Graphs and Their Complements With Equal Total Domination Numbers

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Abstract

A set S of vertices in a graph G is a total dominating set of G if every vertex of G is adjacent to some vertex in S. The minimum cardinality of a total dominating set of G is the total domination number of G. We study graphs having the same total domination number as their complements. In particular, we characterize the cubic graphs having this property. Also we characterize such graphs with total domination numbers equal to two or three, and we determine properties of the ones with larger total domination numbers.

1 Introduction

Let $G = (V, E)$ be a graph with order $|V(G)| = n$. The *open neighborhood* of $v \in V(G)$ is $N_G(v) = \{u \in V \mid uv \in E\}$, and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N_G(v)$. If the graph G is clear from the context, then we simply write $N(v)$ and $N[v]$ rather than $N_G(v)$ and $N_G[v]$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \bigcup_{v \in S} N(v)$, and its *closed neighborhood* is $N[S] = N(S) \cup S$. For a vertex v, the subgraph induced by $N(v)$ is called the *link* of v. We denote the subgraph of G induced by S as $G[S]$.

For two vertices u and v in a connected graph G, the *distance* $d_G(u, v)$ between u and v is the length of a shortest u–v path in G. The maximum distance between any pair of vertices of G is the *diameter* of G, denoted diam(G). We say that G is a diameter-k graph if $\text{diam}(G) = k$. Let C_n denote the cycle on n vertices. The girth $g(G)$ is defined for graphs with cycles and is the shortest length of a cycle of G. We say that a graph is F-free if it has no induced subgraph F. In particular, a graph is *claw-free* if it has no induced $K_{1,3}$, is triangle-free if it has no induced K_3, and is quadrilateral-free if it has no induced C_4. The minimum and maximum degrees of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

A set $S \subseteq V(G)$ is a *dominating set* of G, denoted DS, if every vertex not in S is adjacent to a vertex in S, that is, if $N[S] = V(G)$. A *total dominating set*, abbreviated TDS, of a graph G is a set S of vertices of G such that every vertex in G is adjacent to a vertex in S, that is, $N(S) = V(G)$. Every isolate-free graph G has a TDS, since $V(G)$ is such a set. The *domination number* $\gamma(G)$ is the minimum cardinality of a DS of G, and the *total domination number* $\gamma_t(G)$ is the minimum cardinality of a TDS of G. A TDS of G of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set. Total domination was introduced by Cockayne, Dawes, and Hedetniemi [4] and is now well studied in graph theory. The literature on the subject of domination parameters in graphs has been surveyed and detailed in the two books [5, 6]. A recent survey of total domination in graphs can be found in [7].

In this paper, we investigate graphs having the same total domination number as their complements. Graphs having this property for the domination number were studied in [2]. If $\gamma_t(G) = \gamma_t(\overline{G})$, then clearly G and $\overline{G}$ are isolate-free graphs. Thus no vertex dominates G (respectively, $\overline{G}$). Hence we make the following observation.

Observation 1 *If G is a graph with $\gamma_t(G) = \gamma_t(\overline{G})$, then*

(i) *$\text{diam}(G) \geq 2$ and $\text{diam}(\overline{G}) \geq 2$, and*

(ii) $1 \leq \delta(G) \leq \Delta(G) \leq n-2$ *and* $1 \leq \delta(\overline{G}) \leq \Delta(\overline{G}) \leq n-2$.

2 Graphs G with $\gamma_t(G) = \gamma_t(\overline{G}) = 2$

Since any pair of vertices at a distance three or more apart in G total dominates $\overline{G}$, we have another useful observation.

Observation 2 *If G is a graph with diam$(G) \geq 3$, then $\gamma_t(\overline{G}) = 2$.*

We note that if G is disconnected, then $\gamma_t(G) \geq 4$ and $\gamma_t(\overline{G}) = 2$ (since two vertices in different components of G total dominate $\overline{G}$). Hence if $\gamma_t(G) = \gamma_t(\overline{G}) = k$, then G and $\overline{G}$ are connected, and the graphs G having $\gamma_t(G) = \gamma_t(\overline{G}) = 2$ are precisely the graphs for which diam$(G) \geq 3$ and diam$(\overline{G}) \geq 3$.

Proposition 3 *A graph G has $\gamma_t(G) = \gamma_t(\overline{G}) = 2$ if and only if diam$(G) \geq 3$ and diam$(\overline{G}) \geq 3$.*

Proof. For the necessity, assume that $\gamma_t(G) = \gamma_t(\overline{G}) = 2$. Let $\{x, y\}$ be a $\gamma_t(G)$-set. In $\overline{G}$, x and y are non-adjacent vertices with no common neighbor so diam$(\overline{G}) \geq 3$. By a symmetric argument, diam$(G) \geq 3$ as well. The sufficiency follows from Observation 2. □

Since the only tree with diameter at most two is a star and the complement of a star has an isolated vertex, it follows that if $\gamma_t(T) = \gamma_t(\overline{T})$ for a tree T, then diam$(T) \geq 3$ and so $\gamma_t(\overline{T}) = 2$. We characterize the trees T for which $\gamma_t(T) = \gamma_t(\overline{T})$. A *double star* $S_{r,s}$ is a tree with exactly two non-leaf vertices, one of which is adjacent to r leaves and the other is adjacent to s leaves.

Proposition 4 *For a tree T, $\gamma_t(T) = \gamma_t(\overline{T})$ if and only if T is the double star $S_{r,s}$ where $1 \leq r \leq s$.*

Proof. Assume that $\gamma_t(T) = \gamma_t(\overline{T})$. It follows from Observation 1 that T is not a star and so diam$(T) \geq 3$. By Observation 2, $\gamma_t(\overline{T}) = 2$ and so $\gamma_t(T) = 2$. Let $\{x, y\}$ be a $\gamma_t(T)$-set. Then $xy \in E(T)$, $N(x) \cup N(y) = V(T)$, and since T is a tree $N(x) \cap N(y) = \emptyset$. Moreover $(N(x) \cup N(y)) \setminus \{x, y\}$ is an independent set, for otherwise T has a cycle. Thus, T is a double star $S_{r,s}$ for $1 \leq r \leq s$. If $T = S_{r,s}$ for $1 \leq r \leq s$, then $\gamma_t(T) = \gamma_t(\overline{T}) = 2$. □

Our next result follows from Proposition 3.

Proposition 5 *Let G be a graph such that G and $\overline{G}$ are isolate-free graphs. Then $\gamma_t(G) \geq 3$ and $\gamma_t(\overline{G}) \geq 3$ if and only if $diam(G) = diam(\overline{G}) = 2$.*

3 Graphs G with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$

Henceforth we consider graphs G having $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 3$. Thus, by Propositions 4 and 5, diam(G) = diam$(\overline{G}) = 2$. We begin by showing that such graphs exist for all $k \geq 3$. In [2], the authors gave a construction for a family of graphs G where $\gamma(G) = \gamma(\overline{G}) = k$ for any $k \geq 3$. We note that these graphs also have $\gamma_t(G) = \gamma_t(\overline{G}) = k$ and list the construction here for completeness.

Example of Existence [2]: Let $V(G) = A \cup B \cup C$ where A consists of $\binom{k^2}{k-1}$ vertices labeled by the distinct subsets of cardinality $k-1$ taken from the first k^2 positive integers $\{1, 2, \ldots k^2\}$. Let $B = \{b_l \mid 1 \leq l \leq k\}$, and let $C = \{c_j \mid 1 \leq j \leq k^2\}$. Add edges as follows to obtain G. Make $G[A \cup B]$ a complete subgraph. For $1 \leq j \leq k^2$, c_j is adjacent to the $\binom{k^2-1}{k-2}$ vertices in A which contain j in their labels. Make vertex b_l, $1 \leq l \leq k$, adjacent to vertex c_s, $1 \leq s \leq k^2$, if and only if s and l are congruent modulo k.

It is shown in [2] that $\gamma(G) = \gamma(\overline{G}) = k$, so $\gamma_t(G) \geq k$ and $\gamma_t(\overline{G}) \geq k$. Note that B is a TDS of G and $\{c_1, c_2, \ldots, c_k\} \subset C$ is a TDS of $\overline{G}$, so $\gamma_t(G) \leq |B| = k$ and $\gamma_t(\overline{G}) \leq k$. Hence $\gamma_t(G) = \gamma_t(\overline{G}) = k$.

3.1 Properties

We list some known upper bounds on the total domination number and show that these bounds can be improved for graphs having the same total domination number as their complement.

Theorem 6 *Let G be a graph with minimum degree $\delta(G)$. Then*

$$\begin{array}{llllll} \delta(G) \geq 1 & \textit{implies} & \gamma_t(G) \leq 2n/3 & \textit{if } n \geq 3 \textit{ and } G \textit{ is connected} & ([4]), \\ \delta(G) \geq 2 & \textit{implies} & \gamma_t(G) \leq 4n/7 & \textit{if } n \geq 11 \textit{ and } G \textit{ is connected} & ([7]), \textit{and} \\ \delta(G) \geq 3 & \textit{implies} & \gamma_t(G) \leq n/2 & & ([1,\ 3,\ 8]). \end{array}$$

Note that for any vertex v of a diameter-2 graph G, the $N[v]$ is a TDS of G. Hence if G is a diameter-2 graph, we make the following observation.

Observation 7 *For any diameter-2 graph G with minimum degree $\delta(G)$, $\gamma_t(G) \leq \delta(G) + 1$.*

By Proposition 5 and Observation 7, we have the following.

Corollary 8 *If $3 \leq \gamma_t(G) = \gamma_t(\overline{G}) = k$, then $k \leq \delta(G) + 1$.*

Note that the self-complementary 5-cycle attains the bound of Corollary 8. Our next bound on the total domination number of graphs G having $\gamma_t(G) = \gamma_t(\overline{G})$ is a significant improvement over known bounds. Let $\delta^* = \min\{\delta(G), \delta(\overline{G})\}$.

Theorem 9 *If G is a graph of order n with $3 \leq \gamma_t(G) = \gamma_t(\overline{G}) = k$, then $k \leq \sqrt{\delta^*} + 2$.*

Proof. Let G be any graph satisfying the hypothesis. For a vertex v of minimum degree in G, let $A = N_G(v)$, and let $B = V(G) \backslash N_G[v]$. If a subset S of at most $k-2$ vertices of A dominates B, then $S \cup \{v\}$ is a TDS of G with cardinality less than k, a contradiction. We partition A into $a = \left\lceil \frac{|A|}{k-2} \right\rceil$ sets $A_1, A_2, \ldots, A_a$ such that $|A_i| \leq k-2$ for $1 \leq i \leq a$. Thus no A_i, $1 \leq i \leq a$, dominates B in G implying that in $\overline{G}$, for each A_i there exists a vertex $b_i \in B$ where b_i dominates A_i. Then $\{b_i \mid 1 \leq i \leq a\} \cup \{v\}$ is a TDS of $\overline{G}$. Since $\gamma_t(\overline{G}) = k$, it follows that $a \geq k-1$. Thus, $|B| \geq k-1$, and we have $\left\lceil \frac{|A|}{k-2} \right\rceil \geq k-1$, which implies that $\frac{|A|}{k-2} \geq k-2$ and $\delta(G) = |A| \geq (k-2)^2$. A similar argument holds for $\overline{G}$, so $\delta^* \geq (k-2)^2$ or equivalently, $k \leq \sqrt{\delta^*} + 2$.
□

Corollary 10 *If G is a graph of order n with $3 \leq \gamma_t(G) = \gamma_t(\overline{G}) = k$, then $k \leq \sqrt{\frac{n-1}{2}} + 2$.*

Since graphs G with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$ have $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, it follows that the girth $g(G) \leq 5$. Proposition 4 implies that G has a cycle, so we make another observation.

Observation 11 *If G is a graph with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$, then $3 \leq g(G) \leq 5$.*

In general the total domination number of a graph can be arbitrarily larger than its domination number. For example, if G is formed from a star $S_{1,r}$ by subdividing each edge of the star exactly twice, then $\gamma_t(G) = 2r$ while $\gamma(G) = r+1$. However, our next result shows that the total domination number of a graph G having $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$ differs from its domination number by at most one.

Theorem 12 *If $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 3$, then $k-1 \leq \gamma(G) = \gamma(\overline{G}) \leq k$.*

Proof. Clearly, $\gamma(G) \leq \gamma_t(G)$ for any isolate-free graph G. For the lower bound, assume to the contrary that $\gamma_t(G) = k$ and that $\gamma(G) \leq k-2$. Let X be a $\gamma(G)$-set. Since $\gamma_t(G) = k$, X is not a TDS of G implying that $G[X]$ has an isolated vertex. Thus in $\overline{G}$, $\overline{G}[X]$ has no isolates. But since $|X| < \gamma_t(G) = \gamma_t(\overline{G})$, X does not dominate $\overline{G}$. Thus there exists a vertex, say v, in $V(\overline{G}) \setminus X$ that has no neighbor in X, implying that $X \subseteq N_G(v)$. Hence $X' = X \cup \{v\}$ is a TDS of G with $|X'| \leq k-1 < \gamma_t(G)$, a contradiction. Thus, $\gamma(G) \geq k-1$. A similar argument shows that $\gamma(\overline{G}) \geq k-1$. □

A graph G is *j-vertex-connected* (or simply, *j-connected*) if $n \geq j+1$ and deletion of any $j-1$ or fewer vertices leaves a connected graph.

Proposition 13 *If $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 3$, then G and $\overline{G}$ are $(k-1)$-connected.*

Proof. Since $\gamma_t(G) = \gamma_t(G) = k \geq 3$, $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Suppose S is a cutset of G with cardinality at most $k-2$. Since $\mathrm{diam}(G) = 2$, every vertex in $G - S$ is adjacent to a vertex in S, so $\gamma(G) \leq |S| \leq k-2$. But since $\gamma_t(G) = k$, Theorem 12 implies that $\gamma(G) \geq k-1$, a contradiction. □

3.2 $\gamma_t(G) = \gamma_t(\overline{G}) = 3$

We use the bounds on the domination number of Theorem 12 to determine the total domination number of claw-free graphs having the same total domination numbers as their complements.

Theorem 14 *If G is a claw-free graph with $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 3$, then $k = 3$.*

Proof. Let G be a claw-free graph, and assume that $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 3$. Note that every triangle of $\overline{G}$ total dominates $\overline{G}$, otherwise, the vertices of the triangle along with an undominated vertex in $\overline{G}$ induce a claw in G. Thus, if $\overline{G}$ has a triangle, $\gamma_t(\overline{G}) \leq 3$ and so $k = 3$. Assume that $\overline{G}$ is triangle-free. This implies that the largest independent set of vertices of G has cardinality two. But every maximal independent set is a dominating set, so $\gamma(G) \leq 2$. By Theorem 12, $k - 1 \leq \gamma(G) \leq 2$, so $k = 3$. □

Lemma 15 *If G is a diameter-2 graph and $\overline{G}$ has a vertex with an isolate in its link, then $\gamma_t(G) \leq 3$.*

Proof. Assume that $\mathrm{diam}(G) = 2$ and that in $\overline{G}$ a vertex v has an isolate, say u, in its link $\overline{G}[N(v)]$. Then $\{u, v\}$ is a DS of G. Moreover, since $\mathrm{diam}(G) = 2$, u and v have a common neighbor, say x, in G. Hence $\{u, v, x\}$ is a TDS of G, and so $\gamma_t(G) \leq 3$. □

The next corollary follows directly from Lemma 15 and Proposition 5.

Corollary 16 *Let G be a graph with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$. If G or $\overline{G}$ has a vertex with an isolate in its link, then $\gamma_t(G) = \gamma_t(\overline{G}) = 3$.*

By Theorem 12, if $\gamma_t(G) = \gamma_t(\overline{G}) = 3$, then $\gamma(G)$ (respectively, $\gamma(\overline{G})$) is either 2 or 3. Next we characterize the graphs having $\gamma_t(G) = \gamma_t(\overline{G}) = 3$ and $\gamma(G) = \gamma(\overline{G}) = 2$.

Theorem 17 *A graph G has $\gamma_t(G) = \gamma_t(\overline{G}) = 3$ and $\gamma(G) = \gamma(\overline{G}) = 2$ if and only if $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$ and each of G and $\overline{G}$ has a vertex with an isolate in its link.*

Proof. Let $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$, and let each of G and $\overline{G}$ have a vertex with an isolate in its link. By Proposition 5 and Lemma 15, it follows that $\gamma_t(G) = \gamma_t(\overline{G}) = 3$. By Theorem 12, $\gamma(G) \in \{2, 3\}$ and $\gamma(\overline{G}) \in \{2, 3\}$. Let u be a vertex in G with an isolate, say v, in its link. Then $\{u, v\}$ is a DS for $\overline{G}$ and so $\gamma(\overline{G}) \leq 2$ implying that $\gamma(\overline{G}) = 2$. A similar argument shows that $\gamma(G) = 2$.

For the necessity, assume that $\gamma_t(G) = \gamma_t(\overline{G}) = 3$ and $\gamma(G) = \gamma(\overline{G}) = 2$. By Proposition 5, $\mathrm{diam}(G) = \mathrm{diam}(\overline{G}) = 2$. Let $S = \{a, b\}$ be a $\gamma(\overline{G})$-set. Since $\gamma_t(\overline{G}) = 3$, S is not a TDS of $\overline{G}$, that is, $ab \notin E(\overline{G})$. Thus a and b are adjacent in G, and $N_G(a) \cap N_G(b) = \emptyset$. Hence in G, b is an isolate in the link of a. A similar argument shows that $\overline{G}$ has a vertex with an isolate in its link. □

We note that $\gamma(G) = \gamma(\overline{G}) = 2$ is a necessary condition for Theorem 17. For example, the following family $\mathcal{G}$ of graphs G has $\gamma_t(G) = \gamma_t(\overline{G}) = 3$, $\gamma(G) = 3$, and no vertex in G has an isolate in its link. Let $\mathcal{G}$ denote the family of graphs that can be obtained from a 5-cycle $v_1v_2v_3v_4v_5v_1$ by replacing each vertex v_i, $1 \leq i \leq 5$, with a clique A_i and adding all edges between A_i and A_{i+1}, where addition is taken modulo 5. A graph in the family $\mathcal{G}$ is illustrated in Figure 1.

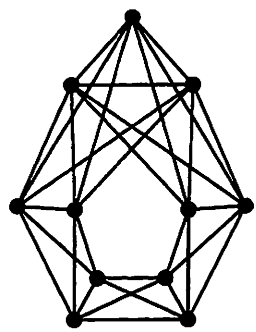

Figure 1: A graph G in $\mathcal{G}$.

We characterize the graphs having $\gamma_t(G) = \gamma_t(\overline{G}) = 3$.

Theorem 18 *A graph G has $\gamma_t(G) = \gamma_t(\overline{G}) = 3$ if and only if diam(G) = diam$(\overline{G}) = 2$ and in each of G and $\overline{G}$, there exist three vertices whose closed neighborhood intersection is empty.*

Proof. Let $\text{diam}(G) = \text{diam}(\overline{G}) = 2$, and let u, x, and y be vertices of G satisfying the theorem condition. Since $\text{diam}(G) = 2$, $\{u, x, y\}$ is a TDS of $\overline{G}$ and so $\gamma_t(\overline{G}) \leq 3$. Since $\text{diam}(\overline{G}) = 2$, $\gamma_t(\overline{G}) \geq 3$ and hence $\gamma_t(\overline{G}) = 3$. A similar argument shows that $\gamma_t(G) = 3$.

For the necessity, assume that $\gamma_t(G) = \gamma_t(\overline{G}) = 3$. By Proposition 5, $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Let $S = \{u, x, y\}$ be a $\gamma_t(G)$-set. Then $G[S] \in \{P_3, K_3\}$. We may assume, relabeling the vertices if necessary, that u is adjacent to both x and y. Hence in $\overline{G}$, $d_{\overline{G}}(u, x) = d_{\overline{G}}(u, y) = 2$ and $N_{\overline{G}}(u) \cap N_{\overline{G}}(x) \cap N_{\overline{G}}(y) = \emptyset$. Therefore $N_{\overline{G}}[u] \cap N_{\overline{G}}[x] \cap N_{\overline{G}}[y] = \emptyset$. A similar argument shows that there exist three vertices of G with this property. □

Recall that for a graph G with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$, we have $3 \leq g(G) \leq 5$. We characterize the graphs G with this property having girth 5 and show that in this case $\gamma_t(G) = \gamma_t(\overline{G}) = 3$. We begin with a lemma.

Lemma 19 *If G and $\overline{G}$ are diameter-2 graphs and $g(G) = 5$, then $\gamma_t(\overline{G}) = 3$.*

Proof. By Proposition 5, $\gamma_t(\overline{G}) \geq 3$. Since $g(G) = 5$, it follows that G is triangle-free. Thus $N(u)$ is an independent set for all $u \in V(G)$, and so $u \in V(G)$ has an isolate in its link. Lemma 15 implies that $\gamma_t(\overline{G}) \leq 3$, and hence $\gamma_t(\overline{G}) = 3$. □

Theorem 20 *A triangle-free, quadrilateral-free graph G has $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$ if and only if $G = C_5$.*

Proof. Let G be a triangle-free, quadrilateral-free graph with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 3$. Proposition 4 implies that G is not a tree and so $g(G) \geq 5$. By Proposition 5, $\text{diam}(G) = \text{diam}(\overline{G}) = 2$ implying that $g(G) = 5$. It follows by Lemma 19 that $\gamma_t(\overline{G}) = 3$, and so $\gamma_t(G) = 3$. Let $S = \{s_1, s_2, s_3\}$ be a $\gamma_t(G)$-set, and let $S_i = N(s_i) \setminus S$ for all $s_i \in S$. Since G is triangle-free and quadrilateral-free, it follows that S_i is an independent set for $1 \leq i \leq 3$ and that $S_i \cap S_j = \emptyset$ for $i \neq j$. Also $G[S]$ induces a P_3. By the minimality of S, $S_1 \neq \emptyset$ and $S_3 \neq \emptyset$. If $S_2 \neq \emptyset$, then our diameter-constraint implies that the vertex in S_2 has a neighbor in S_1 and a C_4 is formed with these two vertices and $\{s_1, s_2\}$. Hence $S_2 = \emptyset$. To show that $G = C_5$, it suffices to show that $|S_1| = |S_3| = 1$. Without loss of generality, assume to the contrary, that x and y are vertices in S_1. Since $\text{diam}(G) = 2$, for every vertex $z \in S_3$, $xz, yz \in E(G)$. But then $\{s_1, x, z, y\}$ induces a C_4 in G, contradicting that $g(G) = 5$. It follows that $G = C_5$. For the sufficiency, it is easy to see that the self-complementary 5-cycle satisfies the theorem. □

3.3 $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$

We can improve the upper bound of Corollary 8 for graphs G having $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$.

Theorem 21 *If a graph G has $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 4$, then $k \leq \delta(G)$.*

Proof. Let G be a graph with $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 4$. By Proposition 5, $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Let v be a vertex of minimum degree. Lemma 15 implies that there is no isolate in the link of v, and since $\text{diam}(G) = 2$, $N(v)$ is a TDS of G. Hence $\gamma_t(G) \leq \delta(G)$. □

By Theorem 21 for graphs with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$, we have $\delta(G) \geq 4$ and $\delta(\overline{G}) \geq 4$. Proposition 5, Observation 11, and Lemma 19 imply that for such graphs G, $3 \leq g(G) \leq 4$ (respectively, $3 \leq g(\overline{G}) \leq 4$). But we also know that no vertex of G (respectively, $\overline{G}$) has an isolate in its link, for otherwise, $\gamma_t(\overline{G}) = 3$. Since $\delta(G) \geq 4$ and no vertex has an isolate in its link, we make the following observations.

Observation 22 *If a graph G has $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$, then $g(G) = g(\overline{G}) = 3$.*

Observation 23 *If a graph G has $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$, then every vertex of G lies on a triangle in G and on a triangle in $\overline{G}$.*

Proposition 24 *If G is a graph with $\gamma_t(G) = \gamma_t(\overline{G}) \geq 4$ and $v \in V(G)$, then the link of v has diameter at most 2 in G and in $\overline{G}$.*

Proof. For a vertex $v \in V(G)$, let $A = N_G(v)$ and let $B = V(G) \setminus N_G[v]$. Suppose $\text{diam}(G[A]) \geq 3$ or $G[A]$ is not connected. In either case, $\gamma_t(\overline{G}[A]) \leq 2$. Since v dominates B in $\overline{G}$, it follows that $\gamma_t(\overline{G}) \leq 4$, a contradiction. A similar argument holds for $\overline{G}$. □

Proposition 25 *If G is a graph with $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 4$, then for every pair of non-adjacent vertices u and v in G (respectively, $\overline{G}$), $|N_G(u) \cap N_G(v)| \geq k - 2$.*

Proof. Let $\gamma_t(G) = \gamma_t(\overline{G}) = k \geq 4$. By Proposition 5, $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Let u and v be non-adjacent vertices in G, and let $X = N_G(u) \cap N_G(v)$ and $Y = V(G) \setminus (X \cup \{u, v\})$. In $\overline{G}$, $\{u, v\}$ total dominates $V(G) \setminus X$. Since $\text{diam}(\overline{G}) = 2$, each vertex in X has a neighbor in $Y \subset N_{\overline{G}}(u) \cup N_{\overline{G}}(v)$. Form a set X' by selecting for each vertex in X one of its neighbors in Y in $\overline{G}$. Then $X' \cup \{u, v\}$ is a TDS of G, implying that $k = \gamma_t(G) \leq |X' \cup \{u, v\}| \leq |X| + 2$. A similar argument holds for any non-adjacent pair of vertices in $\overline{G}$. □

4 Cubic Graphs

We note that the only 2-regular graph having the same total domination number as its complement is the 5-cycle. In this section we characterize

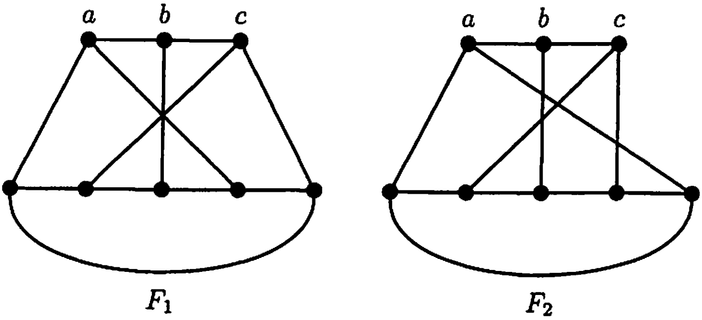

Figure 2: Cubic Graphs in $\mathcal{F}$.

cubic graphs having this property.

We will use a straightforward lower bound on $\gamma_t(G)$.

Observation 26 *If G is an isolate-free graph of order n, then $\gamma_t(G) \geq n/\Delta(G)$.*

We also need the following definitions and lemma.

An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S.

Definition 1 *Let $\mathcal{F}$ be the set containing the two cubic graphs on eight vertices shown in Figure 2 and constructed in the following manner. Begin with a cycle $C_5 = v_1v_2v_3v_4v_5v_1$ and a path $P_3 = abc$. Add the edge bv_3. To form F_1, add edges av_1, av_4, cv_2 and cv_5. To form F_2, add edges av_1, av_5, cv_2 and cv_4.*

Lemma 27 *If G is a cubic graph with $\gamma_t(G) = \gamma_t(\overline{G}) = k$, then $k \geq 3$.*

Proof. Let G be any cubic graph of order n satisfying the hypothesis. Assume for the purpose of a contradiction that $k = 2$. Since G is cubic and $\gamma_t(G) = 2$, two adjacent vertices dominate G implying that $4 \leq n \leq 6$. Since G is cubic, n is even and so $n \in \{4, 6\}$. If $n = 4$, then $G = K_4$ and $\overline{G}$ has an isolate, a contradiction. If $n = 6$, then $\overline{G}$ is 2-regular and so $\overline{G} = C_6$ or $\overline{G} = C_3 \cup C_3$. In either case, $\gamma_t(\overline{G}) = 4$ and $\gamma_t(G) = 2$, a contradiction. □

We now give our characterization.

Theorem 28 *For any cubic graph G, $\gamma_t(G) = \gamma_t(\overline{G})$ if and only if $G \in \mathcal{F}$.*

Proof. Let G be any cubic graph of order n such that $\gamma_t(G) = \gamma_t(\overline{G}) = k$. By Lemma 27, $k \geq 3$. By Proposition 5, $\text{diam}(G) = \text{diam}(\overline{G}) = 2$. Then from Observation 7, we have $\gamma_t(G) \leq 4$, and from Observation 26, $\gamma_t(G) \geq n/\Delta(G) = n/3$. Combining inequalities yields $n \leq 12$. Since G is cubic, it is necessary that $n \geq 4$ and n is even. Thus $n \in \{4, 6, 8, 10, 12\}$. From the proof of Lemma 27, we have $n \notin \{4, 6\}$.

It follows by Observations 7 and 26 that $\lceil \frac{n}{\Delta(G)} \rceil \leq \gamma_t(G) \leq 4$. Note that $\overline{G}$ is r-regular with $r = n - 3 - 1$, so for $8 \leq n \leq 12$, we have $r \geq 4$. Let $v \in V(G)$ and $N_G(v) = \{a, b, c\}$. Then in $\overline{G}$, it follows by the pigeonhole principle that two vertices of $\{a, b, c\}$, say a and b, have a common neighbor $x \in N_{\overline{G}}(v)$. The set $\{v, x, y\}$, where $y \in N_{\overline{G}}(v) \cap N_{\overline{G}}(c)$, is a TDS for $\overline{G}$. Hence $\gamma_t(\overline{G}) \leq 3$, and so $\gamma_t(\overline{G}) = 3$. Now Observation 26 implies that $n = 8$.

Let $S = \{a, b, c\}$ be any $\gamma_t(G)$-set. Then $G[S] = P_3$ or $G[S] = K_3$. If $G[S] = K_3$, then since G is cubic, $n \leq 6$, a contradiction. Thus $G[S] = P_3 = abc$. By the minimality of S, each of a and c has an S-external private neighbor. Let $V(G) \setminus S = \{v_1, v_2, v_3, v_4, v_5\}$. Since G is cubic and each vertex of $V(G) \setminus S$ is adjacent to at least one vertex of S, it follows that b has exactly one S-external private neighbor, and both a and c have two S-external private neighbors. Since G is cubic, $G[V(G) \setminus S]$ is 2-regular, that is $G[V(G) \setminus S]$ induces a 5-cycle. Relabeling the vertices if necessary, let the 5-cycle be $v_1v_2v_3v_4v_5v_1$. We may assume that $bv_3 \in E(G)$. Now, G can be (up to isomorphism) one of three possibilities. If $av_1, av_2, cv_4, cv_5 \in E(G)$, then $\text{diam}(G) = 3$, a contradiction. If $av_1, av_4, cv_2, cv_5 \in E(G)$, then $G = F_1 \in \mathcal{F}$. If $av_1, av_5, cv_2, cv_4 \in E(G)$, then $G = F_2 \in \mathcal{F}$. Hence $G \in \mathcal{F}$. This proves the necessity. For each graph in $\mathcal{F}$, $\{a, b, c\}$ is a TDS. Consequently, $\gamma_t(G) \leq 3$. By Observation 26, $\gamma_t(G) \geq 3$, and so $\gamma_t(G) = 3$. It is an easy exercise to show that for every $G \in \mathcal{F}$, $\gamma_t(\overline{G}) = 3$ as well. □

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