Roman fractional bondage number of a graph

N. Dehgardi, B. Kheirfam[1] and S.M. Sheikholeslami[2]
Department of Mathematics
Azarbaijan University of Tarbiat Moallem
Tabriz, I.R. Iran
[1]b.kheirfam@azaruniv.edu
[2]s.m.sheikholeslami@azaruniv.edu

O. Favaron
Univ Paris-Sud, LRI, UMR 8623, Orsay, F-91405, France;
CNRS, Orsay, F-91405.
Odile.Favaron@lri.fr

Abstract

A *Roman dominating function* on a graph G is a labeling $f : V(G) \longrightarrow \{0, 1, 2\}$ such that every vertex with label 0 has a neighbor with label 2. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u\in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted by $\gamma_R(G)$. The *Roman bondage number* of a graph G is the cardinality of a smallest set of edges whose removal results in a graph with Roman domination number greater than that of G.

In this paper we initiate the study of the Roman fractional bondage number, and we present different bounds on Roman fractional bondage. In addition, we determine the Roman fractional bondage number of some classes of graphs.

Keywords: Roman domination number, Roman bondage number, Roman fractional bondage number.
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1 Introduction

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V(G) \mid uv \in E(G)\}$

and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *open neighborhood* of a set $S \subseteq V$ is the set $N(S) = \cup_{v \in S} N(v)$, and the *closed neighborhood* of S is the set $N[S] = N(S) \cup S$. The independent domination number $i(G)$, is the cardinality of the smallest maximal independent set of G. Let $S \subseteq V$ be a subset of vertices of G and x a vertex of S. A vertex y is a *private neighbor* of x with respect of S, or S-private neighbor, if $y = x$ in the case x is isolated in $G[S]$, or $y \in V \setminus S$ and x is the unique neighbor of y in S. The private neighbors of the second kind are called *external private neighbors*.

We write K_n for the *complete graph* of order n, P_n for a path on n vertices and C_n for a *cycle* of length n. Consult [7, 12] for the notation and terminology which are not defined here.

A *Roman dominating function* on a graph G is a labelling $f : V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The weight of a Roman dominating function is the value $f(V(G)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the *Roman domination number*, denoted by $\gamma_R(G)$. A graph G of order n satisfies $\gamma_R(G) = n$ if and only $\Delta(G) \leq 1$, i.e., each of its component is a K_1 or a K_2. A *$\gamma_R(G)$-function* is a Roman dominating function on G with weight $\gamma_R(G)$. A Roman dominating function $f : V \to \{0,1,2\}$ can be represented by the ordered partition (V_0, V_1, V_2) (or (V_0^f, V_1^f, V_2^f) to refer to f) of V, where $V_i = \{v \in V \mid f(v) = i\}$. In this representation, its weight is $\omega(f) = |V_1| + 2|V_2|$. In [3], some properties of $\gamma_R(G)$-functions are given. In particular every vertex of V_2 of a $\gamma_R(G)$-function has at least two V_2-private neighbors, one of them being possibly internal and the other ones in V_0. If an isolated vertex x of V_2 has exactly one private neighbor y in V_0, we can also put x and y in V_1. To avoid this ambiguity, we choose in this case to put x and y in V_1 and we call *good $\gamma_R(G)$-function* of G a $\gamma_R(G)$-function such that $|V_2|$ is minimum. Then every vertex of V_2^f has at least two external V_2^f-private neighbors, obviously all in V_0. When we delete an edge e of G, the Roman domination number cannot decrease and $\gamma_R(G - e) \geq \gamma_R(G)$ for every edge of G.

The definition of the Roman dominating function was given implicitly by Stewart [11] and Revelle et al. [10]. Cockayne, Dreyer, Hedetniemi and Hedetniemi [3] as well as Chambers, Kinnersley, Prince and West [1] have given a lot of results on Roman domination.

Let G be a graph with maximum degree at least two. The *Roman bondage number* $b_R(G)$ of G is the minimum cardinality of all sets $E' \subseteq E(G)$ for which $\gamma_R(G - E') > \gamma_R(G)$. The Roman bondage number was introduced by Jafari Rad and Volkmann in [8], and has been further studied for example in [4, 9]. Since the Roman domination number of the graph K_2 does not change when its only edge is deleted, in the study of Roman bondage number we must assume that one of the components of the graph has order at least 3 (equivalently, $\gamma_R(G) < n$) or that G is connected of order $n \geq 3$.

In [2], Chvátal and Cook gave a characterization of the bondage number of a graph as the entire optimal solution of a linear program and called fractional bondage number the solution of the relaxation to R of this program. They also defined the discipline number as the entire optimal solution of the dual problem. Our purpose is to introduce and study in a similar way the fractional Roman bondage number and the Roman discipline number of a graph. We keep the same terminology as in [2].

Definition 1. A *whip* associated to a good $\gamma_R(G)$-function f is a spanning forest of G with $|V_1^f| + |V_2^f|$ components whose K_1-components are the vertices of V_1^f and the other components are stars of center in V_2^f and leaves in V_0^f. The set of all the whips of G is denoted $W(G)$.

Several whips can be associated to the same good $\gamma_R(G)$-function f but for all of them, all the V_2^f-private neighbors of a vertex u of V_2^f are leaves of the star centered at u. Therefore each star has order at least 3. Let us call *star-forest* of G a spanning forest whose components are n_1 K_1's and n_2 stars of order at least 3. Then a star-forest of G satisfies $n_1 + 2n_2 \geq \gamma_R(G)$ and is a whip if and only if $n_1 + 2n_2 = \gamma_R(G)$.

Proposition 2. Let $E' = \{e_1, e_2, \cdots, e_p\}$ be a set of edges of G. Then $\gamma_R(G - E') > \gamma_R(G)$ if and only if each whip of G contains at least one edge of E'.

Proof. Suppose each whip of G contains at least one edge of E'. Let W be a whip of $\gamma_R(G - E')$ and let V_0, V_1, V_2 be respectively the set of isolated vertices, leaves and centers of stars of W. Then $2|V_2| + |V_1| = \gamma_R(G - E')$. As W is a star-forest but not a whip of G, $2|V_2| + |V_1| > \gamma_R(G)$. Hence $\gamma_R(G - E') > \gamma_R(G)$.
Suppose some whip W of G contains no edge of E' and let V_0, V_1, V_2 be respectively the set of isolated vertices, leaves and centers of stars of W. Then W is a star-spanning forest of $G - E'$ and $\gamma_R(G - E') \leq 2|V_2| + |V_1| = \gamma_R(G)$. As $\gamma_R(G - E') \geq \gamma_R(G)$, we get $\gamma_R(G - E') = \gamma_R(G)$. □

From Proposition 2, $b_R(G)$ is the optimal solution of the problem

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E(G)} x_e & \\ \text{subject to} & \sum_{e \in E(F)} x_e \geq 1 & \text{for all } F \text{ in } W(G), \\ & x_e \geq 0 & \text{for all } e \text{ in } E(G), \\ & x_e = \text{integer} & \text{for all } e \text{ in } E(G). \end{array} \qquad (1)$$

By the Roman fractional bondage number $b_R^*(G)$ we shall mean the optimal value of the linear programming relaxation of (1),

$$\begin{array}{lll} \text{minimize} & \sum_{e\in E(G)} x_e & \\ \text{subject to} & \sum_{e\in E(F)} x_e \geq 1 & \text{for all } F \text{ in } W(G), \\ & x_e \geq 0 & \text{for all } e \text{ in } E(G). \end{array} \tag{2}$$

By the duality theorem of linear programming, $b_R^*(G)$ equals the optimal value of the dual of (2),

$$\begin{array}{lll} \text{maximize} & \sum_{F\in W(G)} y_F & \\ \text{subject to} & \sum_{e\in E(F)} y_F \leq 1 & \text{for all } e \text{ in } E(G), \\ & y_F \geq 0 & \text{for all } F \text{ in } W(G). \end{array} \tag{3}$$

Equation (3) can be seen as the linear programming relaxation of

$$\begin{array}{lll} \text{maximize} & \sum_{F\in W(G)} y_F & \\ \text{subject to} & \sum_{e\in E(F)} y_F \leq 1 & \text{for all } e \text{ in } E(G), \\ & y_F \geq 0 & \text{for all } F \text{ in } W(G), \\ & y_F = \text{integer} & \text{for all } F \text{ in } W(G). \end{array} \tag{4}$$

Problems (1) and (4) are in a sense dual. We refer to the optimal value of (4) as the *Roman discipline number* $\mathrm{dis}_R(G)$ of G and to the optimal value of (3) as the *fractional Roman discipline number* $\mathrm{dis}_R^*(G)$ of G. A particular feasible solution of (4) is obtained by taking $y_{F_0} = 1$ for one whip F_0 of W and $y_F = 0$ for $F \neq F_0$. Then for all graphs G we have

$$1 \leq \mathrm{dis}_R(G) \leq \mathrm{dis}_R^*(G) = b_R^*(G) \leq b_R(G). \tag{5}$$

Apart from establishing upper bounds on $b_R(G)$, Rad and Volkmann computed the Roman bondage number of cycles, paths, and complete bipartite graphs and studied the Roman bondage number of trees. Our purpose in this paper is to provide ties with analogous results for the Roman fractional number and for discipline number.

To end this section, we present a lower bound on the size of a graph G with given order and Roman bondage number. We make use of the following results.

Theorem A. **(Chambers et al. [1] 2009)** If G is a connected n-vertex graph, then $\gamma_R(G) \leq \frac{4n}{5}$, with equality if and only if G is C_5 or is the union of $\frac{n}{5}P_5$ with a connected subgraph whose vertex set is the set of centers of the components of $\frac{n}{5}P_5$.

Theorem B. **(Rad, Volkmann [8])** If G is a graph of order $n \geq 3$ and uvw a path of length 2 in G, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 3.$$

If u and w are adjacent, then

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - |N(u) \cap N(v)| - 4.$$

For any connected graph G, let $\deg_a(G)$ represent the value of the expression $\sum_{v \in V(G)} \deg(v)/|V(G)|$.

Theorem C. (Hartnell, Rall [6] 1999) For any connected graph G, there exists a pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that $\deg(u) + \deg(v) \leq 2\deg_a(G)$.

Theorem 3. Let G be a connected graph of order $n \geq 3$, average degree $\deg_a(G)$ and bondage number $b_R(G)$. Then

$$b_R(G) \leq 2\deg_a(G) + \Delta(G) - 3 \text{ and } |E(G)| \geq (n/4)(b_R(G) - \Delta(G) + 3).$$

Proof. Let G be a graph satisfying the hypothesis. By Theorem C we know there is at least one pair of vertices, say u and v, that are either adjacent or at distance 2 from each other, with the property that $\deg(u) + \deg(v) \leq 2\deg_a(G)$. Since G is connected and $n \geq 3$, there is a path such as uvw or uwv. In either case by Theorem B we have

$$b_R(G) \leq \deg(u) + \deg(v) + \deg(w) - 3 \leq 2\deg_a(G) + \Delta(G) - 3.$$

Since $2|E(G)| = n\deg_a(G)$, we have $4|E(G)| = 2n\deg_a(G) \geq n(b_R(G) - \Delta + 3)$. Hence

$$|E(G)| \geq (n/4)(b_R(G) - \Delta + 3).$$

□

We observe that the two bounds are sharp for the cycle C_n when $n \equiv 2 \pmod 3$ [5, 8].

2 The Roman fractional bondage number

In this section, we first give some bounds on the Roman fractional bondage number and then we determine the Roman fractional bondage number of some classes of graphs.

Theorem 4. Let G be a connected graph of order $n \geq 3$ and size m such that each whip has at least k edges. Then $b_R^*(G) \leq m/k$.

Proof. It is easy to verify that the constraints of (2) are fulfilled by $x_e = 1/k$ for all e. □

Theorem 5. For any connected graph G of order $n \geq 3$ and size m such that $\gamma_R(G) < n$,

$$b_R^*(G) \leq \frac{m}{n - \gamma_R(G) + 1}.$$

Proof. Let $F = \{F_1, \ldots, F_t\}$ be a whip of G. Let $F_1, \ldots, F_r$ be the stars of order at least 3 and $F_{r+1}, \ldots, F_t$ the trivial components of F if $t > r$. Then $\gamma_R(G) = 2r + (t - r) = r + t \geq t + 1$ and the number of edges in the forest F with t components is equal to $n - t$. The constraints of (2) are fulfilled by $x_e = 1/(n - \gamma_R(G) + 1)$ for all e. This completes the proof. □

Theorem 6. For any connected graph G of order $n \geq 3$ such that $i(G) \leq n/2$,

$$b_R^*(G) \leq \frac{\Delta(G)(n - i(G))}{n - 2i(G) + 1}$$

.

Proof. Let S be a minimum maximal independent set of G. Clearly every vertex outside S has at least one neighbor in S and hence $f = (V(G) - S, \emptyset, S)$ is a Roman dominating function of G. Thus $\gamma_R(G) \leq 2i(G)$. On the other hand, the number of edges in G is at most the sum of the degrees of all the vertices in $V(G) - S$. Thus, $|E(G)| \leq \Delta(G)(n - i(G))$. It follows from Theorem 5 that

$$b_R^*(G) \leq \frac{\Delta(G)(n - i(G))}{n - 2i(G) + 1}.$$

□

When $\Delta(G) = n - 1$, Ebadi and PushpaLatha [5], Rad and Volkmann [8] proved that if G has $k \geq 1$ vertices of degree $n - 1$, then $b_R(G) = \lceil \frac{k}{2} \rceil$.

Theorem 7. Let G be a connected graph of order $n \geq 3$. If G has exactly $k \geq 1$ vertices of degree $n - 1$, then $b_R^*(G) = 1$ if $k = 1$ and $b_R^*(G) = k/2$ if $k \geq 2$.

Proof. We may assume that $k \geq 2$, for otherwise the result follows from $b_R(G) = 1$ and (5). Let S be the set of vertices of degree $n - 1$. The whips are the spanning stars centered at a vertex of S and $\gamma_R(G) = 2$. Setting $x_e = 1/(k - 1)$ if both endpoints of e have degree $n - 1$, and $x_e = 0$ otherwise, we obtain a feasible solution of (2) and hence $b_R^*(G) \leq k/2$. On the other hand, there are exactly k whips. Setting $y_F = \frac{1}{2}$ for each whip F, we obtain a feasible solution of (3) of value $k/2$. Hence by (5), $k/2 \leq \mathrm{dis}_R^*(G) = b_R^*(G)$ and the proof is complete. □

Corollary 8. For $n \geq 3$, $b_R^*(K_n) = n/2$.

In dealing with complete multipartite graphs, we can now suppose that each part contains at least two vertices and we distinguish two cases depending on the size of the smallest part. Ebadi and PushpaLatha [5] proved that for the complete bipartite graph K_{n_1,n_2} with $2 \le n_1 \le n_2$, $b_R(K_{3,3}) = 5$ and $b_R(K_{n_1,n_2}) = n_1$ in the other cases. The particular case $b_R(K_{2,n_2}) = 2$ was also proved by Rad and Volkmann [8].

Theorem 9. Let $G = K_{n_1,\ldots,n_t}$ be the complete t-partite graph with $2 = n_1 \le \ldots \le n_t$ and exactly k partite sets of size 2. Then $b_R^*(G) = 2$ if $k = 1$ and $b_R^*(G) = k$ when $k \ge 2$.

Proof. Let $n = \sum_{i=1}^{t} n_i$. It is proved in [3] that $\gamma_R(G) = 3$. First let $k = 1$ and let $\{v_1, v_2\}$ be the partite set of size 2. There are two whips, each of them consists of a star with $n-2$ leaves centered at v_i and one isolated vertex $v_j, 1 \le i \ne j \le 2$. Setting $x_e = 1/(n-2)$ if one of the endpoints of e is v_1 or v_2, and $x_e = 0$ otherwise, we obtain a feasible solution of (2). Hence $b_R^*(G) \le 2$. On the other hand, each edge is contained in at most one whip. Setting $y_F = 1$ for each whip gives a feasible solution of (3). Hence $b_R^*(G) \ge 2$ and so $b_R^*(G) = 2$.

Let now $k \ge 2$. Let $\{v_1^\ell, v_1^\ell\}, 1 \le \ell \le k$, be the k partite sets of size 2. There are exactly $2k$ whips. Each of them consists of a star with $n-2$ leaves centered at v_i^ℓ and one isolated vertex $v_j^\ell, 1 \le i \ne j \le 2$, for $1 \le \ell \le k$. Each whip contains $2(k-1)$ edges, both endpoints of which have degree $n-2$, and there are $2k(2k-1)$ such edges. Setting $x_e = 1/(2k-2)$ if both endpoints of e have degree $n-2$, and $x_e = 0$ otherwise, we obtain a feasible solution of (2). Hence $b_R^*(G) \le k$. On the other hand, each edge is contained in at most 2 whips. Setting $y_F = \frac{1}{2}$ for each whip F gives a feasible solution of (3). Hence $b_R^*(G) \ge k$ and the proof is complete. □

Theorem 10. Let $G = K_{n_1,\ldots,n_t}$ be the complete t-partite graph with $3 \le n_1 \le \ldots \le n_t$. Then $b_R^*(G) = \frac{m}{n-2}$ where $n = \sum_{i=1}^{t} n_i$ and $|E(G)| = m$.

Proof. Let S_i, $1 \le i \le t$, be the partite sets with $|S_i| = n_i \ge 3$. By [3], $\gamma_R(G) = 4$. For each $\gamma_R(G)$-function, V_2 is a set of two adjacent vertices and $V_1 = \emptyset$. Each whip consists in two stars of order at least 3 with centers in different partite sets and has exactly $n-2$ edges. By Theorem 4, $b_R^*(G) \le \frac{m}{n-2}$.

Now we exhibit an appropriate feasible solution of (3). The whip is of type (i, j) if its two centers belong to S_i and S_j. Denote by $W(u_i, u_j)$ the set of whips of centers $u_i \in S_i$ and $u_j \in S_j$. For each whip in $W(u_i, u_j)$, all the vertices of $S_j \setminus \{u_j\}$ are leaves of the star centered at u_i and all the vertices of $S_i \setminus \{u_i\}$ are leaves of the star centered at u_j. There are $2^{n-(n_i+n_j)}$ ways to distribute the remaining vertices between the two stars. Hence $|W(u_i, u_j)| = 2^{n-(n_i+n_j)}$. As there are n_i choices for u_i in S_i and n_j choices for u_j in S_j, there are precisely $n_i n_j 2^{n-(n_i+n_j)}$ whips of type (i,j). Also, each edge with one endpoint u_i in S_i and the other endpoint u_j in S_j belongs to exactly $(n_i+n_j-2)2^{n-(n_i+n_j)}$ whips of type (i,j) (one star is centered at u_i or u_j and there are n_i+n_j-2 choices for the second center), to precisely $n_k 2^{n-(n_i+n_k)-1}$ whips of type (i,k) with $k \neq i,j$ (n_k choices for the center of the star centered in S_k, and $2^{n-(n_i+n_k)-1}$ ways to distribute between the two stars the vertices of $V \backslash (S_i \cup S_k \cup \{u_j\})$, and to precisely $n_k 2^{n-(n_j+n_k)-1}$ whips of type (k,j) with $k \neq i,j$. Let $z_H = \frac{1}{n-2} 2^{n_i+n_j-n}$ for each whip H of type (i,j) and $y_F = \sum\{z_H \mid H \text{ is a whip such that } E(H) = E(F)\}$. Let e be an edge of G. We can suppose without loss of generality $e = u_i u_j$ with $u_i \in S_i$ and $u_j \in S_j$. By considering all the whips F containing e we get

$$\begin{aligned}
\textstyle\sum_{e \in E(F)} y_F &= \tfrac{2^{n_i+n_j-n}}{n-2}(n_i+n_j-2)2^{n-(n_i+n_j)} + \\
&\quad \textstyle\sum_{k \neq i,j}\left(\tfrac{2^{n_i+n_k-n}}{n-2} n_k 2^{n-(n_i+n_k)-1} + \right. \\
&\quad \left. \tfrac{2^{n_j+n_k-n}}{n-2} n_k 2^{n-(n_j+n_k)-1}\right) \\
&= \tfrac{1}{n-2}(n_i+n_j-2+\textstyle\sum_{k \neq i,j} n_k) \\
&= 1.
\end{aligned}$$

Hence we get in this way a feasible solution of (3). For this solution,

$$\sum\{y_F \mid F \text{ is of type } (i,j)\} = n_i n_j 2^{n-(n_i+n_j)} \frac{2^{n_i+n_j-n}}{n-2} = \frac{n_i n_j}{n-2}$$

and

$$\sum_{F \in W} y_F = \frac{\sum_{1 \leq i \neq j \leq t} n_i n_j}{n-2} = \frac{m}{n-2}.$$

This completes the proof. □

Recall that a graph G is edge-transitive if for every pair e_1, e_2 of edges, some automorphism of G sends e_1 to e_2.

Theorem 11. If G is a connected edge-transitive graph on $n \geq 3$ vertices with m edges and each whip has precisely k edges, then

$$b_R^*(G) = m/k.$$

Proof. Since G is edge-transitive, (2) has an optimal solution in which all the values of x_e are equal. Hence b_R^* is the optimal value of the problem

$$\text{minimize } mx \quad \text{subject to } kx \geq 1,\ x \geq 0,$$

from which the result follows. □

For cycles, it was shown in [3] that $\gamma_R(C_n) = \lceil 2n/3 \rceil$. In addition, we know by [5, 8] that for $n \geq 3$, $b_R(C_n) = 3$ if $n \equiv 2 \pmod 3$ and $b_R(C_n) = 2$ otherwise. As the cycle is edge-transitive, we can apply Theorem 11.

Theorem 12. For $k \geq 1$, $b_R^*(C_{3k}) = 3/2, b_R^*(C_{3k+1}) = (3k+1)/2k$ and $b_R^*(C_{3k+2}) = (3k+2)/(2k)$.

Proof. First let $n \equiv 0 \pmod 3$. Then $n = 3k$ for some positive integer and $\gamma_R(C_{3k}) = 2k$. For each $\gamma_R(C_{3k})$-function, V_2 is an independent set of size k and $V_1 = \emptyset$. Each whip consists in k stars of order 3 and has exactly $2k$ edges. It follows from Theorem 11 that $b_R^*(G) = 3/2$.

Now let $n = 3k+1$ for some positive integer k. Then $\gamma_R(C_{3k+1}) = 2k+1$. For each $\gamma_R(G)$-function, V_2 is an independent set of size k and $|V_1| = 1$. Each whip consists in k stars of order 3 and one isolated vertices, and has exactly $2k$ edges. It follows from Theorem 11 that $b_R^*(G) = (3k+1)/2k$.

Finally, let $n = 3k+2$ for some positive integer k. Then $\gamma_R(C_{3k+2}) = 2k+2$. For each good $\gamma_R(G)$-function, V_2 is an independent set of size k and $|V_1| = 2$ (the two vertices of V_1 may or not be adjacent). Each whip has exactly $2k$ edges and it follows from Theorem 11 that $b_R^*(G) = (3k+2)/(2k)$. □

The determination of b_R^* for paths is more difficult because they are not edge-transitive. If P_n is the path of order n, then it was shown in [3] that $\gamma_R(P_n) = \lceil 2n/3 \rceil$. In addition, we find in [5, 8] that for $n \geq 3$, $b_R(P_n) = 2$ if $n \equiv 2 \pmod 3$ and $b_R(P_n) = 1$ otherwise.

Theorem 13. If $n \geq 3$, then $b_R^*(P_5) = 2$, $b_R^*(P_n) = \frac{3}{2}$ if $n = 3k+2$ for some integer $k \geq 2$, and $b_R^*(P_n) = 1$ otherwise.

Proof. We denote $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and for $1 \leq i \leq n-1$, $v_i v_{i+1} = e_i$. If $n \not\equiv 2 \pmod 3$, then the result follows from (5) and $b_R(P_n) = 1$. Let $n = 3k+2$ for some positive integer k. Then $\gamma_R(P_n) = 2k+2$ and for each good $\gamma_R(G)$-function, V_2 is an independent set of k vertices and V_1 contains two adjacent or nonadjacent vertices. As $\Delta(G) = 2$, each star of each whip has order 3 and exactly one whip is associated to each good $\gamma_R(G)$-function. Each whip has $2k$ edges and is determined by the place of the two isolated vertices. We denote by $F(p,q)$ the whip corresponding to $V_1 = \{v_p, v_q\}$. Since the vertices of the stars form paths P_3, the vertices v_p, v_q are such that $p = 3i+1$ and $q = 3j+2$ for $0 \leq i \leq j \leq k$. The indices i and j are equal when v_p and v_q are adjacent.

First let $k = 1$, i. e., $G = P_5$. The three whips have edge-sets $E(F(1,2)) = \{e_3, e_4\}$, $E(F(1,5)) = \{e_2, e_3\}$, $E(F(4,5)) = \{e_1, e_2\}$. The values $x(e_2) = x(e_3) = 1$ and $x(e_2) = x(e_3) = 0$ are a feasible solution of (2). Hence $b_R^*(P_5) \leq 2$. On the other hand, e_1 belongs to the unique whip $F(4,5)$, e_4 belongs to the unique whip $F(1,2)$, e_2 belongs to $F(1,5)$

and $F(4,5)$ and e_3 belongs to $F(1,2)$ and $F(1,5)$. Therefore the values $y(F(1,2)) = y(F(4,5)) = 1$ and $y(F(1,5)) = 0$ are a feasible solution of (3), and $b^*_R(P_5) \geq 2$. Hence $b^*_R(P_5) = 2$.

Now let $k \geq 2$. If the two vertices of V_1 are adjacent, then $j = i$ and

$$E(F(3i+1,3i+2)) = \bigcup_{0\leq s\leq i-1 \text{ if } i\neq 0}\{e_{3s+1,3s+2}\} \cup \bigcup_{i+1\leq s\leq k \text{ if } i\neq k}\{e_{3s}, e_{3s+1}\}. \tag{6}$$

If the two vertices of V_1 are not adjacent, then $j > i$ and

$$E(F(3i+1,3j+2)) = \bigcup_{0\leq s\leq i-1 \text{ if } i\neq 0}\{e_{3s+1}, e_{3s+2}\} \cup \bigcup_{i\leq s\leq j-1}\{e_{3s+2}, e_{3s+3}\} \cup \bigcup_{j+1\leq s\leq k \text{ if } j\neq k}\{e_{3s}, e_{3s+1}\}. \tag{7}$$

The unique whip not containing e_2 is $F(1,2)$ and $F(1,2)$ contains e_4 and e_{3k}. The unique whip not containing e_{3k} is $F(3k+1,3k+2)$ and $F(3k+1,3k+2)$ contains e_2 and e_4. Therefore each whip contains at least two edges among e_2, e_4, e_{3k}. Hence putting $x_{e_2} = x_{e_4} = x_{e_{3k}} = \frac{1}{2}$ and $x_{e_s} = 0$ for $s \notin \{2,4,3k\}$ gives a feasible solution of (2). Thus $b^*_R(P_{3k+2}) \leq \frac{3}{2}$.

To construct a feasible solution of (3) we observe that by (6),

$$E(F(1,2)) \bigcap E(F(3k+1,3k+2)) = \bigcup_{1\leq s\leq k-1} \{e_{3s+1}\}$$

and by (7),

$$F(4,3k+2) = \{e_1,e_2\} \bigcup \bigcup_{1\leq s\leq k-1} \{e_{3s+2}, e_{3s+3}\}$$

and

$$F(4,3k-1) = \{e_1,e_2\} \bigcup \bigcup_{1\leq s\leq k-2} \{e_{3s+2}, e_{3s+3}\} \bigcup \{e_{3k}, e_{3k+1}\}.$$

Hence $E(F(4,3k+2)) \cap E(F(1,2)) \cap E(F(3k+1,3k+2)) = E(F(4,3k-1)) \cap E(F(1,2)) \cap E(F(3k+1,3k+2)) = \emptyset$. Let us put $y(F(1,2)) = y(F(3k+1,3k+2)) = 1/2$, $y(F(4,3k+2)) = y(F(4,3k-1)) = 1/4$ and $y(F) = 0$ for the other whips. Then, denoting the symmetrical difference of two sets by Δ,

$$\sum_{e\in E(F)} y(F) = 1 \text{ if } e \in E(F(1,2) \cap E(F(3k+1,3k+2)),$$

$$\sum_{e\in E(F)} y(F) \leq 1 \text{ if } e \in E(F(1,2)\Delta E(F(3k+1,3k+2)).$$

Therefore y is a feasible solution of (3). Thus $b^*_R(P_{3k+2}) \leq \frac{3}{2}$, which completes the proof when $k \geq 2$. Figure 1 gives a concrete illustration of the proof for $k = 2$. □

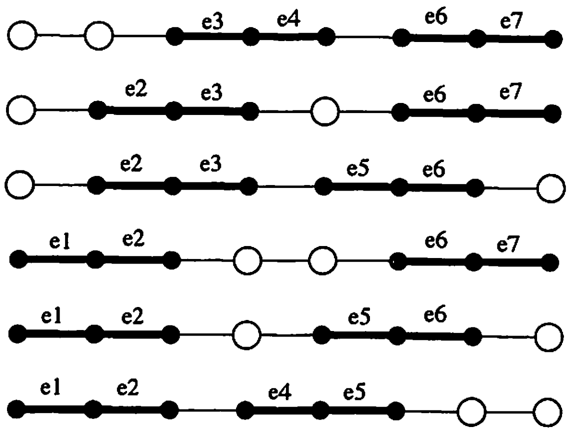

Figure 1: The whips of P_8

As an application of Theorems A, 5 and 13 we obtain the following result.

Corollary 14. For any graph T of order $n \geq 3$,

$$b_R^*(T) \leq \frac{5(n-1)}{n+5}.$$

Furthermore, the bound is sharp for P_5.

We conclude this section with an open problem.
Problem. Characterize the trees achieving the bound of corollary 14.

3 Roman discipline number

For any feasible solution of (4), $y_F = 0$ or 1 for each whip F, and the whips F such that $y_F = 1$ are edge-disjoint. Hence the optimal value of (4) is the maximal value of edge-disjoint whips. We first determine the discipline number of cycles and paths.

Proposition 15. For $n \geq 3$, $\text{dis}_R(C_n) = 1$ if $n \notin \{4, 5, 8\}$ and $\text{dis}_R(C_4) = \text{dis}_R(C_5) = \text{dis}_R(C_8) = 2$.

Proof. If $n \notin \{4, 5, 8\}$, this is a consequence of (5) and Theorem 12.
If $n = 4$ or 5, each whip consists of one star of order 3 and one or two isolated vertices, and has two edges. Each of C_4, C_5 admits two edge-disjoint whips. Then $\text{dis}_R(C_4) = \text{dis}_R(C_5) = 2$.

If $n = 8$, each whip consists of two stars of order 3 and two isolated vertices, and contains four edges. Hence there can be at most two edge-disjoint whips. As the two whips of edge-sets $\{e_1, e_2, e_5, e_6\}$ and $\{e_3, e_4, e_7, e_8\}$ are edge-disjoint, $\text{dis}_R(C_8) = 2$. □

Proposition 16. For $n \geq 3$, $\text{dis}_R(P_n) = 1$ if $n \neq 5$ and $\text{dis}_R(P_5) = 2$.

Proof. If $n \neq 5$, this is a consequence of (5) and Theorem 13. If $n = 5$, each whip consists of one star of order 3 and two isolated vertices, and has two edges. Hence there are at most two edge-disjoint whips. The whips of edge-sets $\{e_1, e_2\}$ and $\{e_3, e_4\}$ show that $\text{dis}_R(P_5) = 2$. □

Proposition 17. Let G be a connected graph of order $n \geq 3$. If $\gamma_R(G) = 2$ then $\text{dis}_R(G) = 1$, and if $\gamma_R(G) = 3$ then $\text{dis}_R(G) \leq 2$.

Proof. If $\gamma_R(G) = 2$, then $\Delta(G) = n - 1$ and each whip is a spanning star centered at a vertex of degree $n - 1$. Two such stars always share an edge. Hence $\text{dis}_R(G) = 1$.

If $\gamma_R(G) = 3$, then each whip, say F_1, consists in a star centered at a vertex u of degree $n - 2$ and an isolated vertex v. Every star of order $n - 2$ centered at a vertex different from u, v, if any, shares an edge with F_1. If v has degree $n - 2$, then v is the center of the star of a second whip, which is edge-disjoint from F_1, and $\text{dis}_R(G) = 2$. Otherwise, $\text{dis}_R(G) = 1$. □

The previous proposition and Corollary 8 show that $b_R^*(G)$ can be much larger than $\text{dis}_R(G)$. However, we show below that as soon as $\gamma_R(G) \geq 4$, $\text{dis}_R(G)$ can be arbitrary large and even equal to $b_R(G)$.

Consider the complete multipartite graph with r partite sets of size two, that is, $H = K_{2,2,\dots,2}$. Let $S_1 = \{u_1, v_1\}, S_2 = \{u_2, v_2\}, \dots, S_r = \{u_r, v_r\}$ be the partite sets of H. Let G be the graph obtained from H by adding the vertices u_{r+j}, v_{r+j} for $1 \leq j \leq s$ ($s \geq 4$) and adding the edges $u_i u_{r+j}, v_i v_{r+j}$ for $1 \leq i \leq r, 1 \leq j \leq s$.

Theorem 18. If G is the graph above, then $\text{dis}_R(G) = b_R(G) = r$.

Proof. The graph G has order $n = 2(r + s)$. The vertices u_i and v_i have degree $2(r-1)+s$ when $1 \leq i \leq r$ and r when $r+1 \leq i \leq r+s$. The number of edges of G is thus $m = \frac{2r(2r+s-2)+2sr}{2} = r(n-2)$. Since $\Delta(G) < n - 2$, $\gamma_R(G) > 3$, and since $\{u_1, v_1\}$ is a dominating set of G, $\gamma_R(G) = 4$. For $1 \leq i \leq r$, let F_i be the subgraphs induced by the edge sets

$$E(F_1) = \{u_1 u_j, v_1 v_j \mid 2 \leq j \leq r + s\} \text{ and for } 2 \leq i \leq r - 1,$$

$$E(F_i) = \{u_i v_j, v_i u_j \mid 1 \leq j < i\} \cup \{u_i u_j, v_i v_j \mid i < j \leq r + s\}.$$

Each F_i consists of two disjoint stars $K_{1,r+s-1}$ centered at u_i and v_i. Thus F_i is a whip for $1 \leq i \leq r$. Moreover the unique whip of this family containing e with $e = u_iu_j$ or $e = v_iv_j$ is $F_{\min\{i,j\}}$ and the unique whip of this family containing $e = u_iv_j$ is $F_{\max\{i,j\}}$. Hence the r whips F_i are edge-disjoint and $\mathrm{dis}_R(G) \geq r$.

On the other hand, the whip F_i contains the edge u_iu_{r+1} for $1 \leq i \leq r$. Therefore setting $x_{u_iu_{r+1}} = 1$ if $1 \leq i \leq r$ and $x_e = 0$ otherwise gives a feasible solution of (1). Hence $b_R(G) \leq r$ and the result follows by (5). This completes the proof. □

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