Unimodality and Dyck paths

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Abstract

We propose an original approach to the problem of rank-unimodality for Dyck lattices. It is based on a well known recursive construction of Dyck paths originally developed in the context of the ECO methodology, which provides a partition of Dyck lattices into saturated chains. Even if we are not able to prove that Dyck lattices are rank-unimodal, we describe a family of polynomials (which constitutes a polynomial analog of ballot numbers) and a succession rule which appear to be useful in addressing such a problem. At the end of the paper, we also propose and begin a systematic investigation of the problem of unimodality of succession rules.

1 Introduction

In enumerative combinatorics it often happens to discover integer sequences which are unimodal. A finite sequence $a_0, a_1, \ldots a_n$ is said to be *unimodal* when there exists an index $0 \leq i \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_{n-1} \geq a_n$. Proving that a sequence is unimodal is often a very hard task. A few papers illustrating some general techniques to tackle this problem has been published, such as [3, 18] (which provide a very rich account of several methods to prove unimodality). A related property is log-concavity. A finite sequence of integers $a_0, a_1, \ldots a_n$ is said to be *log-concave* whenever $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n$. It is not too difficult to prove that a nonnegative log-concave sequence having no internal zeroes is unimodal. Since proving log-concavity is usually easier than proving unimodality, the above result is often used.

In this paper we will consider unimodality in the context of a particularly interesting and important combinatorial structure, i.e. lattice paths. Another paper dealing with unimodality and lattice paths is [15], but it seems not to be related to what we are studying here. For our purposes, a

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lattice path is a path in the discrete plane starting at the origin of a fixed Cartesian coordinate system, ending somewhere on the x-axis, never going below the x-axis and using only a prescribed set of steps. This definition is extremely restrictive if compared to what is called a lattice path in the literature, but it will be enough for our purposes.

Some very well studied classes of lattice paths are the following:

- *Dyck paths*, i.e. lattice paths using only steps of the type $u = (1, 1)$ and $d = (1, -1)$;
- *Motzkin paths*, i.e. lattice paths using only steps of the type $u = (1, 1)$, $h = (1, 0)$ and $d = (1, -1)$;
- *Schröder paths*, i.e. lattice paths using only steps of the type $u = (1, 1)$, $H = (2, 0)$ and $d = (1, -1)$;

Given a lattice path P, the *area* of P is defined to be the area of the region included between the path P and the x-axis (see figure 1). So we can consider, for instance, the distribution of the parameter "area" over all Dyck paths of a given length. The following conjecture is not at all a new one.

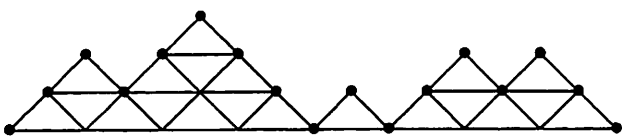

Figure 1: A Dyck path of semilength 8 and area 20. The triangles in the figure have unit area.

Conjecture. The sequence $(a_k^{(n)})_k$ of the number of Dyck paths of semilength n having area $n + 2k$ is unimodal, for all n.

The first few lines of the matrix of the $a_k^{(n)}$'s are the following:

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	3	3	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	4	6	7	7	5	5	3	2	1	1	0	0	0	0	0	0	0	0	0	0	0
1	5	10	14	17	16	16	14	11	9	7	5	3	2	1	1	0	0	0	0	0	0
1	6	15	25	35	40	43	44	40	37	32	28	22	18	13	11	7	5	3	2	1	1

Observe that, for $n \geq 3$, none of the sequences $(a_k^{(n)})_k$ is log-concave. This is trivially seen by observing that the last three nonzero terms of each sequence are 2,1,1. However, for specific values of n, there are also other terms that prevent $(a_k^{(n)})_k$ from being log-concave (as it is readily seen by inspecting the matrix displayed above). This fact makes even more intriguing (and surely more difficult) the above mentioned conjecture.

This conjecture has first appeared in [19], where it is however stated in a different language. Indeed, there is a bijection between Dyck paths of semilength n and Young diagrams fitting inside the *staircase shape* $(n-1, n-2, \ldots 2, 1)$ (see figure 2). Moreover, this bijection maps the area of a Dyck path into the difference between the total area of the staircase shape and the area of the Young diagram associated with the path. Thus the above conjecture is formulated by Stanton in the following form.

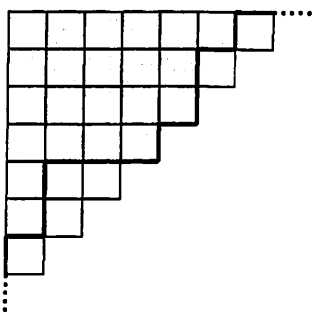

Figure 2: The Young diagram corresponding to the Dyck path in figure 1.

(Equivalent) Conjecture. The sequence $(a_k^{(n)})_k$ of the number of Young diagrams fitting inside the staircase shape $(n-1, n-2, \ldots 2, 1)$ having area k is unimodal, for all n.

The "path version" of the conjecture of Stanton is due to Bonin, Shapiro and Simion, who stated it in [2], together with an analogous conjecture for Schröder paths.

There is still another way to express this unimodality conjecture, which involves lattices.

Given a class of paths $\mathcal{P}$, the set $\mathcal{P}_n$ of all paths in $\mathcal{P}$ having length n can be naturally endowed with a poset structure, by imposing that $P \leq Q$ whenever P lies weakly below Q (weakly meaning that P and Q are allowed to have some points in common). See figure 3 for a "Dyck" example. It turns

out that, for many interesting classes of paths, the resulting poset is indeed a distributive lattice. This happens, for instance, for Dyck, Motzkin and Schröder paths [12]. In all cases, the rank of a path inside the distributive lattice it lives in is related to its area. In particular, according to [12, 10], the rank of a Dyck path of semilength n in his lattice is given by the area enclosed between the path itself and the path $(UD)^n$ divided by 2. This leads us to the following formulation of the above conjecture.

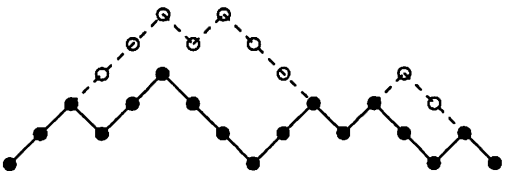

Figure 3: A pair of Dyck paths P (thick) and Q (dashed), with $P < Q$.

(A third form of the same) Conjecture. The distributive lattice $\mathcal{D}_n$ of Dyck paths of semilength n is rank-unimodal, for all n.

This formulation can be found, for instance, in [10], where the conjecture has been extended also to the Motzkin case (there is computational evidence supporting this extension). We wish to point out that there is a case in which rank-unimodality has been proved, namely that of *Grand Dyck paths* (which are like Dyck paths, except for the fact that they are allowed to go below the x-axis). Indeed, this is a special case of the unimodality of Gaussian coefficients (counting integer partitions fitting inside a rectangle with respect to their size), which has been proved in several sources using several different methods [13, 14, 17]. A similar problem concerning compositions inside a rectangle has been considered in [16].

In the present paper we have not been able to solve the above mentioned unimodality conjectures; instead we propose a possible approach which we haven't been able to find elsewhere, nevertheless we are strongly convinced that it could prove useful in tackling these problems.

We start by recalling a particular construction of Dyck paths, falling into the framework of the so-called *ECO methodology*. We then show how such a construction suggests a possible way of decomposing Dyck lattices into saturated chains. This decomposition gives in turn some hints on what to do to prove rank-unimodality.

2 An ECO construction of Dyck paths

Let P be a Dyck path of length $2n$, and suppose that the length of its last descent (i.e., of its last sequence of fall steps) is k (the length of the last descent being simply the number of fall steps belonging to such a descent). Then we construct $k+1$ Dyck paths of length $2n+2$ starting from P (they will be called the *sons* of P) simply by inserting a peak (that is a rise step followed by a fall step) in every point of its last descent. In figure 4 it is shown how this construction works.

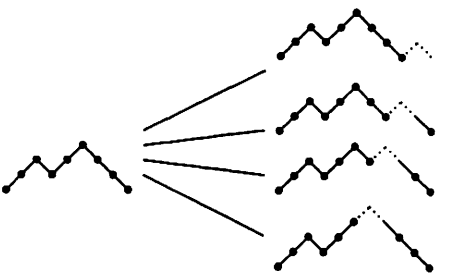

Figure 4: The ECO construction of Dyck paths.

If one performs such a construction on all Dyck paths of length $2n$, then it is not difficult to show that every Dyck path of length $2n+2$ is obtained exactly once.

The above described construction of Dyck paths is well known, and falls into the framework of the so-called *ECO method* (a detailed description of which can be found, for instance, in [1]).

3 A decomposition of Dyck lattices into saturated chains

The above described construction provides a partition of the class $\mathcal{D}_n$ of Dyck paths of semilength n, for all n. From an order-theoretic point of view, it is a partition of $\mathcal{D}_n$ into saturated chains. We would like to employ this decomposition to tackle the problem of unimodality in Dyck lattices. Such a decomposition of the Dyck lattice $\mathcal{D}_n$ will be called the *ECO decomposition* of $\mathcal{D}_n$.

Before starting, we observe that the ECO decomposition of a Dyck lattice does not possess any of the nice properties usually needed in proving unimodality: it is not a symmetric chain decomposition, and it is not even

nested (for the notion of a nested chain decomposition, see for example [7, 8]).

For any fixed n, denote with P_n the $\left(1+\binom{n-1}{2}\right)\times(n-1)$ matrix whose entry (j,k) is the number of saturated chains of cardinality $k+2$ starting at rank j in the ECO decomposition of $\mathcal{D}_n$. So, for small values of n, we have the following matrices:

$$P_2=(1),\quad P_3=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad P_4=\begin{pmatrix}1&0&0\\1&1&0\\0&1&0\\0&0&1\end{pmatrix},\quad P_5=\begin{pmatrix}1&0&0&0\\2&1&0&0\\1&2&0&0\\1&1&1&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

In the sequel we will denote with $P_n^{(k)}$ the columns of P_n, for $k=0,\ldots,n-2$.

In a completely analogous way, we also define the $\left(1+\binom{n}{2}\right)\times(n-1)$ matrices A_n whose entry (j,k) is the number of saturated chains of cardinality $k+2$ ending at rank j in the ECO decomposition of $\mathcal{D}_n$. Also in this case, $A_n^{(k)}$ will denote the columns of A_n, for $k=0,\ldots,n-2$.

Now let $P_n^{(k)}(x)$ and $A_n^{(k)}(x)$ be the polynomials associated with the vectors $P_n^{(k)}$ and $A_n^{(k)}$, respectively; moreover, set

$$P_n(x)=\sum_{k=0}^{n-2}P_n^{(k)}(x),\qquad A_n(x)=\sum_{k=0}^{n-2}A_n^{(k)}(x).$$

So the coefficient of x^i in $P_n(x)$ is the total number of saturated chains starting at rank i, whereas the coefficient of x^i in $A_n(x)$ is the total number of saturated chains ending at rank i.

Now, in order to link the polynomials $P_n(x)$ and $A_n(x)$ to our unimodality conjecture, we need to introduce a few more notations.

Let $r_n^{(k)}$ denote the number of elements having rank k in $\mathcal{D}_n$. The polynomial $r_n(x)=\sum_k r_n^{(k)}x^k$ is called the *rank polynomial* of $\mathcal{D}_n$. We now introduce the polynomials $s_n(x)$ as follows:

$$s_n(x)=\sum_k s_n^{(k)}x^k=\sum_k(r_n^{(k-1)}-r_n^{(k)})x^k,$$

where, by definition, $r_n^{(-1)}=0$, for all n. Obviously, the unimodality conjecture is equivalent to the following:

(Again the same) Conjecture. For every n, there exists $\bar{k}$ such that $s_n^{(k)} \leq 0$ for all $k \leq \bar{k}$ and $s_n^{(k)} \geq 0$ for all $k > \bar{k}$.

In view of this fact, a possible approach to our problem (which actually holds in general) consists then of the investigation of the sign of the coefficients of $s_n(x)$. In this respect, the following result seems to be of some interest.

Proposition 3.1 *For all* $n \in \mathbf{N}$, $s_n(x) = xA_n(x) - P_n(x)$.

Proof. The set of elements having ranks $k-1$ and k in $\mathcal{D}_n$ can be partitioned into three sets (some of which could be empty): the set of those elements belonging to some saturated chain crossing both ranks, the set of the top elements of all chains ending at rank $k-1$ and the set of the bottom elements starting at rank k. The coefficient of x^k in $s_n(x)$ is clearly given by the difference between the cardinalities of the second and the third sets described above, whence the proposition immediately follows. ■

There is also a link between the polynomials $P_n^{(k)}(x)$ and $A_n^{(k)}(x)$ which is easy to show and is recorded in the next proposition.

Proposition 3.2 *For all* $n, k \in \mathbf{N}$, $A_n^{(k)}(x) = x^{k+1}P_n^{(k)}(x)$.

Proof. The coefficient of x^i in $x^{k+1}P_n^{(k)}(x)$ is the number of saturated chains of cardinality $k+2$ starting at rank $i-k-1$. This is clearly the same as the number of saturated chains of cardinality $k+2$ ending at rank i, which is the coefficient of x^i in $A_n^{(k)}(x)$. ■

As a consequence of the above propositions, we have the following expression for $s_n(x)$.

Corollary 3.1 *For all* $n \in \mathbf{N}$,

$$s_n(x) = \sum_{k=0}^{n-2} (x^{k+2} - 1)P_n^{(k)}(x).$$

Everything has been said until this point is valid for *any* partition into saturated chains of $\mathcal{D}_n$ (and in fact of any ranked poset). So corollary 3.1, together with a deep knowledge of the polynomials $P_n^{(k)}(x)$, could be helpful in dealing with unimodality.

From now on, we will suppose to work with the ECO decomposition of $\mathcal{D}_n$ described in the previous section.

We can prove an interesting recurrence for the polynomials $P_n^{(k)}(x)$, which allows us to interpret the $P_n^{(k)}(x)$'s as a polynomial analog of ballot numbers.

Proposition 3.3 *For all* $n, k \in \mathbf{N}$,

$$P_n^{(k)}(x) = x^k \cdot (P_{n-1}^{(k-1)}(x) + \cdots + P_{n-1}^{(n-3)}(x)).$$

Proof. Recall that, for any i, the coefficient of x^i in $P_n^{(k)}(x)$ is the number of saturated chains having cardinality $k+2$ and starting at rank i in $\mathcal{D}_n$. Each of these saturated chains can be uniquely represented by its minimum, which is a Dyck path P of semilength n ending with the sequence of steps $UD^{k+1}UD$ (that is, a peak at level 0 preceded by a sequence of $k+1$ consecutive down steps, which is in turn preceded by an up step). Moreover, as it is shown in [12], the area $\alpha(P)$ of P is given by $\alpha(P) = 2i + n$. If we remove the last peak, we obtain a bijection with the set of Dyck paths of semilength $n-1$ ending with a sequence of precisely $k+1$ D steps and having area $2i + n - 1$. Each of these paths belongs to a different saturated chains (since they all have the same rank). For any such path Q, the minimum of the saturated chain Q belongs to can be obtained by simply replacing the last $k+2$ steps of Q (i.e. the sequence of steps UD^{k+1}) with the sequence of steps D^kUD. Observe that, performing such an operation, we are left with a path R in $\mathcal{D}_{n-1}$ of area $\alpha(R) = \alpha(Q) - 2k = 2i + n - 1 - 2k$. This implies that the rank of R in $\mathcal{D}_{n-1}$ is given by

$$r(R) = \frac{\alpha(R) - (n+1)}{2} = \frac{2i + n - 1 - n + 1 - 2k}{2} = i - k.$$

Thus we can conclude that the total number of saturated chains having cardinality $k+2$ and starting at rank i in $\mathcal{D}_n$ equals the number of saturated chains having cardinality at least $k+1$ (since its minimum has at least k D steps before the final peak) and starting at rank $i-k$ in $\mathcal{D}_{n-1}$, whence the thesis follows. ∎

As an immediate corollary, we also have the following recursion.

Corollary 3.2 *For all* $n, k \in \mathbf{N}$, *with* $k \neq 0$,

$$P_n^{(k)}(x) = x \cdot (P_n^{(k-1)}(x) - x^{k-1} P_{n-1}^{(k-2)}(x)).$$

Proof. A direct application of the above proposition yields:

$$\begin{aligned} P_n^{(k)}(x) - x P_n^{(k-1)}(x) &= x^k \cdot \sum_{i=k-1}^{n-3} P_{n-1}^{(i)}(x) - x^k \cdot \sum_{i=k-2}^{n-3} P_{n-1}^{(i)}(x) \\ &= -x^k P_{n-1}^{(k-1)}(x). \end{aligned}$$

∎

The polynomials $P_n^{(k)}(x)$'s are not new. They have been first studied by Carlitz and others [4, 5], and subsequently considered also by Krattenthaler [9]. They also found recursions which look similar to the ones shown in the two above propositions, however our combinatorial setting is slightly different. In fact, in the above cited works the combinatorial meaning of the $P_n^{(k)}(x)$'s is somehow related to the distribution of Dyck paths of semilength n with respect to the area. It is however easy to relate the two approaches. Indeed, referring to the proof of proposition 3.3, there is a bijection mapping a saturated chain of the ECO decomposition of $\mathcal{D}_n$ starting at rank j into a path of $\mathcal{D}_{n-1}$ at rank j. This enables us to interpret the polynomials $P_n^{(k)}(x)$'s as describing the distribution of Dyck paths in $\mathcal{D}_{n-1}$ with respect to rank (i.e. area) and length of the final descent (i.e. sequence of down steps).

We propose here something new concerning these polynomials, namely we describe the recursion given by proposition 3.3 using a succession rule with two labels, in the spirit of the ECO methodology.

In the vector space of polynomials in two indeterminates over the reals, to be denoted $\mathbf{R}[x,t]$, define the linear operator L as follows on the canonical basis $(x^\alpha t^\beta)_{\alpha,\beta\in\mathbf{N}}$:

$$\begin{aligned} L : \mathbf{R}[x,t] &\longrightarrow \mathbf{R}[x,t] \\ : x^\alpha t^\beta &\longmapsto x^\alpha \cdot \sum_{i=0}^{\beta+1} x^i t^i. \end{aligned}$$

We start by noticing an algebraic property of L that will be useful in what follows.

Lemma 3.1 *L is a homomorphism of the $\mathbf{R}[x]$-module of polynomials $\mathbf{R}[x][t]$.*

Proof. This is immediate, since

$$L(x^\alpha t^\beta) = x^\alpha \cdot \sum_{i=0}^{\beta+1} x^i t^i = x^\alpha L(t^\beta).$$

■

Now define $P_n(x,t) = L^{n-2}(1)$ (here L^h denotes the composition of L with itself h times). The following proposition justifies the introduction of the operator L in our context.

Proposition 3.4 *For all $n \geq 2$,*

$$P_n(x,t) = \sum_{k=0}^{n-2} P_n^{(k)}(x)t^k. \tag{1}$$

Proof. From the definition of $P_n(x,t)$ it obviously follows that $L(P_n(x,t)) = P_{n+1}(x,t)$. Also, observe that the degree of $P_n(x,t)$ with respect to t is $n-2$. Set $P_n(x,t) = \sum_{k=0}^{n-2} Q_n^{(k)}(x)t^k$, also thanks to the above lemma, we then obtain:

$$\begin{aligned}
\sum_{k=0}^{n-1} Q_{n+1}^{(k)}(x)t^k &= L\left(\sum_{k=0}^{n-2} Q_n^{(k)}(x)t^k\right) \\
&= \sum_{k=0}^{n-2} Q_n^{(k)}(x)L(t^k) \\
&= \sum_{k=0}^{n-2} Q_n^{(k)}(x)\left(\sum_{i=0}^{k+1} x^i t^i\right) \\
&= \sum_{i=0}^{n-2} Q_n^{(i)}(x) + \sum_{k=1}^{n-1}\left(\sum_{i=k-1}^{n-2} Q_n^{(i)}(x)\right) x^k t^k.
\end{aligned}$$

Thus the polynomial sequence $(Q_n^{(k)}(x))_n$ satisfies the recursion of proposition 3.3, and of course $Q_2^{(0)}(x) = P_2^{(0)}(x) = 1$, whence $Q_n^{(k)}(x) = P_n^{(k)}(x)$ which gives the thesis. ∎

So the operator L encodes the ballot-like recursive generation of $P_n^{(k)}(x)$. In the language of the ECO method, L is a *rule operator* [11]. The succession rule described by L (which turns out to be a two-labelled one) is easily seen to be the following:

$$\Omega : \begin{cases} (0_0) \\ (\alpha_\beta) \rightsquigarrow (\alpha_0)((\alpha+1)_1)\cdots((\alpha+\beta)_\beta)((\alpha+\beta+1)_{\beta+1}) \end{cases}. \tag{2}$$

The first levels of the generating tree of this rule are depicted in figure 5.

The infinite matrix describing the distribution of the labels at the various levels of the generating tree is called the *ECO matrix* of the succession rule in [6]. In our case, for the rule in (2), the first lines of such a matrix are the following:

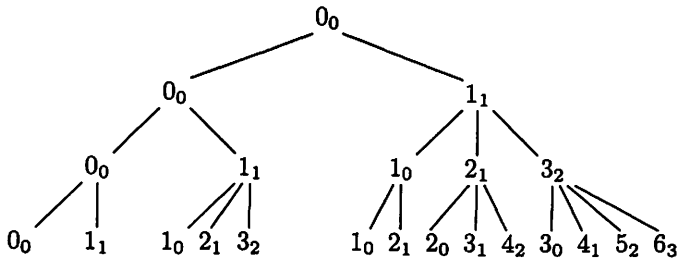

Figure 5: The generating tree associated with Ω.

labels→ at level ↓	0_0	1_0	1_1	2_0	2_1	3_0	3_1	3_2	4_0	4_1	4_2	5_0	5_1	5_2	6_0	...
0	1															
1	1	0	1													
2	1	1	1	0	1	0	0	1								
3	1	2	1	1	2	1	1	1	0	1	1	0	0	1	0	...
4	1	3	1	3	3	3	3	1	2	3	2	1	2	2	1	...
5	1	4	1	6	4	7	6	1	7	7	3	5	7	4	5	...
6	1	5	1	10	5	14	10	1	17	14	4	16	17	7	16	...
7	1	6	1	15	6	25	15	1	35	25	5	40	35	11	43	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

We can also find a recursion for the columns of the above ECO matrix. Since it directly depends on the succession rule (2), the proof is left to the reader.

Proposition 3.5 *Denote with $C_{k_i}(x)$ the generating function of the column associated with the label k_i. Then the following recursion holds:*

$$C_{k_i}(x) = x \cdot \sum_{j \geq i-1} C_{(k-i)_j}(x).$$

4 Conclusions and further work

We have addressed the problem of the rank-unimodality of Dyck lattices and, as announced at the beginning, we have not been able to solve it. Nevertheless, we hope to have provided some interesting insight to the problem, as well as an original way to tackle it. We hope that someone more skillful than us will be able to further develop these ideas to eventually find the desired unimodality proof (maybe by finding more structural and enumerative properties of the ECO decomposition of Dyck lattices).

We remark that a similar approach can be considered, in which we make use of a different decomposition of Dyck lattices. More precisely, for any Dyck lattice $\mathcal{D}_n$, one can consider the sublattices $D_{n,k}$ consisting of all paths starting with exactly k up steps (for $1 \leq k \leq n$). This is not of course a decomposition into chains. What is interesting about this decomposition is that, for any k, $\mathcal{D}_{n,k}$ has a natural embedding into $\mathcal{D}_{n,k-1}$ (see [12]), a fact that could be useful in proving unimodality.

We also notice that, in order to prove unimodality for an integer sequence, an extremely useful information is the position of the maximum. In our case, we even do not know where the maximums are located, and how they depend on n. This is a related open problem that deserves to be solved.

We close this paper by observing that the approach we have presented here to study unimodality of Dyck lattices is suggested by the ECO construction of Dyck paths, and that, in the end, to prove rank-unimodality of Dyck lattices it would be enough to prove that, at each level of the associated generating tree, the distribution of the labels k of the succession rule (2) is unimodal (where $k = \sum_i k_i$). We are thus led to formulate a *unimodality problem* for succession rules in general: given a succession rule, is it possible to find some (necessary and/or sufficient) conditions on the rule itself for the unimodality of the integer sequences describing the distribution of the labels at the various levels of the associated generating tree? This is a problem of independent interest, which seems not to have been previously considered.

To give just the flavor of this kind of investigations, we prove a simple result concerning finite succession rules. We will say that a succession rule is *unimodal* when the integer sequences appearing in the rows of the associated ECO matrix are all unimodal. Moreover, a succession rule deprived of its axiom will be called a *family of succession rules* (since each choice of the axiom determines a distinct succession rule).

Theorem 4.1 *Fix $n \in \mathbf{N}$ and let*

$$\Omega : (k) \rightsquigarrow (1)^{a_1^{(k)}} (2)^{a_2^{(k)}} \cdots (n)^{a_n^{(k)}}$$

be a family of succession rules. For any $b \in \mathbf{N}$, denote with $\Omega^{(b)}$ the succession rule of the family Ω having axiom (b). If $\Omega^{(b)}$ is unimodal for all $b \leq n$, then the sequences $(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ are unimodal for all $k \leq n$. Vice versa, if, for all $k \leq n$, the sequences $(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ are unimodal and all have the maximum in the same position, then $\Omega^{(b)}$ is unimodal for all $b \leq n$.

Proof. Suppose that all the rules of the family Ω are unimodal and, for any given $k \leq n$, consider the succession rule $\Omega^{(k)}$: the sequence $(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ appears in the first row of its ECO matrix, so such a sequence is unimodal (since $\Omega^{(k)}$ is unimodal by hypothesis). On the other hand, suppose that, for all $k \leq n$, the sequences $(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ are unimodal and also that all have maximum at index i. Fixed $b \leq n$, let $\beta_1, \beta_2, \ldots, \beta_n$ be the l-th row of the ECO matrix of Ω_b. The recursion determined by Ω_b allows to express any row of the ECO matrix in terms of the previous one. Therefore, if $\alpha_1, \alpha_2, \ldots, \alpha_n$ is the $l-1$-th row, we have:

$$\begin{aligned} \beta_1 &= \alpha_1 a_1^{(1)} + \cdots + \alpha_n a_1^{(n)} \\ &\vdots \\ \beta_n &= \alpha_1 a_n^{(1)} + \cdots + \alpha_n a_n^{(n)} \end{aligned}$$

The unimodality hypotheses on the sequences $(a_1^{(k)}, a_2^{(k)}, \ldots, a_n^{(k)})$ immediately implies that $\beta_1 \leq \cdots \leq \beta_{i-1} \leq \beta_i \geq \beta_{i+1} \geq \cdots \geq \beta_n$. ∎

Remark. We wish to point out that, in the above theorem, the hypothesis concerning the position of the maximum is essential. Indeed, consider the following family of finite succession rules:

$$\Omega : \begin{cases} (1) \rightsquigarrow (3) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (3)(4)(4) \\ (4) \rightsquigarrow (1)(1)(2)(2) \end{cases}.$$

One immediately notices that, for each $k \leq 4$, the sequence of the productions of (k) is unimodal. However, it is not difficult to realize that none of the succession rules $\Omega^{(b)}$ is unimodal (for $b \leq 4$).

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