Representation number for complete graphs minus stars

Anurag Agarwal

Manuel Lopez

School of Mathematical Sciences, RIT, Rochester, NY 14623-5604

axasma@rit.edu, malsma@rit.edu

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Abstract

Using the definition of the representation number of a graph modulo integers given by Erdős and Evans we establish the representation number of a complete graph minus a set of disjoint stars. The representation number of a graph G is the smallest positive integer n for which there is a labeling of every vertex of G with a distinct element of $\{0, 1, 2, \ldots, n-1\}$ such that two vertices are adjacent if and only if the difference of their labels is relatively prime to n. We apply known results to a complete graph minus a set of stars to establish a lower bound for the representation number; then show a systematic labeling of the vertices producing a representation that attains that lower bound. Thus showing that for complete graphs minus a set of disjoint stars the established lower bound of the representation number modulo n is indeed the representation number of the graph. Since the representation modulo an integer for a complete graph minus disjoint stars is attained using the fewest number of primes allowed by the lower bound, it follows that the corresponding Prague dimension will be determined by the largest star removed from the complete graph.

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1 Introduction

Erdős and Evans [6] showed that every finite graph has a positive integer representation in the following way:

A graph $G = (V, E)$ with vertex set $V = \{v_i\}_{i=1}^{r}$ and edge set $E = \{\overline{v_i v_j}\}_{(i,j)\in B\subset V\times V}$ is said to have a *representation* modulo a positive integer n if there exist distinct integers $a_1, a_2, \ldots, a_r$ such that $0 \leq a_i < n$, and $\gcd(a_i - a_j, n) = 1$ if and only if v_i and v_j are adjacent ($\overline{v_i v_j} \in E$). We say that $\{a_1, a_2, \ldots, a_r\}$ is a *representation* of G modulo n.

We define the representation number of a graph G (denoted $rep(G)$) as the smallest value of such a representation of G. The reader will undoubtedly realize that a complete graph K_r requires $rep(K_r)$ to be at least as large as its chromatic number, and with a little more effort that $rep(K_r)$ will have to be a prime number. The converse statement, that a prime number is the the representation number of G only if representation G is a complete graph on a finite number of vertices, is also true. This is perhaps the single most useful result behind general results establishing lower bounds for the representation number.

The impetus behind the creation of such a representation in [6] was to give a simpler proof of a result of Lindner, Mendelsohn, Mendelsohn, and Wolk [12] that any finite graph can be realized as an orthogonal Latin square graph. A proof that was later simplified further by Narayan [14]. However in recent years, as the title of Narayan's article [14] reveals, the problem of determining the exact representation number of a graph or at least finding bounds for its value has surpassed in importance any possible application. In a reversal of roles between the question about Latin squares and the question of the representation number modulo n, Evans, Isaak, and Narayan [9] showed that the determination of representation numbers for disjoint unions of complete graphs is dependent upon the existence of sets of mutually orthogonal Latin squares.

It is not surprising that the representation number is studied for its own sake given the conceptual link that exists between the representation number and another integral value attached to graphs, known as the *Prague dimension* of the graph and denoted $\dim_P(G)$. Computing $\dim_P(G)$ has been shown to be NP-Complete [11]. In some cases like the one we here consider, the determination of $\dim_P(G)$ can be made directly from $rep(G)$. Determining the representation number of a graph seems destined not to yield to an all-encompassing approach. Instead it seems finding $rep(G)$ must proceed by identifying classes of graphs that share narrowly defined properties. Representation numbers for several families of graphs including complete graphs, and graphs of the form $K_m - P_l$, $K_m - C_l$, $K_m - K_{1,l}$ (each along with a set of isolated vertices) were determined in [8] and [9]. Evans [7] used linked matrices and difference-covering matrices to obtain new results involving representation numbers for the disjoint union of complete graphs. Narayan and Urick [15] investigated representation numbers for split graphs, their complements, stars, and hypercubes. Evans, Isaak, and Narayan determined the representation number of a complete graph minus a path [9]. Agarwal, Lopez and Narayan determined the representation number of a complete graph minus a disjoint union of two paths [1] and later extended this result to the representation number and the Prague dimension of a complete graph minus a disjoint union of arbitrarily many paths (see [2]). Akhtar, Evans and Pritikin [3] obtained results involving representation numbers of stars. The purpose of this paper is to establish the representation number of the complement of a (finite) disjoint union of stars in a complete graph.

Besides defining the representation number of a graph G we should also address what the *Prague dimension* of a graph G is. The Prague dimension (also known as the product dimension) was introduced by Nešetřil and Pultr [16] and has been extensively studied [13], [4], and [5]. We say a graph G has a *product representation of length d* if each vertex v of G can be assigned a ordered $d-$tuple so that the vertices v and w are adjacent if and only if their vectors differ in every coordinate. The *Prague dimension* of the graph G, $\dim_P G$, is the minimum possible length d of such a representation.

Our use of the Chinese Remainder Theorem (CRT) will make it quite clear how *rep*(G) and $\dim_P G$ are related. Suppose G has a representation modulo a positive, square-free integer n. Let $n = p_1 p_2 \cdots p_d$, where p_i's for $1 \leq i \leq d$ are distinct primes. We obtain a product representation of G (of length d) as follows: Suppose the vertex v has label a, then the vector for v is $(v_1, v_2, \ldots, v_d)$, where $v_i \equiv a \pmod{p_i}$ and $0 \leq v_i < p_i$ for $1 \leq i \leq d$. If vertex v with label a has vector representation $(v_1, v_2, \ldots, v_d)$ and vertex w with label b has vector representation $(w_1, w_2, \ldots, w_d)$, then $\gcd(a - b, n) = 1$ implies that v and w are adjacent if and only if $v_i \neq w_i$ for all $1 \leq i \leq d$, making this assignment a product representation. On the other hand given a product representation choose distinct primes for the coordinates, each prime larger than the largest value used in that coordinate. The numbers assigned to the vertices can then be computed using the Chinese Remainder Theorem.

2 Some known results

In this section, we restate some previously known results from [8] involving the representations modulo an integer and the representation numbers of graphs. These results together with the Chinese Remainder Theorem is all that we require. The reader should consult [10] for an in-depth treatment of representation number computation techniques.

Theorem 1. *A graph has a representation modulo a prime if and only if it is a complete graph.*

The disjoint union of graphs G and H will be denoted $G + H$. That is, $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$.

Theorem 2. *A graph has a representation modulo a product of some pair of distinct primes if and only if it does not contain an induced subgraph isomorphic to* $K_2 + 2K_1$, $K_3 + K_1$ *or the complement of a chordless cycle of length at least five.*

The following results deal with the size of the prime divisors of the representation numbers.

Theorem 3. *If* G *has a representation modulo* n, *and* p *is the smallest prime divisor of* n *then* $p \geq \chi(G)$.

We have the following corollary where $\omega(G)$ is the size of the largest complete subgraph in G.

Corollary 3.1. *If* G *has a representation modulo* n, *and* p *is a prime divisor of* n *then* $p \geq \omega(G)$.

We restate Lemma 2.10 and Corollary 2.12 from Evans, Isaak, and Narayan [9].

Lemma 4. *If* G *contains a* $K_m + K_1$ *as an induced subgraph and* G *is representable modulo* n, *then* n *contains at least* m *distinct prime factors.*

Corollary 4.1. *If* G *contains a* $K_m + K_1$ *and* p_i *is the smallest prime satisfying* $p_i \geq \chi(G)$ *then* $rep(G) \geq p_i p_{i+1} \cdots p_{i+m-1}$, *where* $p_{i+1}, p_{i+2}, \ldots, p_{i+m-1}$ *are the next* $m - 1$ *primes larger than* p_i.

3 Complete Graphs minus disjoint set of stars

3.1 Lower bound for the representation number

A star $K_{1,m}$ is a graph with center vertex v_0 connected to m vertices $\{v_1, v_2, \ldots, v_m\}$ and having no other edges. When we consider a complete graph minus one or more disjoint stars, we only remove from the complete graph the edges that correspond to the stars. The set of vertices is not diminished. In other words, if we let $G_* = K_r - \sum_{1\leq j\leq s} K_{1,m_j}$ then $V(G_*) = V(K_r)$ and $E(G_*) = E(K_r) \setminus \bigcup_{1\leq j\leq s} E(K_{1,m_j})$. Indexing the stars removed from large to small facilitates computing the lower bound for the representation number. That is, we work with $K_r - \sum_{1\leq j\leq s} K_{1,m_j}$ such that $m_1 \geq m_2 \geq \cdots \geq m_s \geq 2$ and satisfying $V(K_{1,m_j}) \cap V(K_{1,m_l}) = \emptyset$ for $1 \leq l \neq j \leq r$.

Set $M_1 = 0$ and denote by $M_{i_0} = \sum_{i=1}^{i_0-1} m_j$ for $1 < i_0 \leq s+1$. Re-index if necessary the vertices of G_* so that the set of centers of the stars excluded from K_r is $C = \{v_{1+M_1}, v_{2+M_2}, v_{3+M_3}, \ldots, v_{s+M_s}\}$ and the vertices of any of the stars $K_{1,m_{i_0}} = \{v_{i_0+M_{i_0}}, v_{1+i_0+M_{i_0}}, \ldots, v_{m_{i_0}+i_0+M_{i_0}}\}$. Observe that the index for any vertex v_q that is not part of any of the excluded stars will have to be greater than $s + M_{s+1}$; that is $(s+1) + M_{s+1} \leq q \leq r$ if and only $v_q \in V(G_*) \setminus V\left(\sum_{1\leq j\leq s} K_{1,m_j}\right)$.

Observe that the set of vertices $V(G_*)\setminus C$ does induce a complete subgraph in G_* because for every pair of vertices in $V(G_*)\setminus C$ there is an edge in the graph G_*. This induced complete subgraph has $r - s > 0$ vertices. Moreover, any complete subgraph induced by a vertex set that contains a vertex $v_{i_0+M_{i_0}} \in C$ must exclude the vertices $\{v_{1+i_0+M_{i_0}}, \ldots, v_{m_{i_0}+i_0+M_{i_0}}\}$. Since $m_j \geq 2$ for all $1 \leq j \leq s$ and the stars are disjoint then any vertex set intersecting C must induce a complete subgraph in G_* whose order is strictly less than $r-s$. Therefore the smallest prime factor in the prime factorization of $rep(G_*)$ must be $p_i \geq r - s$ and $rep(G_*) \geq p_i p_{i+1} \cdots p_{i+m_1-1}$ according to Corollary 4.1. Observe that if the Prague dimension of G_* were less than m_1 then, as discussed above, the corresponding vector representation would yield a representation modulo an integer n with fewer than m_1-many primes. That would contradict this result obtained from Corollary 4.1. Therefore the Prague dimension of G_* must be at least m_1. A direct result of the theorem we prove in the next section is that the Prague dimension of G_* is exactly m_1.

3.2 Representation of $K_r - \sum_{1\leq j\leq s} K_{1,m_j}$ using the lower bound

Keeping the same set-up and notation as above, we now prove our main Theorem.

Theorem 5. *Let $G = K_r - \sum_{1\leq j\leq s} K_{1,m_j}$ be a complete graph (of size r) minus a disjoint union of ($s-$many) stars with $m_1 \geq m_2 \geq \cdots \geq m_s$. Then G has representation number $n = p_i p_{i+1} \cdots p_{i+m_1-1}$, where p_i is the smallest prime such that $p_i \geq r - s$.*

Remarks: The strategy behind our proof is to assign m_1-tuples in $\mathbb{Z}/p_i\mathbb{Z}\times \mathbb{Z}/p_{i+1}\mathbb{Z}\times\cdots\times\mathbb{Z}/p_{i+m_1-1}\mathbb{Z}$ to the vertices of $K_r-\sum_{1\leq j\leq s} K_{1,m_j}$ as follows:

- Label $v_{i_0+M_{i_0}} \in C$ with the m_1-tuple $(M_{i_0}, M_{i_0}, \ldots, M_{i_0})$ if $v_q \in C$; and if $v_q \in V(G_*) \setminus V\left(\sum_{1\leq j\leq s} K_{1,m_j}\right)$. attach to this vertex the m_1-tuple $(q-s-1, q-s-1, \ldots, q-s-1)$.
- Label $v_{j+i_0+M_{i_0}}$, where $1 \leq j < m_{i_0}$, with the m_1-tuple $(a_1, \ldots, a_{m_1})$ where each of the entries with index less than j are assigned the value $(j-1)+M_{i_0}$, the entry indexed by j is assigned the value M_{i_0}, and the last (m_1-j)-many entries of the m_1-tuple are assigned the value $j+M_{i_0}$.
- Label $v_{m_{i_0}+1+M_{i_0}}$ with the m_1-tuple $(a_1, \ldots, a_{m_1})$ where each of the entries with index less than m_{i_0} is assigned the value $M_{i_0}+m_{i_0}-1 = M_{i_0+1}-1$ and the last $(m_1-m_{i_0}+1)$-many entries of the m_1-tuple are assigned the value M_{i_0}.

Using this labeling scheme and the standard isomorphism $\mathbb{Z}/p_i\mathbb{Z}\times\mathbb{Z}/p_{i+1}\mathbb{Z}\times\cdots\times\mathbb{Z}/p_{i+m_1-1}\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ we produce a representation of $K_r-\sum_{1\leq j\leq s} K_{1,m_j}$ modulo n. To help the reader visualize the workings of this labeling scheme we organize the data in the following table:

vertices	mod p_i	mod p_{i+1}	mod p_{i+2}	$\cdots$	mod p_{i+m_2-1}	$\cdots$	mod p_{i+m_1-1}
v_{1+M_1}	$0 = M_1$	0	0	$\cdots$	0	$\cdots$	0
v_{1+M_1+1}	0	$1 = 1 + M_1$	1	$\cdots$	1	$\cdots$	1
v_{1+M_1+2}	1	0	2	$\cdots$	2	$\cdots$	2
v_{1+M_1+3}	2	2	0	$\cdots$	3	$\cdots$	3
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$
$v_{1+M_1+m_1}$	$m_1 - 1$	$m_1 - 1$	$m_1 - 1$	$\cdots$	$m_1 - 1$	$\cdots$	0
v_{2+M_2}	$m_1 = M_2$	M_2	M_2	$\cdots$	M_2	$\cdots$	M_2
v_{2+M_2+1}	M_2	$1 + M_2$	$1 + M_2$	$\cdots$	$1 + M_2$	$\cdots$	$1 + M_2$
v_{2+M_2+2}	$1 + M_2$	M_2	$2 + M_2$	$\cdots$	$2 + M_2$	$\cdots$	$2 + M_2$
v_{2+M_2+3}	$2 + M_2$	$2 + M_2$	M_2	$\cdots$	$3 + M_2$	$\cdots$	$3 + M_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$
$v_{2+M_2+m_2}$	$m_2 - 1 + M_2$	$m_2 - 1 + M_2$	$m_2 - 1 + M_2$	$\cdots$	M_2	$\cdots$	M_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_{s+M_s}	M_s	M_s	M_s	$\cdots$	M_s	$\cdots$	M_s
v_{s+M_s+1}	M_s	$1 + M_s$	$1 + M_s$	$\cdots$	$1 + M_s$	$\cdots$	$1 + M_s$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\cdots$	$\vdots$
$v_{s+M_s+m_s}$	$m_s - 1 + M_2$	$m_s - 1 + M_2$	$\cdots$	$\cdots$	M_2	$\cdots$	M_2
$v_{(s+1)+M_{s+1}}$	M_{s+1}	M_{s+1}	M_{s+1}	$\cdots$	M_{s+1}	$\cdots$	M_{s+1}
$v_{(s+1)+M_{s+1}+1}$	$M_{s+1} + 1$	$M_{s+1} + 1$	$M_{s+1} + 1$	$\cdots$	$M_{s+1} + 1$	$\cdots$	$M_{s+1} + 1$
$v_{(s+1)+M_{s+1}+2}$	$M_{s+1} + 2$	$M_{s+1} + 2$	$M_{s+1} + 2$	$\cdots$	$M_{s+1} + 2$	$\cdots$	$M_{s+1} + 2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
v_r	$r - s - 1$	$r - s - 1$	$r - s - 1$	$\cdots$	$r - s - 1$	$\cdots$	$r - s - 1$

The last rows in the above table with indices greater than $s + M_{s+1}$ exist only if there are vertices not belonging to any of the stars in the complete graph K_r. With the above table as an aide, we prove our labeling scheme produces a representation for complete graphs minus a set of stars.

Proof. Observe that for every $1 \leq l \leq m_1$, the l^{th} entry of every m_1-tuple has a value in the set $\{0, 1, \ldots, r - s - 1\} \subset \{0, 1, \ldots, p_{i+l-1}\}$. Also, for two different vertices with corresponding labels $(a'_1, \ldots, a'_{m_1})$ and $(a''_1, \ldots, a''_{m_1})$, if $a'_1 = a''_1$ then $a'_{m_1} \neq a''_{m_1}$. This follows from the labeling scheme and the fact that $m_1 \geq 2$. Therefore invoking the Chinese remainder theorem one concludes that the corresponding labeling of the vertices via the isomorphism $\mathbb{Z}/p_i\mathbb{Z} \times \mathbb{Z}/p_{i+1}\mathbb{Z} \times \cdots \times \mathbb{Z}/p_{i+m_1-1}\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ assigns distinct labels from $\{0, 1, 2, \ldots, n - 1\}$ to the vertices of G_*.
It remains to show this labeling scheme for G_* satisfies the adjacency condition. Fix a value $1 \leq l \leq m_1$ and consider the sequence $\{a_l(i)\}_{i=1}^{r}$ where $a_l(i)$ is the l^{th} entry of the m_1-tuple assigned to the i^{th} vertex (this sequence corresponds to the l^{th} column in the above table). Then (by construction) for $i' < i''$, $a_l(i') \equiv a_l(i'') \mod p_{i+l-1}$ if and only if $i' = i_0 + M_{i_0}$ for some $1 \leq i_0 \leq s$ and $i'' = \min\{i' + l,\ M_{i_0} + m_{i_0}\}$. These "if-and-only-if" statement can be restated as the following sequence of equivalent statements:

The difference between the values in labels from $\{0, 1, 2, \ldots, n - 1\}$ assigned to vectors $v_{i'}$ and $v_{i''}$ is divisible by p_{i+l-1} for some $1 \leq l \leq m_1$

$$\Longleftrightarrow$$
$$v_{i'} = v_{i_0+M_{i_0}} \text{ for some } 1 \le i_0 \le s \text{ and } v_{i''} = v_{i'+l} \text{ for some } 1 \le l \le m_{i_0}$$
$$\Longleftrightarrow$$
$$v_{i'},\, v_{i''} \in K_{1,m_{i_0}} \text{ for some } 1 \le i_0 \le s$$

We have established a representation for $G = K_r - \sum_{1 \le j \le s} K_{1,m_j}$ modulo $n = p_i p_{i+1} \cdots p_{i+m_1-1}$. Thus we can infer that $rep(G) \le n$. Combining this and the remarks made in subsection 3.1 completes the proof. ■

Corollary 5.1. *The Prague dimension of* $K_r - \sum_{1 \le j \le s} K_{1,m_j}$ *is* m_1.

Example 3.1. Consider the case of a complete graph K_m plus an isolated vertex. This can be represented as $K_{m+1} - K_{1,m}$. If p_i is a prime integer greater than or equal to m then $rep(K_{m+1} - K_{1,m}) = p_i p_{i+1} \cdots p_{i+m}$.

The reader might want to consider the complete graphs minus disjoint stars(2). Here we denote star(2) a graph with a central vertex to which paths with two edges are attached and such that there are no closed paths or loops. It seems that this case should still have Prague dimension determined by the largest star(2) removed. However, the largest complete subgraph might not provide a large enough smallest prime factor of the representation number modulo an integer for these type of graphs.

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