A note on the lower bound for the Ramsey number $R(7, 9)$

Fei Deng
School of Information Science and Technology,
Chengdu University of Technology, Chengdu, 610059, China
dengfei@cdut.cn

Abstract. For integers $s, t \geq 1$, the Ramsey number $R(s, t)$ is defined to be the least positive integer n such that every graph on n vertices contains either a clique of order s or an independent set of order t. In this note, the lower bound for the Ramsey number $R(7, 9)$ is improved from 241 to 242. The new bound is obtained by searching the maximum common induced subgraph between two graphs with a depth variable local search technique.

1 Introduction

In this note, only undirected graphs without multiple edges or loops are considered. If G is a graph, then the set of vertices of G is denoted by $V(G)$, the set of edges of G by $E(G)$, the cardinality of $V(G)$ by $|V(G)|$, and the complementary graph of G by $\overline{G}$. The subgraph of G induced by $S \subseteq V(G)$ is denoted by $G[S]$. A cycle of order n is denoted by C_n. Given a positive integer n, $Z_n = \{0, 1, 2, \cdots, n-1\}$, and $S \subseteq \{1, 2, \cdots, \lfloor n/2 \rfloor\}$, let G be a graph with the vertex set $V(G) = Z_n$ and the edge set $E(G) = \{(x, y) : \min\{|x-y|, n-|x-y|\} \in S\}$, then G is called a cyclic graph of order n, denoted by $G_n(S)$. For integers $s, t \geq 1$, the Ramsey number $R(s, t)$ is defined to be the least positive integer n such that every graph on n vertices contains either a clique of order s or an independent set of order t. G is called a (p, q)-graph if G contains neither a complete graph on p vertices nor an independent set of order q. A (p, q)-graph on n vertices is called a $(p, q; n)$-graph. A recent summary of the state of the art for Ramsey numbers can be found in the Dynamic Survey [1].

JCMCC **87** (2013), pp. 297-302

In the area of the Ramsey numbers, constructive and probabilistic methods play an important role in the literature. However, a number of lower bounds have been also found by computer search techniques. For example, the lower bounds for $R(4,6)$, $R(3,10)$, and $R(5,5)$, which are the smallest unsettled cases for two color classical Ramsey numbers, were improved by Exoo. The lower bounds for $R(6,8)$, $R(7,9)$, and $R(8,17)$ were improved in [6, 7].

In [7], the following configuration is set to compute lower bounds for Ramsey numbers: Let G be a $(k,s;p)$-graph, H be a $(k,t;q)$-graph. If M is a common induced subgraph of G and H, and a partition of the vertex set V_M into $W_1 \cup W_2$ so that neither of $M[W_1]$ nor $M[W_2]$ contains a clique of order $k-2$. Then a $(k, s+t-1, p+q+|M|)$-graph can be constructed and therefore, $R(k, s+t-1) > p+q+|M|$.

In order to improve the lower bounds by using this approach, we first need to search for a common induced subgraph M (between two graphs G and H) with the order as large as possible, then we use a procedure to partition the vertex set V_M into $W_1 \cup W_2$ so that neither of $M[W_1]$ nor $M[W_2]$ contains a clique of order $k-2$. However, for large graphs G and H, an exhaustive search usually can not obtain satisfactory solutions. Therefore, heuristic search may be considered. We use a depth variable local search heuristic to construct a $(7,9;241)$-graph successfully, and have the following result:

Theorem 1 $R(7,9) \geq 242$.

2 The approach

Local search is a common tool for finding approximation solutions in reasonable time for combinatorial optimization problems. Usually the current solution x is repeatedly replaced by a better solution from the neighborhood of x until no better solution can be obtained. When no better solution from the neighborhood of x can be found, the current solution is called *locally optimal.* In many cases, local search can be applied into heuristic algorithms such as simulated annealing, ant colony, and particle swarm optimizations. In many cases, it is hard to obtain the best solution, or even an approximate solution with high quality, for a certain combinatorial optimization problems. Therefore, various local search methods, such as phased local search [8] and variable depth local search , were proposed. The variable depth local search was initially used to solve graph partitioning problem (GPP) [9] and the traveling salesman problem (TSP) [10]. Then it was applied to other heuristic algorithms [11, 12, 16, 14, 15]. In [16], it was applied to solve the maximum clique problem and successfully obtained good solutions.

We introduce the basic variable depth local search algorithm (Algorithm 2) for finding a common induced subgraph between two graphs with the order as large as possible. The main framework of this algorithm is taken from KLS in [16].

We will use the following notations to describe the algorithms.

S: the current vertex set of common induced subgraph.

PA: the possible vertex set of addition, i.e., $PA = \{v \in \overline{S} : H$ contains an induced subgraph of $G[\{v\} \cup S]\}$.

The variable depth local search consists of two phases: *addition* and *drop*. Let G and H be two graphs , $S \subseteq V(G)$. We denote by $\overline{S}$ the subset $V(G) \setminus S$. We take one vertex $S \subseteq V(G)$ as an initial common induced subgraph. Now we consider if there exists a vertex $v \in S$ such that H contains an induced subgraph of $G[\{v\} \cup S]$. If so, then $PA \neq \emptyset$. Otherwise $PA = \emptyset$.

- *addition phase*:
  If $PA \neq \emptyset$, we choose a vertex $v \in \overline{S}$ to make $S \leftarrow S \cup \{v\}$. The vertex v is selected such that the resulting PA is maximum. If multiple vertices are found, choose one randomly.

- *drop phase*:
  If $PA = \emptyset$, we choose a vertex $v \in S$ to make $S \leftarrow S \setminus \{v\}$. The vertex v is selected such that the resulting PA is maximum. If multiple vertices are found, choose one randomly.

Procedure 1 VDS(G, H, S, PA)

Require:

S: the current common induced subgraph between G, H;

PA: the possible vertex set of addition;

```
repeat
  S_Prev ← S; D ← S_Prev; P ← {1, 2, ···, n}; g ← 0; g_max ← 0
  repeat
    if PA ∩ P ≠ ∅ then
      find a vertex v from PA ∩ P that maximizes the resulting PA.
      if multiple vertices are found, select one randomly.
      S ← S ∪ {v}; g ← g + 1; P ← P \ {v};
      if g > gmax then
        g_max ← g, S_best ← S;
      end if
    else
      if S ∩ P ≠ ∅ then
        find a vertex v from S ∩ P that maximizes the resulting PA.
        if multiple vertices are found, select one randomly.
        S ← S \ {v}; g ← g − 1; P ← P \ {v};
        if v is contained in S_Prev then
          D ← D \ {v};
        end if
        if g_max > 0 then
          S ← S_best;
        else
          S ← S_prev;
        end if
      end if
    end if
  until D = ∅
until g_max ≤ 0
```

We fist use a procedure whose pseudocode is shown in Procedure 1. The variable S_{best} records the vertex set of common subgraph with the maximum cardinality found so far. The variable g_{max} records the difference of the current solution and the initial solution. So $g_{max} > 0$ means a larger common subgraph is found.

Procedure 1 starts with an initial S, then changes the set S by repeatedly adding or removing vertices in G. If we only consider *addition* and *drop* operation, the procedure may be trapped. Therefore a forbidden table P is used. If a vertex v is added to S, at the next recent iterations, v can not be removed immediately; a vertex v is removed from S, at the next recent iterations, v can not be added immediately.

We can see that the input of Procedure 1 is a small common induced subgraph and the output is a maximum common induced subgraph found so far, which is actually obtained by adding or removing several vertices from the initial common induced subgraph. We start with S being only one vertex, then call Procedure 1 many times and record the best solution during the search process to obtain the common induced subgraph.

References

[1] S. P. Radziszowski, Small Ramsey numbers, Electron. J. Combin., DS1, 2011. http://www.combinatorics.org/issue/view/Surveys.

[2] G. Exoo, A lower bound for $R(5,5)$, J. Graph Theory 13 (1989) 97-98.

[3] G. Exoo, On two classical Ramsey numbers of the form $R(3,n)$, SIAM J. Discrete Math., 2 (1989) 488-490.

[4] G. Exoo, Announcement: On the Ramsey numbers $R(4,6), R(5,6)$, and $R(3,12)$, Ars Combin., 35 (1993) 85.

[5] G. Exoo, On the Ramsey Number $R(4,6)$, Electron J. Combin., 19 (2012), #P66.

[6] Z. Shao, J. Xu and L. Pan, Lower bounds on Ramsey numbers $R(6,8), R(7,9)$ and $R(8,17)$, Ars Combinatoria, 94 (2010) 55-59.

[7] X. Xu, Z. Shao and S. P. Radziszowski, More Constructive Lower Bounds on Classical Ramsey Numbers, SIAM J. on Discrete Math. 25 (2011) 394-400.

[8] W. Pullan, Phased local search for the maximum clique problem. J. Comb. Optim. 12 (2006) 303-323.

[9] B. W. Kernighan, S. Lin, An efficient heuristic procedure for partitioning graphs. Bell System Techn. J. 49 (1970) 291-307.

[10] S. Lin, B. W. Kernighan, An effective heuristic algorithm for the traveling salesman problem. Oper. Res. 21 (1973) 498-516.

[11] D. Applegate, W. Cook, A. Rohe, Chained Lin-Kernighan for large traveling salesman problems. INFORMS J. Comput. 15 (1) (2003) 82-92.

[12] D. S. Johnson, Local optimization and the traveling salesman problem. in: Proc. 17th Internat. Colloquium on Automata, pp. 446-461, Languages and Programming, 1990.

[13] K. Katayama, H. Narihisa, Iterated local search approach using genetic transformation to the traveling salesman problem, in: Proc. Genetic and Evolutionary Computation Conf. : pp. 321-328, (1999).

[14] P. Merz, B. Freisleben, Memetic algorithms for the traveling salesman problem. Complex Systems 13 (4) (2001) 297-345.

[15] P. Merz, B. Freisleben,Fitness landscapes, memetic algorithms and greedy operators for graph bipartitioning, Evolutionary Comput. 8 (1) (2000) 61-91.

[16] K. Katayama, A. Hamamoto, H. Narihisa, An effective local search for the maximum clique problem, Information Processing Letters 95 (2005) 503-511.