Binding Number, Minimum Degree for Connected $(g, f+1)$-factors in Graphs *

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Abstract

Let G be a graph of order n. The *binding number* of G is defined as $\text{bind}(G) := \min\left\{\frac{|N_G(X)|}{|X|} \mid \emptyset \neq X \subseteq V(G) \text{ and } N_G(X) \neq V(G)\right\}$. A (g, f)-factor is called a connected (g, f)-factor if it is connected. A (g, f)-factor F is called a Hamilton (g, f)-factor if F contains a Hamilton cycle. In this paper, several sufficient conditions related to binding number and minimum degree for graphs to have connected $(g, f+1)$-factors or Hamilton (g, f)-factors are given.

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1. Introduction

All graphs under considering are finite, undirected and simple. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges, respectively. For $x \in V(G)$, we denote by $d_G(x)$ the degree of x in G and by $N_G(x)$ the set

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of vertices adjacent to x in G. The *minimum degree* of $V(G)$ is denoted by $\delta(G)$. For any subset $S \subseteq V(G)$, we denote by $N_G(S)$ the union of $N_G(x)$ for every $x \in S$, and by $G[S]$ the subgraph of G induced by S, by $G-S$ the subgraph obtained from G by deleting the vertices in S together with the edges incident to the vertices in S. If $S, T \subseteq V(G)$, then we write $e_G(S,T)$ for the number of edges in G joining a vertex in S to a vertex in T. The *binding number* of G is defined as

$$\text{bind}(G) := \min\left\{\frac{|N_G(X)|}{|X|} \mid \emptyset \neq X \subseteq V(G) \text{ and } N_G(X) \neq V(G)\right\}.$$

Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions from $V(G)$ to Z^+ such that $g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. Then a spanning subgraph F of G is called a (g,f)-factor if $g(x) \leq d_F(x) \leq f(x)$ holds for each $x \in V(G)$. If $g(x) = f(x)$ for every $x \in V(G)$, we say the (g,f)-factor to be an f-factor. For two constants a and b, if $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g,f)-factor is called an $[a,b]$-factor. An $[a,b]$-factor is called a k-factor if $a = b = k$, which is a regular factor. A (g,f)-factor is called a connected (g,f)-factor if it is connected. A (g,f)-factor F is called a Hamilton (g,f)-factor if F contains a Hamilton cycle. Obviously a Hamilton (g,f)-factor is a connected (g,f)-factor. The other terminologies and notations can be found in [1].

2. Binding number and connected $(g, f+1)$-factors

Many authors have investigated (g,f)-factors [6, 7, 8, 9] and factor critical graphs [14, 15]. The concept of the connected factors of a graph was first proposed by M. Kano. The problem is closely related to hamiltonian problem since a connected 2-factor is just a Hamiltonian cycle. By now there are no non-trivial necessary and sufficient conditions for a connected graph to contain a connected (g,f)-factor in general. Cai and Liu [2] gave a degree and stability number condition for the existence of connected factors in graphs. Cai, Liu and Hou [3] studied the stability number condition for connected $[k,k+1]$-factor in graphs. Zhou [16] gave a minimum degree condition for a graph to have a connected (g,f)-factor. In [17], Zhou gave two neighborhood conditions for a Hamilton graph G to have a Hamilton factor.

Tokushige [11] gave the following sufficient conditions in terms of binding number and minimum degree for the existence of k-factors.

Theorem A. *Let G be a graph of order n, $k \geq 2$ be an integer, $n > 4k + 1 - 4\sqrt{k+2}$ and $kn \equiv 0 (mod\, 2)$. Suppose that G satisfies conditions (A.1) and (A.2). Then G has a k-factor.*
$(A.1)$ $bind(G) \geq 2 - \frac{1}{k}$.
$(A.2)$ $\delta(G) \neq \lfloor \frac{(k-1)n+2k-3}{2k-1} \rfloor$.

In [4], Chen showed the following result on binding number and minimum degree for the existence of $[a, b]$-factors.

Theorem B. *Let G be a graph of order n, a and b be integers such that $1 \leq a < b$, and $n \geq \frac{(a+b-1)(a+b-2)}{b}$. Suppose that G satisfies conditions (B.1) and (B.2). Then G has an $[a, b]$-factor.*
$(B.1)$ $bind(G) \geq 1 + \frac{a-1}{b}$.
$(B.2)$ $\delta(G) \neq \lfloor \frac{(a-1)n+a+b-2}{a+b-1} \rfloor$.

Furthermore, it is pointed out that the result is best possible in some sense in [4].

Li [5] gave a sufficient condition for a graph to have a connected $(g, f + 1)$-factor.

Theorem C. *Let G be a graph, and $g(x)$ and $f(x)$ be two positive integral functions from $V(G)$ to Z^+ such that $2 \leq g(x) \leq f(x) \leq d_G(x)$ for all $x \in V(G)$. If G has both a (g, f)-factor and a hamilton path, then G contains a connected $(g, f + 1)$-factor.*

We now prove the following result, which is a binding number and minimum degree condition for a graph to have a connected $(g, f + 1)$-factor.

Theorem 2.1. *Let G be a connected graph of order n, and let a, b and n be nonnegative integers such that $2 \leq a \leq b$ and $n \geq \frac{(a+b)(a+b-1)}{a}$. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \leq g(x) \leq f(x) \leq b$ for any $x \in V(G)$ and $f(V(G))$ is even. If $bind(G) \geq \frac{a+b-1}{a}$ and $\delta(G) \geq 1 + \frac{bn}{a+b-1}$, then G has a connected $(g, f + 1)$-factor.*

In Theorem 2.1, if $g(x) \equiv a$ and $f(x) \equiv b$, then we obtain the following corollary.

Corollary 2.1. *Let G be a connected graph of order n, and let a, b and n be nonnegative integers such that $2 \leq a \leq b$, $n \geq \frac{(a+b)(a+b-1)}{a}$ and bn is* *even. If bind(G)* $\geq \frac{a+b-1}{a}$ *and* $\delta(G) \geq 1 + \frac{bn}{a+b-1}$, *then G has a connected* $[a, b+1]$*-factor.*

In Theorem 2.1, if $g(x) \equiv f(x) \equiv k$, then we obtain the following corollary.

Corollary 2.2. *Let G be a connected graph of order n, and let k and n be nonnegative integers such that* $k \geq 2$, $n \geq 4k-2$ *and kn is even. If bind(G)* $\geq 2 - \frac{1}{k}$ *and* $\delta(G) \geq 1 + \frac{kn}{2k-1}$, *then G has a connected* $[k, k+1]$*-factor.*

Now we prove Theorem 2.1. The following three lemmas are very useful to our proof. Lemma 2.1 and Lemma 2.2 are fundamental results in factor theory due to Lovász [10].

Lemma 2.1 *Let G be a graph, and* $g(x)$ *and* $f(x)$ *be two integer-valued functions defined on* $V(G)$ *such that* $g(x) \leq f(x)$ *for all* $x \in V(G)$. *Then G has a* (g, f)*-factor if and only if for all disjoint subsets S and T of* $V(G)$

$$\delta_G(g, f, S, T) = f(S) - g(T) + d_{G-S}(T) - h_G(g, f, S, T) \geq 0,$$

where $h_G(g, f, S, T)$ *denotes the number of components C of* $G-(S \cup T)$ *such that* $g(x) = f(x)$ *for all* $x \in V(C)$ *and* $f(V(C) + e_G(V(C), T) \equiv 1 (mod\, 2)$.

We call $h_G(g, f, S, T)$ the number of odd components of $G - (S \cup T)$ when $g(x) = f(x)$ for all $x \in V(G)$ and write $h_G(g, f, S, T) = h_G(f, S, T)$.

Lemma 2.2 *(1) Let G be a graph and f be a non-negative integer-valued function defined on* $V(G)$. *Then G has an f-factor if and only if for all disjoint subsets S and T of* $V(G)$

$$\delta_G(f, S, T) = f(S) - f(T) + d_{G-S}(T) - h_G(f, S, T) \geq 0,$$

where $h_G(f, S, T)$ *denotes the number of odd components of* $G - (S \cup T)$.

(2) $\delta_G(f, S, T) \equiv f(V(G))$ $(mod\ 2)$.

Lemma 2.3 [12]. *Let G be a simple graph of order n. If bind(G)* $\geq t$, *then*

$$|N_G(X)| \geq \frac{(t-1)n + |X|}{t}$$

for all $\emptyset \neq X \subseteq V(G)$, *and in particular,*

$$\delta(G) \geq n - \frac{n-1}{t}.$$

The proof of Theorem 2.1. Suppose that G satisfies the hypothesis of Theorem 2.1. Since $\delta(G) \geq 1 + \frac{bn}{a+b-1} \geq \frac{n}{2}$, graph G has a Hamilton cycle. Thus according to Theorem C, to prove Theorem 2.1, it suffices to prove that graph G also has a (g, f)-factor.

By contradiction, we assume that graph G has no (g, f)-factors. Then by Lemma 2.1, there exist disjoint subsets S and T of $V(G)$ such that

$$\delta_G(g, f, S, T) = f(S) + d_{G-S}(T) - g(T) - h_G(g, f, S, T) < 0. \tag{1}$$

Let $s = |S|$ and $t = |T|$. Then by (1) we get

$$as + d_{G-S}(T) - bt - \omega \leq -1, \tag{2}$$

where ω denotes the number of components of $G - (S \cup T)$. It is clear that

$$\omega \leq n - s - t. \tag{3}$$

Let m denote the minimum order of components of $G - (S \cup T)$. Then we get that

$$m \leq \frac{n - s - t}{\omega}. \tag{4}$$

In view of the definition of m, we obtain

$$\delta(G) \leq m - 1 + s + t. \tag{5}$$

We first prove that the following claim holds.

Claim 1. $T \neq \emptyset$.

Proof. We prove Claim 1 by contradiction. Suppose that $T = \emptyset$. We shall consider two cases and derive a contradiction in each case.

Case 1. $S = \phi$.

As G is connected and $\delta_G(g, f, S, T) < 0$, then $h_G(g, f, S, T) = 1$. Considering the definition of $h_G(g, f, S, T)$, we can easily get that $f(x) = g(x)$ for every $x \in V(G)$. Therefore $\delta_G(g, f, S, T) = -1$. On the other hand, since $f(V(G))$ is even, according to Lemma 2.2, we have that $\delta_G(g, f, S, T)$ is even. This is a contradiction.

Case 2. $S \neq \emptyset$.

By (2) and (3), we obtain

$$as + 1 \leq \omega \leq n - s. \tag{6}$$

Hence according to (4), (5) and (6), we know that

$$
\begin{aligned}
1+\frac{bn}{a+b-1} &\leq \delta(G) \leq m-1+s \leq \frac{n-s}{\omega}-1+s \\
&\leq \frac{n-s}{as+1}-1+s \\
&\leq \frac{n-1}{a}-\frac{(n-1-as-s)(s-1)}{as+1}.
\end{aligned}
$$

Combining this inequality with (6), we get

$$\frac{n}{2} \leq 1+\frac{bn}{a+b-1} \leq \frac{n-1}{a}.$$

This is a contradiction since $a \geq 2$. Thus $T \neq \emptyset$. □

As $T \neq \emptyset$ by Claim 1, we define $h = \min\{d_{G-S}(x) \mid x \in T\}$. Thus

$$\delta(G) \leq h+s. \tag{7}$$

We shall consider various cases according to the value of h and derive contradictions.

Case 1. $h = 0$.

We define $I = \{x \in T \mid d_{G-S}(x) = 0\}$. Then I is an independent vertex subset of G and $I \neq \emptyset$. By Lemma 2.3, we have

$$|N_G(I)| \geq \frac{(b-1)n+a|I|}{a+b-1}. \tag{8}$$

On the other hand, by (2) and (3), we get

$$as-b|I|+(1-b)(t-|I|)-(b-1)(n-s-t) \leq -1.$$

It follows that

$$s \leq \frac{(b-1)n+|I|-1}{a+b-1}.$$

Combining this inequality with (8), we get

$$\frac{(b-1)n+a|I|}{a+b-1} \leq N_G(I)| \leq |S| < \frac{(b-1)n+|I|-1}{a+b-1}.$$

This inequality implies $0<(a-1)|I|<-1$. This is a contradiction.

Case 2. $1 \leq h \leq b-1$.

By (2), (3) and $b-h \geq 1$, we have

$$as + (h-b)t - (b-h)(n-s-t) \leq -1,$$

implying

$$s \leq \frac{(b-h)n-1}{a+b-h}.$$

Then using (7) and the above inequality, we obtain

$$1 + \frac{bn}{a+b-1} \leq \delta(G) \leq s+h \leq \frac{(b-h)n-1}{a+b-h} + h.$$

Let $f(h) = \frac{(b-h)n-1}{a+b-h} + h$. Then in view of $n \geq \frac{(a+b)(a+b-1)}{a}$, $f(h)$ attains its maximum value at $h=1$ since

$$\begin{aligned} f'(h) &= \frac{-n(a+b-h)+(b-h)n-1}{(a+b-h)^2} + 1 \\ &= \frac{-na-1}{(a+b-h)^2} + 1 < 0. \end{aligned}$$

Thus

$$1 + \frac{bn}{a+b-1} \leq \frac{(b-1)n-1}{a+b-1} + 1.$$

This inequality implies $n \leq -1$. That is a contradiction.

Case 3. $h > b$.

Subcase 3.1. $m \geq 2$.

By (2), it follows that

$$as + (h-b)t - \omega \leq -1.$$

Thus

$$\omega \geq as + t + 1. \tag{9}$$

Since $T \neq \emptyset$ by Claim 1, we have $\omega \geq 2$.

According to (5) and (9), we get

$$\begin{aligned} 1 + \frac{bn}{a+b-1} \leq \delta(G) &\leq m-1+s+t \leq m+\omega-2 \\ &\leq m+\omega-2+\frac{(m-2)(\omega-2)}{2} \\ &= \frac{m\omega}{2} \leq \frac{n}{2}. \end{aligned}$$

This is a contradiction.

Subcase 3.2. $m = 1$.

By (3) and (9), we have

$$s + t \leq \frac{n-1}{2}.$$

Let C_1 be the least component of $G - (S \cup T)$. Then C_1 contains only one vertex x. Hence

$$1 + \frac{bn}{a+b-1} \leq \delta(G) \leq d(x) \leq s + t \leq \frac{n-1}{2},$$

which is a contradiction.

Case 4. $h = b$.

By (2) we have

$$\omega \geq as + 1.$$

Combining this with (4), we get that

$$1 \leq m \leq \frac{n-s-t}{\omega} \leq \frac{n-s-t}{as+1} \leq \frac{n-s}{as+1}. \tag{10}$$

By (7), we know that

$$1 + \frac{bn}{a+b-1} \leq \delta(G) \leq h + s = b + s.$$

On the other hand, by (10) we have

$$s \leq \frac{n-1}{a+1}.$$

Thus

$$1 + \frac{bn}{a+b-1} \leq b + \frac{n-1}{a+1}.$$

It follows that $n \leq \frac{(a+b-1)(ab-a+b-2)}{ab-a+1} < 2(a+b-1) \leq \frac{(a+b)(a+b-1)}{a}$. This contradicts the assumption that $n \geq \frac{(a+b)(a+b-1)}{a}$.

From all the argument above, we deduce contradictions. Hence we conclude that G has a (g, f)-factor. Since G has both a Hamilton path and a (g, f)-factor, according to Theorem C, graph G has a connected $(g, f+1)$-factor. Completing the proof of Theorem 2.1. □

3. Binding number and Hamilton (g, f)-factors

In this section, we prove the following theorem, which shows a binding number and minimum degree condition for the existence of a Hamilton (g, f)-factor.

Theorem 3.1. *Let G be a graph of order n, and let a, b and n be nonnegative integers such that $2 \le a < b$ and $n \ge \frac{(a+b-2)(a+b-3)}{a-1}$. Let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for any $x \in V(G)$. If $bind(G) \ge \frac{a+b-3}{a-1}$ and $\delta(G) \ge 1 + \frac{(b-2)n}{a+b-3}$, then G has a Hamilton (g, f)-factor.*

In Theorem 3.1, if $g(x) \equiv a$ and $f(x) \equiv b$, then we obtain the following corollary.

Corollary 3.1. *Let G be a graph of order n, and let a, b and n be nonnegative integers such that $2 \le a < b$ and $n \ge \frac{(a+b-2)(a+b-3)}{a-1}$. If $bind(G) \ge \frac{a+b-3}{a-1}$ and $\delta(G) \ge 1 + \frac{(b-2)n}{a+b-3}$, then G has a Hamilton $[a, b]$-factor.*

In Theorem 3.1, if $g(x) \equiv k$ and $f(x) \equiv k + 1$, then we obtain the following corollary.

Corollary 3.2. *Let G be a graph of order n, and let k and n be nonnegative integers such that $k \ge 2$ and $n \ge 4k - 2$. If $bind(G) \ge 2$ and $\delta(G) \ge 1 + \frac{n}{2}$, then G has a Hamilton $[k, k+1]$-factor.*

We use the following Lemma in our proof. Lemma 3.1 provides a necessary and sufficient condition for a graph to have a (g, f)-factor, which is a special case of Lovász's (g, f)-factor theorem.

Lemma 3.1 [10]. *Let G be a graph, and let $g(x)$ and $f(x)$ be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G has a (g, f)-factor if and only if*

$$\delta_G(g, f, S, T) = f(S) + d_{G-S}(T) - g(T) \ge 0$$

for all disjoint subsets S and T of $V(G)$.

Proof of Theorem 3.1. We shall use a different technique from Theorem 2.1 to prove Theorem 3.1. Suppose that G satisfies the assumption of Theorem 3.1. Since $\delta(G) \ge 1 + \frac{(b-2)n}{a+b-3} \ge \frac{n}{2}$, graph G has a Hamilton cycle C. Let $G' = G - E(C)$. In order to prove Theorem 3.1, we need only to show that G' has a $(g-2, f-2)$-factor. For convenience, let $f'(x) = f(x) - 2, g'(x) = g(x) - 2, c = a - 2$ and $d = b - 2$. Thus in the following we need only to prove that G' has a (g', f')-factor such that $c \le g'(x) < f'(x) \le d$ for each $x \in V(G)$, where $c \ge 0$ and $d > c$. We prove this by contradiction. Suppose that G' satisfies the assumption of Theorem 3.1, but has no (g', f')-factor. Then, by Lemma 3.1, there exist two disjoint subsets S and T of $V(G')$ such that

$$\delta_{G'}(S,T) = f'(S) + d_{G'-S}(T) - g'(T) \le -1. \tag{11}$$

We choose such subsets S and T which satisfy $|T|$ is minimum.

We first prove the following claims.

Claim 2. $d_{G'-S}(x) < g'(x) \le d - 1$ for all $x \in T$.

Proof. If $d_{G'-S}(x) \ge g'(x)$ for some $x \in T$, then the subsets S and $T \setminus \{x\}$ satisfy (11). This contradicts the choice of S and T. Therefore,

$$d_{G'-S}(x) < g'(x) \le d - 1$$

for all $x \in T$ holds. □

Claim 3. $T \neq \phi$.

Proof. If $T = \phi$, then $\delta_{G'}(S,T) = f'(S) \ge 0$. This contradicts (11). Therefore $T \neq \phi$. □

Claim 4. $d_{G-S}(x) \le d_{G'-S}(x) + 2 \le d$ for all $x \in T$.

Proof. Note that $G = G' \cup E(C)$. Thus by Claim 2 $d_{G-S}(x) \le d_{G'-S}(x) + 2 \le d$ for all $x \in T$ holds. □

Since $T \neq \phi$, define

$$h = \min\{d_{G-S}(x) \mid x \in T\}.$$

By Claim 4, we have

$$0 \le h \le d.$$

In the following we shall consider two cases and derive a contradiction in each case.

Case 1. $h = 0$.

We define $I = \{x \in T \mid d_{G-S}(x) = 0\}$. Then I is an independent vertex subset of G and $I \neq \emptyset$. By Lemma 2.3, we have

$$|N_G(I)| \ge \frac{(b-2)n + (a-1)|I|}{a+b-3}. \tag{12}$$

By (11) and Claim 4 we have

$$\begin{aligned}
-1 &\geq \delta_{G'}(S,T) = f'(S) + d_{G'-S}(T) - g'(T) \\
&\geq (c+1)|S| + d_{G-S}(T) - 2|T| - (d-1)|T| \\
&\geq (c+1)|S| + |T-I| - (d+1)|T| \\
&= (c+1)|S| - (d+1)|I| - d|T-I| \\
&\geq (c+1)|S| - (d+1)|I| - d(n-|S|-|I|) \\
&= (c+d+1)|S| - |I| - dn.
\end{aligned}$$

Thus

$$|S| < \frac{|I| + dn}{c+d+1}. \tag{13}$$

By (12) and (13) we have

$$\frac{(b-2)n + (a-1)|I|}{a+b-3} \leq |N_G(I)| \leq |S| < \frac{|I|+dn}{c+d+1} = \frac{|I| + (b-2)n}{a+b-3}.$$

This inequality implies $0 \leq (a-2)|I| < 0$, which is a contradiction.

Case 2. $1 \leq h \leq d$.

Since

$$\begin{aligned}
-1 &\geq \delta_{G'}(S,T) = f'(S) + d_{G'-S}(T) - g'(T) \\
&\geq (c+1)|S| + d_{G-S}(T) - 2|T| - (d-1)|T| \\
&\geq (c+1)|S| + (h-d-1)(n-|S|) = (c+d-h+2)|S| - (d+1-h)n,
\end{aligned}$$

we obtain

$$|S| < \frac{(d+1-h)n}{c+d-h+2}.$$

By considering a vertex $v \in T$ with $d_{G-S}(v) = h$, we get

$$\delta(G) \leq d_G(v) \leq h + |S| < h + \frac{(d+1-h)n}{c+d-h+2}.$$

Define

$$f(h) = h + \frac{(d+1-h)n}{c+d-h+2}, \qquad 1 \leq h \leq d.$$

Considering $n \geq \frac{(a+b-2)(a+b-3)}{a-1}$ and $h \geq 1$, $f(h)$ attains its maximum value at $h = 1$ since its derivative

$$\begin{aligned}
f'(h) &= 1 - \frac{n(c+1)}{(c+d-h+2)^2} \\
&\leq 1 - \frac{n(c+1)}{(c+d+1)^2} < 0.
\end{aligned}$$

Therefore $1 + \frac{(b-2)n}{a+b-3} \leq \delta(G) < f(h) = h + \frac{(d+1-h)n}{c+d-h+2} \leq f(1) = 1 + \frac{dn}{c+d+1} = 1 + \frac{(b-2)n}{a+b-3}$, which implies $0 < 0$. That is a contradiction.

From all the argument above, we deduce contradictions. So we conclude that G' has a (g', f')-factor. Therefore the union of C and (g', f')-factor is the desired Hamilton (g, f)-factor. This completes the proof of Theorem 3.1. □

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