convenient to study the problem by computer.

Problem 15. (Problem of (k,λ)-matchable sets) *Let $U=\{1,2,\dots,\frac{1}{2}p(p+1)\}$ be the universal set for integers $p\geq 2$. For a p-set $V\subset U$, we wish to find a q-set $E\subset U\setminus V$ with respect to $p-1\leq q$ and two integers k and λ ($\neq 0$) such that*

(i) $V\cup E=\{1,2,\dots,p+q\}$; *and*

(ii) for each $c\in E$ there are distinct $a,b\in V$ such that

$$a+b=k+\lambda c. \tag{7}$$

We call V and E a pair of (k,λ)-matchable sets. Furthermore, determine: (a) the possible values of k and $\lambda\neq 0$; (b) the maximum of $|q-p| = \big||V|-|E|\big|$ spanning over all pairs of (k,λ)-matchable sets V and E.

Here are three examples for illustrating the above problem of (k,λ)-matchable sets.

(1) For a 4-set $Q=\{1,2,5,8\}\subset U=[1,10]$, we can foind a 4-set $S=\{3,4,6,7\}\subset U\setminus Q$ and $(k,\lambda)=(3,1)$ such that $Q\cup S=[1,8]$, and the equation (7) holds true.

(2) There are two sets $Q=\{1,4,7,8\}\subset U=[1,10]$ and $S=\{2,3,5,6,9\}\subset U\setminus Q$, and $(k,\lambda)=(6,1)$. Clearly, $Q\cup S=[1,9]$, and the equation (7) holds true.

(3) For a 4-set $Q = \{1, 2, 3, 4\} \subset U = [1, 10]$ and a 3-set $S = \{5, 6, 7\} \subset U \setminus Q$, there are $(k, \lambda) = (-2, 1)$ and $(k', \lambda') = (17, -2)$ for satisfying the equation (7).

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