On Odd-gracefulness of All Symmetric Trees[1]

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Abstract

In 1991 Gnanajothi conjectured: Each tree is odd-graceful. In this paper, we define the edge-ordered odd-graceful labelling of trees and show the odd-gracefulness of all symmetric trees.

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1 Introduction and concepts

Graph labelling is a classic problem in mathematics and computing. The problems of transforming a graph labelling into another one have applications in areas such as bioinformatics, (scale-free, small-world) networks, VLSI, and so on. An example, studied first by Graham and Sloane in [5], is the harmonious graphs of modular versions of additive bases problems stemming from error-correcting codes. The overview of concepts and results of current graph labellings can be found in the survey paper [3] in which the author collects more than 1000 papers on various graph labellings. For graph labellings including $|f(u) - f(v)|$ or $f(u) + f(v)$ ($uv \in E(G)$) there are the following famous conjectures.

Conjecture. Evert tree has at least two vertices.

1. (1966 [7]) *Every tree is graceful.*
2. (1980 [5]) *Every tree is harmonious.*
3. (1991 [4]) *Every tree is odd-graceful.*

These conjectures are the hot topics of graph labellings with an extensive and continuously growing literature [3]. In this article, we will study the odd-gracefulness of symmetric trees. First of all, we will show a method for efficiently connecting a graceful tree with an edge-ordered odd-graceful tree. Standard notation and terminology of graph theory are used here. The undefined terminologies will follow [2] and [3]. All graphs mentioned

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in this article are simple, undirected and finite. For the sake of simplicity, the shorthand symbol $[m, n]$ stands for an integer set $\{m, m+1, \dots, n\}$, where m and n are integers with $0 \le m < n$; the notation $[s, t]^o$ indicates an odd-set $\{s, s+2, \dots, t\}$, where s and t both are odd integers with $1 \le s < t$; and the notation $[k, \ell]^e$ is an even-set $\{k, k+2, \dots, \ell\}$, where k and ℓ both are even integers with $0 \le k < \ell$.

Definition 1. [3] A *labelling* f of a tree T on n vertices is a mapping $f : V(T) \to [0, N]$ such that vertex labels $f(u)$ and $f(v)$ are different for distinct $u, v \in V(T)$, where $N \ge n-1$ is a natural number. The label of each edge uv of T, denoted as $f(uv)$, is defined as $|f(u) - f(v)|$. The notations $f(V(T))$ and $f(E(T))$ denote the set of vertex labels and the set of edge labels, respectively. If $f(V(T)) = [0, n-1]$ and $f(E(T)) = [1, n-1]$, we say that f is a *graceful labelling*, and T is *graceful*. If T admits a labelling h with $h(V(T)) \subseteq [0, 2n-3]$ and $h(E(T)) = [1, 2n-3]^o$, then h is called an *odd-graceful labelling*, and T is *odd-graceful* [4].

Definition 2. Let (V_1, V_2) be the bipartition of a tree T. If T admits an odd-graceful labelling f such that each edge uv with $u \in V_1$ and $v \in V_2$ holds $f(u) < f(v)$, we refer to f as an *edge-ordered odd-graceful labelling* of T, and T as an *edge-ordered odd-graceful tree*.

In a *rooted tree* T with the root w, if distances $d_T(u, w)$ and $d_T(v, w)$ are the same for distinct vertices $u, v \in V(T)$, then we call u and v both to be at the same level. The set of *neighbours* of a vertex v in a graph G is denoted by $N_G(v)$. The *degree* $d_G(v)$ of a vertex v in a graph G is defined as $d_G(v) = |N_G(v)|$. A *leaf* is a vertex of degree one. Our results are related with three particular classes of trees as follows.

1. A *symmetric tree* is a rooted tree in which the vertices in every level have the same degree. A *complete k-ary tree* is a symmetric tree in which the degree of the root is k, the rest non-leaf vertices have degree $k+1$. It is known that each symmetric tree is graceful, and each complete *k-ary tree* is graceful [1].

2. A connected graph H is *k-vertex-symmetric* if H has a set X of k vertices such that $H - X$ has components $H_1, H_2, \dots, H_m$ with $H_i \cong H_j$. We describe a class of 1-vertex-symmetric trees $T_w^m(n)$ as follows. Let $T(n)$ be a tree on n vertices and let $T_i(n)$ be a copy of $T(n)$ for $i \in [1, m]$. For a vertex $w \in V(T(n))$, the vertex $w_i \in V(T_i(n))$ is isomorphic to the vertex w, $i \in [1, m]$. The tree obtained by adding a new vertex w_0 and then joining w_0 with every $w_i \in V(T_i(n))$, $i \in [1, m]$, is denoted as $T_w^m(n)$ [6].

3. A connected graph G is *k-edge-symmetric* if there exists a set S of k edges of G such that the graph $G - S$ has components $G_1, G_2, \dots, G_m$ ($m \ge 2$) with $G_i \cong G_j$.

2 Results

Lemma 1. *Let T be a tree on n vertices.*

(i) [8] Suppose that T admits a graceful labelling f such that $f(z) = n-1$ for a certain vertex $z \in V(T)$. Then T admits a graceful labelling h defined as $h(x) = n - 1 - f(x)$ for each vertex $x \in V(T)$ such that $h(z) = 0$;

(ii) [1] Every symmetric tree T with the root w admits a graceful labelling f such that $f(w) = 0$.

(iii) Suppose that T admits an (edge-ordered) odd-graceful labelling f such that $f(y) = 2n - 3$ for a certain vertex $y \in V(T)$. Then T admits an (edge-ordered) odd-graceful labelling h defined as $h(x) = 2n - 3 - f(x)$ for each vertex $x \in V(T)$ such that $h(y) = 0$.

Very often, we call the labelling h in Lemma 1 the *complementary labelling* of f.

Theorem 2. *Let T be a graceful tree and let T' be a copy of T. Joining any vertex $x \in V(T)$ with its isomorphic vertex $x' \in V(T')$ by an edge yields an edge-ordered odd-graceful tree.*

Proof. Let f be a graceful labelling of a tree T on n vertices. Hence, $f(V(T)) = [0, n-1]$, $f(E(T)) = [1, n-1]$. Let (X, Y) be the bipartition of $V(T)$, where $X = \{x_i : i \in [1, s]\}$ and $Y = \{y_j : j \in [1, t]\}$ with $s + t = |T| = n$. Correspondingly, the bipartition of vertices of a copy T' of T is (X', Y'), where $X' = \{x'_i : i \in [1, s]\}$, $Y' = \{y'_j : j \in [1, t]\}$; x'_i is isomorphic to x_i for $i \in [1, s]$; y'_j is isomorphic to y_j for $j \in [1, t]$; $x'_i y'_j \in E(T')$ is isomorphic to $x_i y_j \in E(T)$ for $i \in [1, s]$, $j \in [1, t]$. A new tree G is obtained by joining a vertex $x \in V(T)$ with its isomorphic vertex $x' \in V(T')$ together. Clearly, $(X \cup Y', X' \cup Y)$ is the bipartition of $V(G)$ and $E(G) = E(T) \cup E(T') \cup \{xx'\}$, $|V(G)| = 2n$. We define a labelling φ of G as follows:

(i) $\varphi(x_i) = 2f(x_i)$, $\varphi(x'_i) = 2f(x_i) + 2n - 1$, $i \in [1, s]$;

(ii) $\varphi(y_j) = 2f(y_j) + 2n - 1$, $\varphi(y'_j) = 2f(y_j)$, $j \in [1, t]$.

Clearly, φ is a labelling from $V(G)$ to $[0, 4n-3]$ since $\varphi(u) \neq \varphi(v)$ for distinct $u, v \in V(G)$, $\varphi(X \cup Y') = [0, 2n-2]^e$, $\varphi(X' \cup Y) = [2n-1, 4n-3]^o$. Therefore, $\varphi(u) < \varphi(v)$ for each edge $uv \in E(G)$ with $u \in X \cup Y'$ and $v \in X' \cup Y$.

According to the definition of the labelling φ, we have $\varphi(x_i y_j) = 2n-1+2(f(y_j) - f(x_i))$ and $\varphi(x'_i y'_j) = 2n-1-2(f(y_j) - f(x_i))$. Hence, whenever $f(x_i) < f(y_j)$ or $f(x_i) > f(y_j)$, the labels of edges $x_i y_j \in E(T) \subset E(G)$ and $x'_i y'_j \in E(T') \subset E(G)$ form the set $\{\varphi(x_i y_j), \varphi(x'_i y'_j)\} = \{2n - 1 + 2f(x_i y_j), 2n - 1 - 2f(x_i y_j)\}$ for $i \in [1, s]$ and $j \in [1, t]$. Since $f(E(T)) =$

$\{f(x_iy_j) : i \in [1,s],\ j \in [1,t]\} = [1, n-1]$, then

$$\begin{aligned}&\varphi(E(G)\backslash\{xx'\}) = \{\varphi(x_iy_j), \varphi(x_i'y_j') : i \in [1,s], j \in [1,t]\}\\ &= \{2n-1+2f(x_iy_j), 2n-1-2f(x_iy_j) : i \in [1,s], j \in [1,t]\}\\ &= [1, 2n-3]^o \cup [2n+1, 4n-3]^o.\end{aligned}$$

Whenever $x \in X$ (or $x \in Y$), we always have $\varphi(xx') = |\varphi(x) - \varphi(x')| = 2n-1$. Thereby, we obtain

$$\varphi(E(G)) = [1, 2n-3]^o \cup [2n+1, 4n-3]^o \cup \{2n-1\} = [1, 4n-3]^o.$$

Thereby, φ is an edge-ordered odd-graceful labelling of G. The proof of the theorem is complete. □

Theorem 3. *(i) Let a tree $T(n)$ on n vertices admit a graceful labelling f such that $f(w) = 0$ for some vertex $w \in V(T(n))$. Then, for even integers $m \geq 2$, the 1-vertex-symmetric tree $T_w^m(n)$ admits an odd-graceful labelling such that w_0 is labelled with 0.*

(ii) Let a tree $T(n)$ on n vertices admit a graceful labelling f and an odd-graceful labelling h such that $f(w) = h(w) = 0$, where $w \in V(T(n))$. Then, for even integers $m \geq 0$, the 1-vertex-symmetric tree $T_w^{m+1}(n)$ admits an odd-graceful labelling such that w_0 is labelled with 0.

Proof. Let (S, U) be the bipartition of $V(T(n))$, where $S = \{x_j : j \in [1,s]\}$, $U = \{y_k : k \in [1,t]\}$ with $s + t = n$, and let f be a graceful labelling of $T(n)$ such that $f(w) = 0$ for a certain vertex $w \in V(T(n))$.

(i) The bipartition of vertex set of each copy $T_i(n)$ of $T(n)$ is denoted as (S^i, U^i), where $S^i = \{x_j^i : j \in [1,s]\}$ and $U^i = \{y_k^i : k \in [1,t]\}$ for $i \in [1,m]$. Here, $x_j^i \in S^i \subset V(T_i(n))$ is isomorphic to x_j for $i \in [1,m]$ and $j \in [1,s]$; $y_k^i \in U^i \subset V(T_i(n))$ is isomorphic to y_k for $i \in [1,m]$, $k \in [1,t]$; $x_j^iy_k^i \in E(T_i(n))$ is isomorphic to x_jy_k for $i \in [1,m]$, $j \in [1,s]$ and $k \in [1,t]$. Clearly, we can compute the cardinalities

$$|V(T_w^m(n))| = \left|\left(\bigcup_{i=1}^{m}\left(\{x_j^i : j \in [1,s]\} \bigcup \{y_k^i : k \in [1,t]\}\right)\right) \cup \{w_0\}\right| = mn+1,$$

$$|E(T_w^m(n))| = \left|\left(\bigcup_{i=1}^{m}\{x_j^iy_k^i : j \in [1,s], k \in [1,t]\}\right) \cup \{w_0w_i : i \in [1,m]\}\right| = mn.$$

We, without loss of generality, may assume that $w \in S$, $f(x_j) < f(x_{j+1})$ for $j \in [1, s-1]$ and $f(y_k) < f(y_{k+1})$ for $k \in [1, t-1]$. Clearly, $f(x_1) = 0$ and $f(y_t) = n-1$, $w_i \in S^i$ for $i \in [1,m]$. Based on the labelling f we define a labelling π of $T_w^m(n)$ as follows:

$$\begin{cases} \pi(w_0) = 2mn - 1; \\ \pi(x_j^i) = 2f(x_j) + 2(i-1)n, & i \in [1,m], j \in [1,s]; \\ \pi(y_k^i) = 2f(y_k) + 2(m-i)n - 1, & i \in [1,\frac{m}{2}], k \in [1,t]; \\ \pi(y_k^i) = 2f(y_k) + 2(m-i)n + 1, & i \in [\frac{m}{2}+1,m], k \in [1,t]. \end{cases}$$

Since f is a graceful labelling of $T(n)$, $f(V(T(n))) = \{f(x_j) : j \in [1,s]\} \bigcup \{f(y_k) : k \in [1,t]\} = [0, n-1]$. According to the definition of the labelling π, each vertex x_j^i has its even label $\pi(x_j^i)$ for $i \in [1,m]$ and $j \in [1,s]$, and furthermore $0 = \pi(x_1^1) < \pi(x_2^1) < \pi(x_3^1) < \ldots < \pi(x_s^1) < \pi(x_1^2) < \pi(x_2^2) < \pi(x_3^2) < \ldots < \pi(x_s^2) < \pi(x_1^3) < \pi(x_2^3) < \pi(x_3^3) < \ldots < \pi(x_s^3) < \ldots < \pi(x_1^m) < \pi(x_2^m) < \pi(x_3^m) < \ldots < \pi(x_s^m) < 2mn - 1$. Analogously, each vertex y_k^i has its odd label $\pi(y_k^i)$ for $i \in [1,m]$ and $k \in [1,t]$. Notice that $2mn - 3 = \pi(y_t^1) > \pi(y_{t-1}^1) > \ldots > \pi(y_2^1) > \pi(y_1^1) > \pi(y_t^2) > \pi(y_{t-1}^2) > \ldots > \pi(y_2^2) > \pi(y_1^2) > \pi(y_t^3) > \pi(y_{t-1}^3) > \ldots > \pi(y_2^3) > \pi(y_1^3) > \ldots > \pi(y_t^m) > \pi(y_{t-1}^m) > \ldots > \pi(y_2^m) > \pi(y_1^m) > 0$. Therefore, π is a labelling from $V(T_w^m(n))$ to $[0, 2mn-1]$.

Next, we will show that the set $\pi(T_w^m(n))$ of edge labels of $T_w^m(n)$ equals to $[1, 2mn-1]^o$. For each edge $x_j^i y_k^i \in E(T_w^m(n))$, where $i \in [1,m]$, $j \in [1,s]$ and $k \in [1,t]$, we can compute edge labels

$$\begin{aligned} \pi(x_j^i y_k^i) &= |2f(x_j) + 2(i-1)n - [2f(y_k) + 2(m-i)n - 1]| \\ &= |2(m-2i+1)n - 1 + 2(f(y_k) - f(x_j))|, \quad i \in \left[1, \frac{m}{2}\right]; \end{aligned}$$

$$\begin{aligned} \pi(x_j^i y_k^i) &= |2f(x_j) + 2(i-1)n - [2f(y_k) + 2(m-i)n + 1]| \\ &= |2(m-2i+1)n + 1 + 2(f(y_k) - f(x_j))|, \quad i \in \left[\frac{m}{2}+1, m\right]. \end{aligned}$$

Since $f(x_j y_k) \in [1, n-1]$ and m is even, whenever $f(x_j) < f(y_k)$ or $f(x_j) > f(y_k)$, the set $\{\pi(x_j^i y_k^i) : i \in [1,m]\}$ of edge labels of m edges $x_j^i y_k^i \in E(T_w^m(n))$ ($i \in [1,m]$) in which each edge is isomorphic to $x_j y_k \in E(T(n))$ equals to

$$\{\pi(x_j^i y_k^i) : i \in [1,m]\} = \Big\{ (2+4r)n - 1 + 2f(x_j y_k), \\ (2+4r)n - 1 - 2f(x_j y_k) : r \in \left[0, \frac{m-2}{2}\right] \Big\}, \tag{1}$$

Based on $f(E(T(n))) = [1, n-1]$ and the form (1), we have

$$\pi(E(T_w^m(n))\setminus\{w_0w_i : i \in [1,m]\}) = \bigcup_{i=1}^{m}\{\pi(x_j^i y_k^i) : j \in [1,s], k \in [1,t]\}$$

$$= \left(\bigcup_{r=0}^{\frac{m-2}{2}} \bigcup_{j\in[1,s],k\in[1,t]} \{(2+4r)n - 1 + 2f(x_j y_k)\}\right) \bigcup$$

$$\bigcup \left(\bigcup_{r=0}^{\frac{m-2}{2}} \bigcup_{j\in[1,s],k\in[1,t]} \{(2+4r)n - 1 - 2f(x_j y_k)\}\right)$$

$$= [1, 2mn-1]^o \setminus \{2n-1, 4n-1, \ldots, 2(m-1)n-1, 2mn-1\}.$$

By $\pi(w_i) = 2f(w) + 2(i-1)n = 2(i-1)n$ for $i \in [1,m]$, we obtain $\pi(w_0w_i) = |2mn - 1 - 2(i-1)n|$ and

$$\pi(\{w_0w_i : i \in [1,m]\}) = \{2n-1, 4n-1, \ldots, 2(m-1)n-1, 2mn-1\}.$$

Furthermore,

$$\pi(E(T_w^m(n)) = \left(\bigcup_{i=1}^{m}\{\pi(x_j^i y_k^i) : j \in [1,s], k \in [1,t]\}\right) \bigcup$$

$$\bigcup \pi(\{w_0w_i : i \in [1,m]\}) = [1, 2mn-1]^o.$$

Thereby, π really is an odd-graceful labelling of $T_w^m(n)$ with $\pi(w_0) = 2mn - 1$. By Lemma 1, the complementary labelling θ of π is an odd-graceful labelling of $T_w^m(n)$ with $\theta(w_0) = 0$.

(*ii*) Here, we apply the bipartition (S^i, U^i) of each copy $T_i(n)$ of $T(n)$ for $i \in [1,m]$ defined in the proof of the assertion (*i*). For $m = 0$, we can extend the odd-graceful labelling h to a labelling π' of $T_w^{m+1}(n)$ as follows: $\pi'(w_0) = 0$; $\pi'(x) = 2n - 1 - h(x)$ for each vertex $x \in V(T_w^{m+1}(n))\setminus\{w_0\}$. It is easy to verify that π' is an odd-graceful labelling of $T_w^{m+1}(n)$ with $\pi'(w_0) = 0$.

For even integers $m \geq 2$, by the definition of $T_w^{m+1}(n)$, we can extend the labelling π of $T_w^m(n)$ and the labelling h of $T_{m+1}(n)$ to a labelling π' of $T_w^{m+1}(n)$ as follows:

$$\begin{cases} \pi'(w_0) = \pi(w_0) + 2n; & \\ \pi'(x_j^i) = \pi(x_j^i), & i \in [1, \frac{m}{2}], j \in [1,s]; \\ \pi'(x_j^i) = \pi(x_j^i) + 2n, & i \in [\frac{m}{2}+1, m], j \in [1,s]; \\ \pi'(y_k^i) = \pi(y_k^i) + 2n, & i \in [1, \frac{m}{2}], k \in [1,t]; \\ \pi'(y_k^i) = \pi(y_k^i), & i \in [\frac{m}{2}+1, m], k \in [1,t]; \\ \pi'(x) = h(x) + mn, & x \in V(T_{m+1}(n)). \end{cases}$$

Notice that h is an odd-graceful labelling of $T_{m+1}(n)$. Each vertex label $\pi'(x_j^i)$ is even for $i \in [1, m+1]$, $j \in [1, s]$; and $0 \le \pi'(x_j^i) < mn$ for $i \in [1, \frac{m}{2}]$, $j \in [1, s]$; $(m+2)n \le \pi'(x_j^i) < 2(m+1)n - 1$ for $i \in [\frac{m}{2}+1, m]$, $j \in [1, s]$; and

$$\pi'(x_1^1) < \pi'(x_2^1) < \cdots < \pi'(x_s^1) < \pi'(x_1^2) < \pi'(x_2^2) < \cdots$$
$$< \pi'(x_s^2) < \cdots < \pi'(x_1^{\frac{m}{2}}) < \pi'(x_2^{\frac{m}{2}}) < \cdots < \pi'(x_s^{\frac{m}{2}}), \text{ and}$$
$$\pi'(x_1^{\frac{m}{2}+1}) < \pi'(x_2^{\frac{m}{2}+1}) < \cdots < \pi'(x_s^{\frac{m}{2}+1}) < \pi'(x_1^{\frac{m}{2}+2}) < \cdots$$
$$< \pi'(x_s^{\frac{m}{2}+2}) < \cdots < \pi'(x_1^m) < \pi'(x_2^m) < \cdots < \pi'(x_s^m).$$

Analogously, each vertex label $\pi'(y_k^i)$ is odd for $i \in [1, m+1]$ and $k \in [1, t]$; and $(m+2)n + 1 \le \pi'(y_k^i) < 2(m+1)n - 1$ for $i \in [1, \frac{m}{2}]$, $j \in [1, s]$; $3 \le \pi'(x_j^i) \le mn - 1$ for $i \in [\frac{m}{2}+1, m]$, $j \in [1, s]$. Since

$$\pi'(y_t^1) > \pi'(y_{t-1}^1) > \cdots > \pi'(y_1^1) > \pi'(y_t^2) > \pi'(y_{t-1}^2) > \cdots >$$
$$\pi'(y_1^2) > \cdots > \pi'(y_t^{\frac{m}{2}}) > \pi'(y_{t-1}^{\frac{m}{2}}) > \cdots > \pi'(y_1^{\frac{m}{2}}),$$
$$\pi'(y_t^{\frac{m}{2}+1}) > \pi'(y_{t-1}^{\frac{m}{2}+1}) > \cdots > \pi'(y_1^{\frac{m}{2}+1}) > \pi'(y_t^{\frac{m}{2}+2}) >$$
$$\pi'(y_{t-1}^{\frac{m}{2}+2}) > \cdots > \pi'(y_2^m) > \pi'(y_1^m),$$

and $mn \le \pi'(x) \le (m+2)n - 3$ for each vertex $x \in V(T_{m+1}(n))$ and $\pi'(w_0) = 2(m+1)n - 1$, we conclude that π' is a labelling from $V(T_w^{m+1}(n))$ to $[0, 2(m+1)n - 1]$.

We, next, show that the set $\pi'(E(T_w^{m+1}(n)))$ of edge labels of $T_w^{m+1}(n)$ equals to $[1, 2(m+1)n - 1]^o$. According to the definitions of π and π', we have $\pi(x_j^i) < \pi(y_k^i)$ for $i \in [1, \frac{m}{2}]$, $j \in [1, s]$ and $k \in [1, t]$, $\pi(x_j^i) > \pi(y_k^i)$ for $i \in [\frac{m}{2}+1, m]$, $j \in [1, s]$ and $k \in [1, t]$. Then for each edge $x_j^i y_k^i \in E(T_w^{m+1}(n))$, there are $\pi'(x_j^i y_k^i) = \pi(x_j^i y_k^i) + 2n$ for $i \in [1, m]$, $\pi'(x_j^i y_k^i) = h(x_j^i y_k^i)$ for $i = m+1$. By $h(E(T_{m+1}(n))) = [1, 2n-3]^o$ and $\pi(E(T_w^m(n)) \setminus \{w_0 w_i : i \in [1, m]\}) = [1, 2mn-1]^o \setminus \{2n-1, 4n-1, \ldots, 2(m-1)n - 1, 2mn - 1\}$, we get

$$\begin{aligned}
&\pi'(E(T_w^{m+1}(n)) \setminus \{w_0 w_i : i \in [1, m+1]\} \\
&= \left(\bigcup_{i=1}^{m} \{\pi'(x_j^i y_k^i) : j \in [1, s], k \in [1, t]\} \right) \bigcup [1, 2n-3]^o \\
&= [1, 2(m+1)n - 1]^o \setminus \{2n-1, 4n-1, \ldots, 2mn - 1, 2(m+1)n - 1\}.
\end{aligned}$$

Since $\pi'(w_0) = \pi(w_0) + 2n = 2(m+1)n - 1$, $\pi'(w_i) = \pi(w_i) = 2(i-1)n$ for $i \in [1, \frac{m}{2}]$, $\pi'(w_i) = \pi(w_i) + 2n = 2ni$ for $i \in [\frac{m}{2}+1, m]$ and $\pi'(w_{m+1}) = h(w_{m+1}) + mn = mn$, thus

$$\pi'(\{w_0 w_i : i \in [1, m+1]\}) = \{2n-1, 4n-1, \ldots, 2mn-1, 2(m+1)n - 1\},$$

and furthermore

$$\pi'(E(T_w^{m+1}(n)) = \left(\bigcup_{i=1}^{m+1}\{\pi(x_j^i y_k^i) : j \in [1,s], k \in [1,t]\}\right)\bigcup$$

$$\bigcup \pi'(\{w_0 w_i : i \in [1, m+1]\}) = [1, 2(m+1)n-1]^o.$$

Thereby, π' is an odd-graceful labelling of $T_w^{m+1}(n)$ with $\pi'(w_0) = 2(m+1)n-1$. By Lemma 1, the complementary labelling of π', also, is an odd-graceful labelling of $T_w^{m+1}(n)$ such that w_0 is labelled with 0. The proof of the theorem is finished. □

Theorem 4. *Suppose that a tree $T(n)$ admits an edge-ordered odd-graceful labelling f such that $f(w) = 0$ for some vertex $w \in V(T(n))$. Then, for odd integers m, the 1-vertex-symmetric tree $T_w^m(n)$ admits an edge-ordered odd-graceful labelling such that w_0 is labelled with 0.*

Proof. Let (S, U) be the bipartition of $V(T(n))$, where $S = \{x_j : j \in [1, s]\}$ and $U = \{y_k : k \in [1, t]\}$ with $s + t = n$; and let f be an edge-ordered odd-graceful labelling of $T(n)$ with $f(w) = 0$, where $w \in S$. Clearly, $f(x_j)$ is even and $f(x_j) \leq 2n-4$ for $j \in [1, s]$, and $f(y_k)$ is odd and $f(y_k) \leq 2n-3$ for $k \in [1, t]$. Write the bipartition of each copy $T_i(n)$ of $T(n)$ as (S^i, U^i), where $S^i = \{x_j^i : j \in [1, s]\}$ and $U^i = \{y_k^i : k \in [1, t]\}$ for $i \in [1, m]$; x_j^i is isomorphic to x_j for $i \in [1, m]$, $j \in [1, s]$); $y_k^i \in V(T_i(n)$ is isomorphic to y_k for $i \in [1, m]$, $k \in [1, t]$; $x_j^i y_k^i \in E(T_i(n))$ is isomorphic to $x_j y_k$ for $i \in [1, m]$, $j \in [1, s]$, $k \in [1, t]$. Let (X, Y) be the bipartition of $V(T_w^m(n))$, where $X = \bigcup_{i=1}^m S^i$, $Y = (\bigcup_{i=1}^m U^i)\bigcup\{w_0\}$. Hence, we have

$$|V(T_w^m(n))| = \left|\left(\bigcup_{i=1}^{m}\left(\{x_j^i : j \in [1,s]\}\bigcup\{y_k^i : k \in [1,t]\}\right)\right)\bigcup\{w_0\}\right| = mn+1,$$

$$|E(T_w^m(n))| = \left|\left(\bigcup_{i=1}^{m}\{x_j^i y_k^i : j \in [1,s], k \in [1,t]\}\right)\bigcup\{w_0 w_i : i \in [1,m]\}\right| = mn.$$

We then extend the labelling f to a labelling π of $T_w^m(n)$ in the way that

$$\begin{cases} \pi(w_0) = 2mn-1; & \\ \pi(x_j^i) = m \cdot f(x_j) + 2i - 2, & i \in [1,m], j \in [1,s]; \\ \pi(y_k^i) = m \cdot f(y_k) + 4i - 3 - m, & i \in [1,m], k \in [1,t]. \end{cases}$$

Clearly, $f(x_j) < f(y_k)$ for each edge $x_j y_k \in E(T(n))$, since f is an edge-ordered odd-graceful labelling of $T(n)$. Notice that $\pi(w_0) - \pi(w_i) = 2mn - 1 - 2(i-1) > 0$ and $\pi(y_k^i) - \pi(x_j^i) = m \cdot (f(y_k) - f(x_j) - 1) + 2i - 1 > 0$.

Therefore, for each edge $xy \in E(T_w^m(n))$ with $x \in X$ and $y \in Y$, we obtain $\pi(x) < \pi(y)$, i.e., π is edge-ordered.

Step 1. We show that π is a labelling from $V(T_w^m(n))$ to $[0, 2mn-1]$. Notice that m is odd. For $m = 1$, it is trivial, thus we consider $m \geq 3$. Without loss of generality, we may assume that $f(x_j) < f(x_{j+1})$ for $j \in [1, s-1]$ and $f(y_k) < f(y_{k+1})$ for $k \in [1, t-1]$. Notice that $f(x_j)$ is even with $f(x_j) \leq 2n-4$ for $j \in [1, s]$; $f(y_k)$ is odd with $f(y_k) \leq 2n-3$ for $k \in [1, t]$. Therefore, $\pi(x_j^i)$ is even with $\pi(x_j^i) \leq 2mn - 2m - 2$ for $j \in [1, s]$ and $i \in [1, m]$; $\pi(y_k^i)$ is odd with $\pi(y_k^i) \leq 2mn-3$ for $k \in [1, t]$ and $i \in [1, m]$; $\pi(x_j^i) < \pi(x_j^{i+1})$ and $\pi(y_k^i) < \pi(y_k^{i+1})$ for $j \in [1, s]$, $k \in [1, t]$, $i \in [1, m-1]$; $\pi(x_j^m) - \pi(x_{j+1}^1) = m \cdot (f(x_j) - f(x_{j+1})) + 2m - 2 < 0$ for $j \in [1, s-1]$. Hence, we obtain $0 = \pi(x_1^1) < \pi(x_1^2) < \pi(x_1^3) < \ldots < \pi(x_1^m) < \pi(x_2^1) < \pi(x_2^2) < \pi(x_2^3) < \ldots < \pi(x_2^m) < \pi(x_3^1) < \pi(x_3^2) < \pi(x_3^3) < \ldots < \pi(x_3^m) < \ldots < \pi(x_s^1) < \pi(x_s^2) < \pi(x_s^3) < \ldots < \pi(x_s^m) < 2mn - 1$.

Suppose that $\pi(y_k^i) = \pi(y_{k'}^{i'})$ for distinct vertices $y_k^i, y_{k'}^{i'} \in Y$, $i, i' \in [1, m]$, $k, k' \in [1, t]$, where only one of $i = i'$ and $k = k'$ holds true. Hence,

$$\pi(y_k^i) = m \cdot f(y_k) + 4i - 3 - m = m \cdot f(y_{k'}) + 4i' - 3 - m = \pi(y_{k'}^{i'}),$$

that is, $m|f(y_k) - f(y_{k'})| = 4|i - i'|$. If $k = k'$, thus, $i = i'$; a contradiction. If $k \neq k'$, since $\gcd(m, 4) = 1$ and $|f(y_k) - f(y_{k'})| \neq 0$, thus, $\frac{1}{m}|i - i'|$ is an integer. However, $|i - i'| \leq m - 1$, which means $i = i'$ and $f(y_k) = f(y_{k'})$. Immediately, $k = k'$; this is absurd. Therefore, we have $\pi(u) \neq \pi(v)$ for each pair of vertices $u, v \in V(T_w^m(n))$.

Step 2. We will show that the set of edge labels of $T_w^m(n)$ equals to $[1, 2mn-1]^o$. According to the definition of the labelling π, for each edge $x_j^i y_k^i \in E(T_w^m(n))$ with $i \in [1, m]$, $j \in [1, s]$ and $k \in [1, t]$, we then have

$$\begin{aligned}\pi(x_j^i y_k^i) &= |m \cdot f(y_k) + 4i - 3 - m - [m \cdot f(x_j) + 2i - 2]| \\ &= |m(f(y_k) - f(x_j) - 1) + 2i - 1| \\ &= m(f(x_j y_k) - 1) + 2i - 1.\end{aligned}$$

Since $f(E(T(n))) = \{f(x_j y_k) : j \in [1, s], k \in [1, t]\} = [1, 2n-3]^o$, then

$$\pi(E(T_w^m(n)) \setminus \{w_0 w_i : i \in [1, m]\}) = \bigcup_{i=1}^{m} \{\pi(x_j^i y_k^i) : j \in [1, s], k \in [1, t]\}$$

$$= \bigcup_{i=1}^{m} \{m(f(x_j y_k) - 1) + 2i - 1 : j \in [1, s], k \in [1, t]\}$$

$$= [1, 2mn-1]^o \setminus \{2mn - 2m + 1, 2mn - 2m + 3, \ldots, 2mn - 1\}.$$

According to

$$\begin{aligned}&\{\pi(w_0 w_i) : i \in [1, m]\} = \{2mn - 1 - 2(i-1) : i \in [1, m]\} \\ &= \{2mn - 2m + 1, 2mn - 2m + 3, \ldots, 2mn - 1\},\end{aligned}$$

we obtain

$$\pi(E(T_w^m(n)) = \left(\bigcup_{i=1}^{m}\{\pi(x_j^i y_k^i) : j \in [1,s], k \in [1,t]\}\right) \bigcup$$
$$\bigcup \pi(\{w_0 w_i : i \in [1,m]\}) = [1, 2mn-1]^o.$$

The above two steps show that π is odd-graceful. Therefore, π is an edge-ordered odd-graceful labelling of $T_w^m(n)$ with $\pi(w_0) = 2mn - 1$. By Lemma 1, the complementary labelling of π is an edge-ordered odd-graceful labelling of $T_w^m(n)$ such that w_0 is labelled with 0. The theorem is covered. □

Theorem 5. *(i) Let T be a symmetric tree with the root w. Then T admits an odd-graceful labelling such that w is labelled with* 0.

(ii) If each of non-leaf vertices (not including the root) of a symmetric tree T with the root w of odd degree has even degree, then T admits an edge-ordered odd-graceful labelling such that w is labelled with 0.

Proof. By induction on orders of symmetric trees. For diameter $\text{diam}(T) = 1$ or 2, T is a star, thus, the assertions (i) and (ii) both hold true. Let diameter $\text{diam}(T) \geq 3$ in the following.

Let w be the root of symmetric trees T and $N_T(w) = \{w_1, w_2, \ldots, w_m\}$, where $m = d_T(w)$. Then $T - w$ has m components $T_1, T_2, \ldots, T_m$, where $m \geq 1$, $T_i \cong T_j$, and each T_i is a symmetric tree with the root w_i, $i \in [1, m]$. By Lemma 1, T_1 admits a graceful labelling f with $f(w_1) = 0$.

If degree $d_T(w) = m$ is even, the assertion (i) follows from Theorem 3. Next, we consider the case of odd degree $m = d_T(w)$. Notice that T_1 is a symmetric tree with the root w_1 and $|V(T_1)| < |V(T)|$. By the induction hypothesis, T_1 has an odd-graceful labelling h such that $h(w_1) = 0$. By Lemma 1 and Theorem 3, we conclude that T has an odd-graceful labelling such that its root w is labelled with 0.

We show the assertion (ii). Notice that T is a symmetric tree with the root w of odd degree, every one of the remaining non-leaf vertices has even degree. If T_1 is a star, we are done. Suppose that T_1 is a symmetric tree with the root w_1 of odd degree, its rest non-leaf vertices have even degrees, and $|V(T_1)| < |V(T)|$. By the induction hypothesis, T_1 has an edge-ordered odd-graceful labelling f such that $f(w_1) = 0$. Thereby, from Theorem 4, we conclude that T has an edge-ordered odd-graceful labelling such that its root w is labelled with 0.

The theorem follows from the principle of induction. □

Based on Theorem 5 we can prove the following results:

(i) Each symmetric trees T with the root w admits an odd-graceful labelling such that w is labelled with 0.

(ii) Each complete k-ary tree admits an odd-graceful labelling such that its root is labelled with 0. Furthermore, each complete $(2m-1)$-ary tree is edge-ordered odd-graceful.

It may be interesting to consider this problem: Show the gracefulness and odd-gracefulness of k-vertex- and k-edge-symmetric trees for $k \geq 2$.

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