Corrections to the article "The metric dimension of graph with pendant edges" [Journal of Combinatorial Mathematics and Combinatorial Computing, 65 (2008) 139–145]

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Abstract

We show that the principal results of the article "The metric dimension of graph with pendant edges" [Journal of Combinatorial Mathematics and Combinatorial Computing, 65 (2008) 139–145] do not hold. In this paper we correct the results and we solve two open problems described in the above mentioned paper.

1 Introduction

Let $G = (V, E)$ be a simple graph of order $n = |V|$. Let $u, v \in V$ be two different vertices of G, the distance $d(u, v)$ between vertices u and v is the length of the shortest path between u and v. Given a set of vertices $S = \{v_1, v_2, ..., v_k\}$ of G, the *metric representation* of a vertex $v \in V$ with respect to S is the vector $r(v|S) = (d(v, v_1), d(v, v_2), ..., d(v, v_k))$. We say that S is a *resolving set* for G if for every pair of different vertices $u, v \in V$, $r(u|S) \neq r(v|S)$. The *metric dimension* of G is the minimum cardinality of any resolving set for G and it is denoted by $dim(G)$. The concept of metric dimension was introduced first independently by Harary and Melter [3] and Slater [5], respectively.

Let G and H be two graphs of order n and m, respectively. The corona product $G \odot H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and then joining by edges all the vertices from the i^{th}-copy of H with the i^{th}-vertex of G.

Given the graphs G and H with set of vertices $V_1 = \{v_1, v_2, ..., v_n\}$ and $V_2 = \{u_1, u_2, ..., u_m\}$, respectively, the Cartesian product of G and H is the graph $G \times H$ formed by the vertices $V = \{(v_i, u_j) : 1 \le i \le n, 1 \le j \le m\}$ and two vertices (v_i, u_j) and (v_k, u_l) are adjacent in $G \times H$ if and only if ($v_i = v_k$ and $u_j \sim u_l$) or ($v_i \sim v_k$ and $u_j = u_l$). The metric dimension of Cartesian product graph is studied in [2].

The following results related to the metric dimension of the graph $(P_n \times P_m) \odot K_1$ and $(K_n \times P_m) \odot K_1$ were published in [4]. We include parts of the proofs appearing in such a paper.

Theorem 1. [4] *For $n \ge 1$ and $1 \le m \le 2$, $dim((P_n \times P_m) \odot K_1) = 2$.*

Proof. Let $v_{ij} = (v_i, v_j)$ be the vertices of $P_n \times P_m \subsetneq (P_n \times P_m) \odot K_1$, where $v_i \in P_n$, $v_j \in P_m$, $1 \le i \le n$, and $1 \le j \le m$. Let u_{ij} be the pendant vertex of v_{ij}.

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Case 2. $m = 2$. Again, by Theorem A[1] (i), we only need to show that $dim((P_n \times P_2) \odot K_1) \le 2$. Choose a resolving set $B = \{u_{11}, u_{12}\}$ in $(P_n \times P_2) \odot K_1$. The representation of vertices $v \in (P_n \times P_2) \odot K_1$ by B are

$r(v_{i1}|B) = (i, i+1)$ and $r(v_{i2}|B) = (i+1, i)$ for $1 \le i \le n$,
$r(u_{i1}|B) = (d(v_{i1}, u_{11}) + 1, d(v_{i1}, u_{12}) + 1)$
and $r(u_{i2}|B) = (d(v_{i2}, u_{11}) + 1, d(v_{i2}, u_{12}) + 1)$, for $2 \le i \le n$.

All of those representations are distinct. Therefore, $dim((P_n \times P_m) \odot K_1) = 2$ □

Counterexample: Let $G = (P_3 \times P_2) \odot K_1$ (See Figure 1), from the above result $dim(G) = 2$ and the set $S = \{u_{11}, u_{12}\}$ is a resolving set for G. Now, for the vertices u_{22} and v_{32} we have that

$$r(u_{22}|S) = (4, 3) = r(v_{32}|S).$$

Thus, the sentence in bold of the above proof is not true.

[1]Theorem A state that for any connected graph G, $dim(G) = 1$ if and only if $G = P_n$.

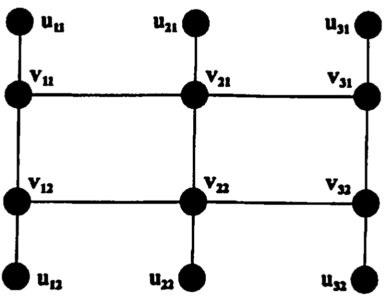

Figure 1: $\{u_{11}, u_{12}\}$ is not a resolving set.

Theorem 2. [4] *For* $n \geq 3$,

$$dim((K_n \times P_m) \odot K_1) = \begin{cases} n-1, & m=1, \\ n, & m=2. \end{cases}$$

Again we have a counterexample, which shows that the above result is also not true. Let the graph $G = (K_4 \times P_2) \odot K_1$. Thus, from the above theorem we get $dim(G) = 4$. Nevertheless, Figure 2 shows that $dim(G) = 3$.

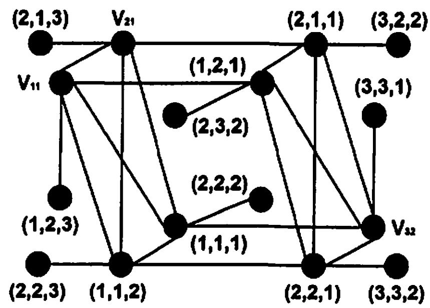

Figure 2: Counterexample for $m = 2$ and $n = 4$. The label of each vertex is its metric representation with respect to the set $\{v_{11}, v_{21}, v_{32}\}$.

In this paper we correct the cases $m = 2$ and $n \geq 3$ of the above results. We also solve the general case $m \geq 2$.

2 Results

Theorem 3. *If $n \geq 3$ and $m \geq 2$, then $dim((P_n \times P_m) \odot K_1) = 3$.*

Proof. Let $\{v_1, v_2, ..., v_n\}$ and $\{u_1, u_2, ..., u_m\}$ be the set of vertices of the graphs P_n and P_m, respectively. The vertices of $P_n \times P_m$ will be denoted by $v_{ij} = (v_i, u_j)$ and the pendant vertex of v_{ij} in $(P_n \times P_m) \odot K_1$ will be denoted by u_{ij}. We will show that $S = \{v_{11}, v_{1m}, v_{nm}\}$ is a resolving set for $(P_n \times P_m) \odot K_1$. The representations of vertices of $(P_n \times P_m) \odot K_1$ with respect to S are given by the following expressions,

$$\begin{aligned} r(v_{ij}|S) &= (d(v_{ij}, v_{11}), d(v_{ij}, v_{1m}), d(v_{ij}, v_{nm})) \\ &= (i + j - 2, m + i - j - 1, m + n - i - j), \end{aligned}$$

$$\begin{aligned} r(u_{ij}|S) &= (d(u_{ij}, v_{11}), d(u_{ij}, v_{1m}), d(u_{ij}, v_{nm})) \\ &= (i + j - 1, m + i - j, m + n - i - j + 1). \end{aligned}$$

Now, let us suppose there exist two different vertices x, y of $(P_n \times P_m) \odot K_1$ such that $r(x|S) = r(y|S)$. If $x = v_{ij}$ and $y = v_{kl}$, then $i \neq k$ or $j \neq l$ and we obtain that

$$(i+j-2, m+i-j-1, m+n-i-j) = (k+l-2, m+k-l-1, m+n-k-l).$$

Which leads to $i = k$ and $j = l$, a contradiction. Analogously we obtain a contradiction if $x = u_{ij}$ and $y = u_{kl}$. On the other hand, if $x = v_{ij}$ and $y = u_{kl}$, then we have

$$(i+j-2, m+i-j-1, m+n-i-j) = (k+l-1, m+k-l, m+n-k-l+1),$$

which leads to $1 = -1$, a contradiction. So, for every different vertices x, y of $(P_n \times P_m) \odot K_1$, we have $r(x|S) \neq r(y|S)$. Therefore, $dim((P_n \times P_m) \odot K_1) \leq 3$.

On the other hand, since $(P_n \times P_m) \odot K_1$ is not a path, $dim((P_n \times P_m) \odot K_1) \geq 2$. Now let us suppose $S' = \{a, b\}$ is a resolving set for $(P_n \times P_m) \odot K_1$. If there exist two different paths of length $d(a, b)$ between a and b, then there exist two different vertices c, d of $(P_n \times P_m) \odot K_1$ such that $d(c, a) = d(d, a)$ and $d(c, b) = d(d, b)$, a contradiction. Let us suppose there is only one path Q, of length $d(a, b)$, between a and b. Thus, all the vertices of Q, except possibly a or b which could be pendant vertices, belong either to a copy of P_n or to a copy of P_m. We consider the following cases.

Case 1: If every vertex belonging to the path Q has degree at most three, then $m = 2$ and $S' \subset \{u_{11}, v_{11}, u_{21}, v_{21}\}$ or $S' \subset \{u_{1n}, v_{1n}, u_{2n}, v_{2n}\}$.

Let us suppose $S' \subset \{u_{11}, v_{11}, u_{21}, v_{21}\}$. Now, for the vertices $u_{1i}, v_{1,i+1}$, $2 \le i \le n-1$ we have that

$$\begin{aligned} d(u_{i1}, a) &= d(u_{i1}, v_{11}) + d(v_{11}, a) \\ &= d(v_{i+1,1}, v_{11}) + d(v_{11}, a) \\ &= d(v_{i+1,1}, a), \end{aligned}$$

$$\begin{aligned} d(u_{i1}, b) &= d(u_{i1}, v_{11}) + d(v_{11}, b) \\ &= d(v_{i+1,1}, v_{11}) + d(v_{11}, b) \\ &= d(v_{i+1,1}, b). \end{aligned}$$

Thus, $r(u_{i1}|S') = r(v_{i+1,1}|S')$, a contradiction. On the contrary, if $S' \subset \{u_{1n}, v_{1n}, u_{2n}, v_{2n}\}$, then for the vertices $u_{i1}, v_{i-1,1}$, $2 \le i \le n-1$ we have

$$\begin{aligned} d(u_{i1}, a) &= d(u_{i1}, v_{1n}) + d(v_{1n}, a) \\ &= d(v_{i-1,1}, v_{1n}) + d(v_{1n}, a) \\ &= d(v_{i-1,1}, a), \end{aligned}$$

$$\begin{aligned} d(u_{i1}, b) &= d(u_{i1}, v_{1n}) + d(v_{1n}, b) \\ &= d(v_{i-1,1}, v_{1n}) + d(v_{1n}, b) \\ &= d(v_{i-1,1}, b). \end{aligned}$$

Thus, $r(u_{i1}|S') = r(v_{i-1,1}|S')$, a contradiction.

Case 2: There exists a vertex v of degree four belonging to the path Q. So, v has two neighbors c, d not belonging to Q, such that $d(c, a) = 1 + d(v, a) = d(d, a)$ and $d(c, b) = 1 + d(v, b) = d(d, b)$. Thus, $r(c|S') = r(d|S')$, a contradiction. Hence, $dim((P_n \times P_m) \odot K_1) \ge 3$. Therefore, the result follows. □

The following lemmas are useful to obtain the next result.

Lemma 4. [2] *If $n \ge 3$ then $dim(K_n \times P_m) = n - 1$.*

Lemma 5. [1] *If G_1 is a graph obtained by adding a pendant edge to a nontrivial connected graph G, then*

$$dim(G) \le dim(G_1) \le dim(G) + 1.$$

Theorem 6. *If $m \ge 2$, then*

$$dim((K_n \times P_m) \odot K_1) = \begin{cases} n-1, & \text{for } n \ge 4, \\ 3, & \text{for } n = 3. \end{cases}$$

Proof. Similarly to the above proof, let $v_{ij} = (v_i, u_j)$ be the set of vertices of $K_n \times P_m$, where v_i, $1 \leq i \leq n$ and u_j, $1 \leq j \leq m$ are vertices of the graphs K_n and P_m, respectively. Let us denote by u_{ij} the pendant vertex of v_{ij}. Assume that $n = 3$. We will show that $S = \{v_{11}, v_{21}, v_{3m}\}$ is a resolving set for $(K_3 \times P_m) \odot K_1$. Let us consider two different vertices x, y of $(K_3 \times P_m) \odot K_1$. We have the following cases.

Case 1: $x = v_{ij}$ and $y = v_{kl}$. If $j = l$, then $i \neq k$ and either $i \neq 3$ or $k \neq 3$, say $i \neq 3$. So, for $v_{i1} \in S$ we have $d(x, v_{i1}) = j - 1 < j = d(y, v_{i1})$. On the contrary, say $j < l$. If $i \neq 3$ or $k \neq 3$, for instance, $i \neq 3$, then for $v_{i1} \in S$ we have $d(x, v_{i1}) = j - 1 < l - 1 \leq d(y, v_{i1})$. Now, if $i = k = 3$, then $d(x, v_{3m}) = m - j > m - l = d(y, v_{3m})$.

Case 2: $x = u_{ij}$ and $y = u_{kl}$. Is analogous to the above case.

Case 3: $x = v_{ij}$ and $y = u_{kl}$. If $j = l$ and $i = k = 3$, then we have $d(x, v_{3m}) = m - j < m - j + 1 = d(y, v_{3m})$. Also, if $j = l$ and ($i \neq 3$ or $k \neq 3$), say $i \neq 3$, then for $v_{i1} \in S$ we have $d(x, v_{i1}) = j - 1 < j \leq d(y, v_{i1})$. On the other hand, if $j \neq l$, we consider the following subcases.

Subcase 3.1: $i = k$ and $i \neq 3$. If $j = l + 1$, then we have that $d(x, v_{3m}) = m - j + 1 = m - l < m - l + 2 = d(y, v_{3m})$. On the other hand, if $j \neq l + 1$, then for $v_{i1} \in S$ we have $d(x, v_{i1}) = j - 1 \neq l = d(y, v_{i1})$.

Subcase 3.2: $i = k = 3$. If $j = l - 1$, then there exists $v_{r1} \in S$, $r \neq 3$ such that $d(x, v_{r1}) = j = l - 1 < l + 1 = d(y, v_{r1})$. On the other hand, if $j \neq l - 1$, then we have that $d(x, v_{3m}) = m - j \neq m - l + 1 = d(y, v_{3m})$.

Subcase 3.3: $i \neq k$. Hence, we have either $i \neq 3$ or $k \neq 3$, for instance $i \neq 3$. If $d(x, v_{i1}) = j - 1 = d(y, v_{i1})$, then there exist $v_{r1} \in S - \{v_{i1}\}$, $r \neq 3$, such that $d(x, v_{r1}) = j > j - 1 \geq d(y, v_{r1})$.

Therefore, $dim((K_3 \times P_m) \odot K_1) \leq 3$.

On the other hand, let $S' = \{a, b\}$ be a resolving set for $(K_3 \times P_m) \odot K_1$. If there exist two different paths of length $d(a, b)$ between a and b, then there exist two different vertices c, d of $(K_3 \times P_m) \odot K_1$ such that $d(c, a) = d(d, a)$ and $d(c, b) = d(d, b)$. Hence, $r(c|S') = r(d|S')$, a contradiction. Moreover, if there is only one path Q, of length $d(a, b)$, between a and b, then there exists a vertex v of degree four belonging to the path Q. So, v has two neighbors c, d not belonging to Q, such that $d(c, a) = 1 + d(v, a) = d(d, a)$ and $d(c, b) = 1 + d(v, b) = d(d, b)$. Thus, $r(c|S') = r(d|S')$, a contradiction. Thus, $dim((K_3 \times P_m) \odot K_1) \geq 3$. Therefore, for $n = 3$, the result follows.

Now, let $n \geq 4$. We will show that $S = \{v_{1m}, v_{31}, v_{41}, ..., v_{n1}\}$ is a resolving set for $(K_n \times P_m) \odot K_1$. Let us consider two different vertices x, y of $(K_n \times P_m) \odot K_1$. We have the following cases.

Case 1: $x = v_{ij}$ and $y = v_{kl}$. If $j = l$, then $i \neq k$. Let us suppose $i = 1$ and $k = 2$. Hence for $v_{1,m} \in S$ we have $d(x, v_{1m}) = m - j < m - j + 1 = d(y, v_{1m})$. Now, if $i \notin \{1, 2\}$ or $k \notin \{1, 2\}$, then we have $v_{i1} \in S$ or $v_{k1} \in S$, say $v_{i1} \in S$. Thus, we have $d(x, v_{i1}) = j - 1 < j = l = d(y, v_{i1})$.

On the other hand, if $j \neq l$, say $j < l$, then there exists $v_{t1} \in S$, $t \in \{3, ..., n\}$, $t \neq k$, such that

$$\begin{aligned} d(x, v_{t1}) &= d(x, v_{i1}) + d(v_{i1}, v_{t1}) \\ &\leq j - 1 + d(v_{k1}, v_{t1}) \\ &< l - 1 + d(v_{k1}, v_{t1}) \\ &= d(y, v_{k1}) + d(v_{k1}, v_{t1}) \\ &= d(y, v_{t1}). \end{aligned}$$

Case 2: $x = u_{ij}$ and $y = u_{kl}$. Since $d(u_{ij}, v) = d(v_{ij}, v) + 1$ for every $v \in S$, we proceed analogously to the above case and we obtain that $r(u_{ij}|S) \neq r(u_{kl}|S)$.

Case 3: $x = v_{ij}$ and $y = u_{kl}$. If $j \leq l$, then for every $v_{t1} \in S$ we have

$$\begin{aligned} d(x, v_{t1}) &= d(x, v_{i1}) + d(v_{i1}, v_{t1}) \\ &= j - 1 + d(v_{i1}, v_{t1}) \\ &< l - 1 + d(v_{i1}, v_{t1}) \\ &\leq l + d(v_{k1}, v_{t1}) \\ &= d(y, v_{k1}) + d(v_{k1}, v_{t1}) \\ &= d(y, v_{t1}). \end{aligned}$$

Now, if $j > l$, then we have

$$\begin{aligned} d(x, v_{1m}) &= d(x, v_{im}) + d(v_{im}, v_{1m}) \\ &= m - j + d(v_{im}, v_{1m}) \\ &< m - l + d(v_{im}, v_{1m}) \\ &\leq m - l + 1 + d(v_{km}, v_{1m}) \\ &= d(y, v_{km}) + d(v_{km}, v_{1m}) \\ &= d(y, v_{1m}). \end{aligned}$$

Therefore, for every two different vertices x, y of $(K_n \times P_m) \odot K_1$ we have, $r(x|S) \neq r(y|S)$ and, as a consequence, S is a resolving set for $(K_n \times P_m) \odot K_1$ of cardinality $n - 1$.

On the other hand, by Lemma 4 and Lemma 5 we have $dim((K_n \times P_m) \odot K_1) \geq n - 1$. Hence, for $n \geq 4$, the result follows. □

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