Domination critical graphs upon edge subdivision

Nader Jafari Rad
Department of Mathematics,
Shahrood University of Technology,
Shahrood, Iran
E-mail: n.jafarirad@shahroodut.ac.ir

Abstract

A set of vertices S in a graph G is a dominating set, if any vertex of $G - S$ is adjacent to some vertex in S. The domination number, $\gamma(G)$, of G is the minimum cardinality of a dominating set of G. The subdivision of an edge uv is the operation of replacing uv with a path uwv throughout a new vertex w. A graph G is domination critical upon edge subdivision if the domination number increases by subdivision of any edge. In this paper we study domination critical graphs upon edge subdivision. We present several properties and bounds for these graphs and then give a constructive characterization of domination critical trees upon edge subdivision.

Keywords: Domination; Subdivision, Critical.

2000 Mathematical subject classification: 05C69.

1 Introduction

For notation and graph theory terminology, we in general follow [3]. Specifically, let G be a graph with vertex set $V(G) = V$ of order $|V| = n$ and size $|E(G)| = m$, and let v be a vertex in V. The *open neighborhood* of v is $N_G(v) = \{u \in V \mid uv \in E(G)\}$ and the *closed neighborhood of* v is $N_G[v] = \{v\} \cup N(v)$. The degree of v is $\deg_G(v) = |N_G(v)|$. If the graph G is clear from the context, we simply write $N(v)$ and $\deg(v)$ rather than $N_G(v)$ and $d_G(v)$, respectively. For a set $S \subseteq V$, its *open neighborhood* is the set $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* is the set $N[S] = N(S) \cup S$. A set of vertices S in G is a *dominating set*, (or just DS), if $N[S] = V(G)$. The *domination number*, $\gamma(G)$, of G is the minimum cardinality of a DS of G. If S is a subset of $V(G)$, then we denote by $G[S]$ the subgraph of G induced by S. A set of vertices S in G is an *independent dominating set*, if S is a DS and the induced subgraph $G[S]$ has no edge. The *independent domination number*, $i(G)$, of G is the minimum cardinality of an independent dominating set of G. A set $S \subseteq V(G)$ is a 2-packing of G if for every two different vertices $x, y \in S$, $N[x] \cap N[y] = \emptyset$. The 2-*packing number* $\rho(G)$ of a graph G is the maximum cardinality of a 2-packing of G.

Let S be a DS in a graph G and let $v \in S$. A vertex $w \in V(G)$ is an S*-private neighbor of* v if $N[w] \cap S = \{v\}$. Further, the S*-private neighborhood of* v, denoted $pn[v, S]$, is the set of all S-private neighbors of v.

A *cycle* on n vertices is denoted by C_n, while a *path* on n vertices is denoted by P_n. We denote by K_n the *complete graph* on n vertices. An r*-partite graph* G is a graph whose vertex set $V(G)$ can be partitioned into r sets of pair-wise non-adjacent vertices. For positive integers $p_1, p_2, \ldots, p_r$, the *complete* r*-partite graph* $K_{p_1,p_2,\ldots,p_r}$ is the r-partite graph with partition $V(G) = V_1 \cup V_2 \cup \ldots \cup V_r$ such that $|V_i| = p_i$ for $1 \leq i \leq r$ and such that every two vertices belonging to different partition sets are adjacent to each other. A *star* is a complete bipartite graph of the form $K_{1,n}$. A vertex of degree one is called a *leaf*, and its neighbor is called a *support vertex.*

The *subdivision* of an edge uv is the operation of replacing uv with a path uwv throughout a new vertex w. In this paper we denote by G^e the graph obtained from G by subdividing the edge $e \in E(G)$.

When a graphical parameter is of interest in an application, often times it is important to know how the parameter behaves when the graph is modified. For instance, the effects of removing or adding an edge and removing a vertex have been considered on parameters such as connectivity, chromatic number and domination number. Several authors studied graphs for which a domination parameter such as domination number, and total domination number changes under removal of a vertex, removal of an edge, contracting of an edge and adding an edge. Thus the are several varieties of critical and stable graphs upon removal of a vertex, removal of an edge, contracting of an edge and addition of an edge. For references of the critical concept on domination see for example [1, 2, 4, 5, 6].

In this paper we will study domination critical graphs upon subdivision of an edge. A graph G is domination critical upon edge subdivision if the domination number increases by subdivision of any edge. We present several properties and bounds for these graphs and then give a constructive characterization of domination critical trees upon edge subdivision.

The 2-*corona* of a graph H, denoted by $H \circ P_2$, is the graph of order $3|V(H)|$ obtained from H by attaching a path of length 2 to each vertex of H so that the resulting paths are vertex-disjoint. The following is useful.

Theorem 1.1. *For any graph G, $\rho(G) \leq \gamma(G)$.*

2 General results and bounds

We begin with investigation of the affection of subdivision of an edge on the domination number.

Proposition 2.1. *For any edge e in a graph G, $\gamma(G) \leq \gamma(G^e) \leq \gamma(G)+1$.*

Proof. Let $e = xy \in E(G)$, and xwy be the subdivision of e. Let S be a $\gamma(G^e)$-set. If $w \notin S$, the S is a DS for G, and if $w \in S$, then $(S-\{w\})\cup\{x\}$ is a DS for G. Consequently $\gamma(G) \leq \gamma(G^e)$. On the other hand if D is a $\gamma(G)$-set, then $D \cup \{w\}$ is a DS for G^e implying that $\gamma(G^e) \leq \gamma(G)+1$. □

We call a graph G, *domination critical upon edge subdivision*, or just γ_{sd}-critical, if $\gamma(G^e) > \gamma(G)$ for any edge $e \in E(G)$. Thus if G is a γ_{sd}-critical, then for any edge e, $\gamma(G^e) = \gamma(G)+1$. If G is a γ_{sd}-critical graph and $\gamma(G) = k$, then we call G, $k-\gamma_{sd}$-critical.

Proposition 2.2. *Any graph G of order $n \geq 3$ with $\gamma(G) = 1$ is γ_{sd}-critical.*

Proof. Assume that G is a graph of order n with $\gamma(G) = 1$. Let $S = \{x\}$ be a $\gamma(G)$-set. Then $\deg(x) = n-1$. Let $e \in E(G)$. If $e = xy$, where $y \in N(x)$, and xwy be the subdivided edge of G^e, then any $\gamma(G^e)$-set intersects $\{x, w, y\}$. If $\gamma(G^e) = 1$ and $D = \{a\}$ is a $\gamma(G^e)$-set, then $a \in \{x, w, y\}$. If $a = x$, then a does not dominate y. Thus $a \neq x$, and similarly $a \neq y$. Thus $a = w$. This implies that $G = P_2$, a contradiction. Thus $\gamma(G^e) \geq 2$. On the other hand $\{x, y\}$ is a DS for G^e, implying that $\gamma(G^e) = 2 = \gamma(G)+1$. Next assume that $e = yz$, where $y, z \in N(x)$. Similarly, we obtain that $\gamma(G^e) = 2 = \gamma(G)+1$. Thus G is γ_{sd}-critical. □

Corollary 2.3. *For any $n \geq 3$, K_n is γ_{sd}-critical.*

In the next lemma we present some classes of γ_{sd}-stable graphs.

Lemma 2.4. *(1) P_n is γ_{sd}-critical if and only if $n \equiv 0 \pmod 3$.*
(2) C_n is γ_{sd}-critical if and only if $n \equiv 0 \pmod 3$.
(3) If $n_1 \leq n_2 \leq \ldots \leq n_k$, then $K_{n_1,n_2,\ldots,n_k}$ is γ_{sd}-critical if and only if $n_1 = 1$.

Proof. Since $\gamma(P_n) = \gamma(C_n) = \lceil \frac{n}{3} \rceil$, (1) and (2) follows immediately.
(3) Let $X_1, X_2, \ldots, X_k$ be the partite sets of $G = K_{n_1,n_2,\ldots,n_k}$. Assume that $n_1 > 1$. Thus $\gamma(G) = 2$. Let $e = xy$ be an arbitrary edge of G. Then $\{x, y\}$ is a DS for G^e implying that $\gamma(G) = \gamma(G^e)$. □

Corollary 2.5. *For any $m \geq 1$, there is a $m-\gamma_{sd}$-critical graph.*

Proof. For $m = 1$, K_n is $1 - \gamma_{sd}$-critical, for $n \geq 3$. For $m \geq 2$, $K_n + \overline{K_{m-1}}$ is $m - \gamma_{sd}$-critical graph, for $n \geq 3$. □

We now give a characterization for γ_{sd}-critical graphs.

Theorem 2.6. *A graph G is γ_{sd}-critical if and only if any $\gamma(G)$-set is a 2-packing.*

Proof. ($\Longrightarrow$) Let G be a γ_{sd}-critical graph, and let S be a $\gamma(G)$-set. If S is not a 2-packing, then there are two vertices $x, y \in S$ such that $d(x, y) \leq 2$. If $d(x, y) = 1$, then S is a DS for G^{xy}, a contradiction. Thus $d(x, y) = 2$. Let $z \in N(x) \cap N(y)$. Then S is a DS for G^{xz}, a contradiction.

($\Longleftarrow$) Let any $\gamma(G)$-set be a 2-packing. Toward a contradiction assume that G is not γ_{sd}-critical. Thus there is an edge $e = xy$ such that $\gamma(G^{xy}) = \gamma(G)$. Let $xy \in E(G)$ be subdivided to $xwy \in E(G^{xy})$, and let S be a $\gamma(G^{xy})$-set. Since w is dominated by S, $S \cap \{x, y, w\} \neq \emptyset$.

If $y \in S$, then x is not a leaf, since otherwise there is a $\gamma(G)$-set which is not a 2-packing. We may assume, without loss of generality, that $\{x, w\} \cap S = \emptyset$, since otherwise we obtain a $\gamma(G)$-set which is not a 2-packing. Since x is dominated by S, $N(x) \cap S \neq \emptyset$. Let $x_1 \in N(x) \cap S$. Then S is a $\gamma(G)$-set which is not a 2-packing, a contradiction. We deduce that $y \notin S$, and Similarly $x \notin S$.

Thus $w \in S$. If $N(y) \cap (S - \{w\}) \neq \emptyset$ or $N(x) \cap (S - \{w\}) \neq \emptyset$, then $(S - \{w\}) \cup \{y\}$ or $(S - \{w\}) \cup \{x\}$, respectively, is a $\gamma(G)$-set which is not a 2-packing, a contradiction. Thus $N(y) \cap (S - \{w\}) = \emptyset$ and $N(x) \cap (S - \{w\}) = \emptyset$. If $|V(G)| = 2$, then G is not γ_{sd}-critical, thus $|V(G)| \geq 3$. Without loss of generality assume that $\deg(y) \geq 2$. Let $y_1 \in N(y) - \{w\}$. There is a vertex $y_2 \in N(y_1) \cap S$, since S is a DS. Now $(S - \{w\}) \cup \{y\}$ is a $\gamma(G)$-set which is not a 2-packing, a contradiction. □

Corollary 2.7. *If G is a γ_{sd}-critical graph, then $\gamma(G) = i(G) = \rho(G)$.*

Proof. Let G be a γ_{sd}-critical graph. Let S be a $\gamma(G)$-set. By Theorem 2.6, S is a 2-packing. This implies that $\rho(G) \geq |S|$. By Theorem 1.1, $\gamma(G) = \rho(G)$. Furthermore, since S is an independent dominating set, we have $i(G) \leq |S|$, which implies that $\gamma(G) = i(G)$. □

Corollary 2.8. *If G is a γ_{sd}-critical graph with $\gamma(G) > 1$, then $diam(G) \geq 3$.*

Proof. Let G be a γ_{sd}-critical graph with $\gamma(G) > 1$. Let S be a $\gamma(G)$-set. By Theorem 2.6, S is a 2-packing. Let $x, y \in S$. Assume that $diam(G) \leq 2$. Then $N[x] \cap N[y] \neq \emptyset$, a contradiction. □

The following is a direct consequence of Theorem 2.6.

Proposition 2.9. *Let G be a γ_{sd}-critical graph and $S = \{x_1, x_2, ..., x_\gamma\}$ be a $\gamma(G)$-set. Then $\{N[x_1], N[x_2], ..., N[x_{\gamma(G)}]\}$ is a partition of $V(G)$, and $\sum_{i=1}^{\gamma}(1 + \deg(x_i)) = n$.*

A subset S of vertices of G is called a *perfect code*, or *efficient dominating set*, if for every $v \in V(G)$, $|N[v] \cap S| = 1$. For the coding theorists the important question is to decide if a graph G has a perfect code.

Corollary 2.10. *If a graph G is γ_{sd}-critical then it has a perfect code.*

Note that the converse of Corollary 2.10 is not correct, since P_4 has a perfect code but it is not γ_{sd}-critical.

Proposition 2.11. *A regular graph G is γ_{sd}-critical if and only if $\gamma(G) = \frac{n}{1+\Delta(G)}$.*

Proof. Let G be a regular graph. If G is γ_{sd}-critical, then by Proposition 2.9, $\gamma(G) = \frac{n}{1+\Delta(G)}$. Conversely assume that $\gamma(G) = \frac{n}{1+\Delta(G)}$. If G is not γ_{sd}-critical, then there is a $\gamma(G)$-set S such that S is not a 2-packing. Then S dominates less than $|S|(1 + \Delta(G)) = n$ vertices of G, a contradiction. □

Observation 2.12. *A graph G of order n is $n - \gamma_{sd}$-critical if and only if $G = \overline{K_n}$.*

Proposition 2.13. *There is no $(n-1) - \gamma_{sd}$-critical graph of order n.*

Proof. Since $\gamma(G) = n - 1$, we have that $G = K_2 + \overline{K_{n-2}}$. Now clearly G is not γ_{sd}-critical. □

Theorem 2.14 (Ore, [3]). *If a graph G of order n has no isolated vertex, then $\gamma(G) \leq \frac{n}{2}$.*

Theorem 2.15. *A graph G of order n is $(n-2) - \gamma_{sd}$-critical if and only if $G \in \{P_3 + \overline{K_{n-3}}, C_3 + \overline{K_{n-3}}\}$.*

Proof. First it is easy to see that $P_3 + \overline{K_{n-3}}$ and $C_3 + \overline{K_{n-3}}$ are $(n-2) - \gamma_{sd}$-critical. Let G be a $(n-2) - \gamma_{sd}$-critical graph of order n. If G is connected, then by Theorem 2.14, we obtain that $G = P_2$, a contradiction, since P_2 is not γ_{sd}critical. Thus G is disconnected. Let G_1 be the component of G with maximum edges. If $|V(G_1)| \geq 5$, then by Theorem 2.14, $\gamma(G) \leq 2 + n - 5 = n - 3$, a contradiction. Thus $|V(G_1)| \leq 4$. Assume that $|V(G_1)| = 4$. Then Theorem 2.14 implies that $G - G_1$ has no edge, and $\gamma(G_1) = 2$. Furthermore, since G is γ_{sd}-critical, we obtain that G_1 is γ_{sd}-critical. But this is a contradiction, since a $2 - \gamma_{sd}$-critical graph has at least 6 vertices. Thus $|V(G_1)| \leq 3$. Similarly we can see that $|V(G_1)| \notin \{1, 2\}$, and so $|V(G_1)| = 3$. If there is a component $G_2 \neq G_1$ such that $E(G_2) \neq \emptyset$, then by Theorem 2.14, $\gamma(G) < n - 2$, a contradiction. Thus $G - G_1$ has no edge. Consequently $G \in \{P_3 + \overline{K_{n-3}}, C_3 + \overline{K_{n-3}}\}$. □

Observation 2.16. *If G is a γ_{sd}-critical graph and S is a $\gamma(G)$-set, then for any vertex $x \in S$, $|pn[x, S]| = \deg(x) + 1$.*

Theorem 2.17. *There is no induced subgraph characterization of γ_{sd}-critical graphs.*

Proof. Let G be a graph of order n, $V(G) = \{v_1, v_2, ..., v_n\}$, and $H = G \circ P_2$. Let $V(H) = \{v1, v_2, ..., v_n\} \cup \{x_{ij} | i = 1, 2, ..., n, j = 1, 2\}$, where for $i = 1, 2, ..., n$, x_{i1} is adjacent to v_i, and x_{i2} is adjacent to x_{i1}. Let S be any DS of H. Then $S \cap \{x_{i1}, x_{i2}\} \neq \emptyset$ for $i = 1, 2, ..., n$. This implies that $|S| \geq n$. On the other hand $\{x_{11}, x_{21}, ..., x_{n1}\}$ is a DS for H, implying that $\gamma(H) = n$. If there is a DS S such that S is not a 2-packing, then $|S| > n$, since $S \cap \{x_{i1}, x_{i2}\} \neq \emptyset$ for $i = 1, 2, ..., n$. Thus H is a γ_{sd}-critical graph. □

Let $L(G)$ be the set of all leaves of G.

Theorem 2.18. *Let G be a γ_{sd}-critical graph of order n and $\delta^*(G) = \min\{\deg(v) | v \in V(G) - L(G)\}$. Then $\gamma(G) \leq \frac{n}{\delta^*(G)+1}$. The equality holds if and only if G has a $\gamma(G)$-set S such that any vertex of S is of degree $\delta^*(G)$.*

Proof. Let S be a $\gamma(G)$-set. By Observation 2.16, for any vertex in S, $\deg(v) > 1$. Any vertex of S dominates at least $\delta^*(G) + 1$ vertices of G (including itself). Thus S dominates at least $|S|(\delta^*(G) + 1)$ vertices of G, and so $\gamma(G) \le \frac{n}{\delta^*(G)+1}$.

Now we prove the equality part. Let G be a γ_{sd}-critical graph with $\gamma(G) = \frac{n}{\delta^*(G)+1}$. Thus $n = (\delta^*(G) + 1)\gamma(G)$. Let S be a $\gamma(G)$-set. By Observation 2.16, any vertex of S dominates at most $\delta^*(G) + 1$ vertices of G (including the vertices of S). If there is a vertex x in S such that $\deg(x) \ge \delta^*(G) + 1$, then x has at least $\delta^*(G) + 1$ external private neighbors. Since $n = (\delta^*(G) + 1)|S|$, we obtain that there are two vertices $u, v \in S$ such that $N(u) \cap N(v) \neq \emptyset$. But then S is not a 2-packing, a contradiction by Theorem 2.6. Thus any vertex of S is of degree $\delta^*(G)$.

Conversely, since any vertex of S is of degree $\delta^*(G)$, we find that S dominate exactly $(\delta^*(G) + 1)|S|$ vertices of G. Consequently, $n = (\delta^*(G) + 1)|S| = (\delta^*(G) + 1)\gamma(G)$. □

Since $\delta^*(G) \ge 2$, we have the following.

Corollary 2.19. *If G is a γ_{sd}-critical graph of order n, then $\gamma(G) \le \frac{n}{3}$, with equality if and only if G has a $\gamma(G)$-set S such that any vertex of S is of degree 2.*

Note that the vertices outside S in the above theorem are not necessarily of degree $\delta^*(G)$. For example let G be obtained from a star $K_{1,n}$ for $n \ge 2$ by subdividing $n - 1$ edge, twice, and subdividing the remaining edge, once. Then $\gamma(G) = \frac{n}{3}$, the vertices of the unique $\gamma(G)$-set are of degree 2, but G has a vertex of degree $n > 2$.

Since for any graph G, $\gamma(G) \ge \frac{n}{1+\Delta(G)}$, we obtain the following which the proof for the equality part is similar to the proof of Theorem 2.18.

Theorem 2.20. *If G is a γ_{sd}-critical graph then $\gamma(G) \geq \frac{n}{1+\Delta(G)}$, with equality if and only if G has a $\gamma(G)$-set S such that any vertex of S is of degree $\Delta(G)$.*

Let $\mathcal{D}$ be the class of all graphs G of diameter $diam(G) \geq 2$, with a diameterical path $P; x_1 x_2 ... x_{diam(G)+1}$, such that for any vertex $v \in V(G) - V(P)$, there is a unique integer $j \geq 0$ such that $v \in N(x_{3j+2})$.

Theorem 2.21. *If G is a γ_{sd}-critical graph then $\gamma(G) \geq \frac{1+diam(G)}{3}$, with equality if and only if $G \in \mathcal{D}$.*

Proof. The lower bound $\gamma(G) \geq \frac{1+diam(G)}{3}$ is obvious, since for any graph G this bound holds. So we prove the next part. Since $P \cong P_{diam(G)+1}$, $\gamma(G) \geq \gamma(P_{diam(G)+1}) = \frac{diam(G)+1}{3}$. On the other hand $\{x_{3i+2} | 0 \leq i < \frac{n}{3}\}$ is a DS for G. Thus $\gamma(G) = \frac{1+diam(G)}{3}$. Conversely, assume that $\gamma(G) = \frac{1+diam(G)}{3}$. Since $P \cong P_{diam(G)+1}$, and $\gamma(P_{diam(G)+1}) = \frac{diam(G)+1}{3}$, any vertex of $G - P$ is adjacent to some vertex x_{3j+2}, where $0 \leq j < \frac{n}{3}$. If there are two different j_1, j_2 such that a vertex $a \in G - P$ is adjacent to x_{3j_1+2} and x_{3j_2+2}, then $diam(G) < diam(P)$, a contradiction. □

3 Trees

In this section we obtain a constructive characterization for all γ_{sd}-critical trees. Let $\mathcal{T}$ be the family of unlabelled trees T that can be obtained from a sequence $T_1, ..., T_j$ $(j \geq 1)$ of trees such that T_1 is a star $K_{1,r}$ for $r \geq 2$, and, if $j \geq 2$, T_{i+1} can be obtained recursively from T_i by the following operation.

- **Operation $\mathcal{O}$.** Let $T_i \in \mathcal{T}$ and v be a vertex of T_i such that v does not belong to a $\gamma(T_i)$-set. Then the tree T_{i+1}

is obtained from T_i by joining v to a leaf of an star $K_{1,m}$ for some $m \geq 2$.

Theorem 3.1. *A tree T is γ_{sd}-critical if and only if $T \in \mathcal{T}$.*

Proof. ($\Longrightarrow$) We proceed by induction on the domination number $\gamma(T)$ of a γ_{sd}-critical tree T to show that $T \in \mathcal{T}$. If $\gamma(T) = 1$, then clearly T is a star. Since P_2 is not γ_{sd}-critical, T is a star of order at least three, and so $T \in \mathcal{T}$. Suppose the result is true for all γ_{sd}-critical trees T with domination number less than $\gamma(T)$. Let T be a γ_{sd}-critical tree with $\gamma(T) > 1$. Since $\gamma(T) \geq 2$, if $diam(T) \leq 4$, then there is a $\gamma(T)$-set which is not a 2-packing, a contradiction. Thus, $diam(T) \geq 5$. Let $x_0 - x_1 - x_2 - ... - x_k$ be a diameterical path in T between two leaves x_0 and x_k, where $k = diam(T)$. We root T at x_0. Let S be a $\gamma(T)$-set containing x_{k-1}. Since S is a 2-packing, $N(x_{k-1}) \cap S = \emptyset$.

If $\deg(x_{k-2}) \geq 3$, then x_{k-2} has some children different from x_{k-1} and x_{k-3}. Let $a \neq x_{k-1}, x_{k-3}$ be a child of x_{k-2}. Since S is a $\gamma(T)$-set, $x_{k-2} \notin S$ and $diam(T) = k$, we may assume that $a \in S$. But then S is not a 2-packing, since $x_{k-1} \in S$. This contradiction implies that $\deg(x_{k-2}) = 2$.

Let T_1 and T_2 be the sub-trees obtained from T by removing the edge $x_{k-3}x_{k-2}$ such that $x_{k-3} \in V(T_1)$ and $x_{k-2} \in V(T_2)$. Note that T_2 is a star of order at least 3. Clearly $S \cap V(T_2) = \{x_{k-1}\}$, and $S \cap V(T_1)$ is a DS for T_1. In particular, $\gamma(T_1) \leq \gamma(T) - 1$. Furthermore, for any $\gamma(T_1)$-set D, $D \cup \{x_{k-1}\}$ is a DS for T. Thus $\gamma(T_1) = \gamma(T) - 1$.

We show that T_1 is γ_{sd}-critical. Assume that T_1 is not γ_{sd}-critical. By Theorem 2.6, there is a $\gamma(T_1)$-set D such that D is not a 2-packing. Then $D \cup \{x_{k-1}\}$ is a $\gamma(T)$-set which is not a 2-packing, contradicting the fact that T is γ_{sd}-critical. Thus T_1 is γ_{sd}-critical.

Since $\gamma(T_1) < \gamma(T)$, by inductive hypothesis, $T_1 \in \mathcal{T}$. We show that no $\gamma(T_1)$-set contains x_{k-3}. Suppose to the contrary that there is a $\gamma(T_1)$-set A such that $x_{k-3} \in A$. Then $A \cup \{x_{k-1}\}$ is a $\gamma(T)$-set which is not a 2-packing, a contradiction. Thus no $\gamma(T_1)$-set contains x_{k-3}. Now T is obtained by joining x_{k-2} to x_{k-3}, and so $T \in \mathcal{T}$.

($\Longleftarrow$) We proceed by induction on the domination number $\gamma(T)$ of a tree $T \in \mathcal{T}$ to show that T is γ_{sd}-critical. If $\gamma(T) = 1$, then T is an star of order at least 3, and obviously by Proposition 2.2, T is γ_{sd}-critical. Hence, the result is true for the base case when $\gamma(T) = 1$. Suppose the result is true for all trees $T \in \mathcal{T}$ with domination number less than $\gamma(T)$. Let $T \in \mathcal{T}$ be a tree with $\gamma(T) > 1$. Then T can be obtained from a sequence $T_1, ..., T_j$ ($j \geq 1$) of trees such that T_1 is a star $K_{1,r}$ for $r \geq 2$, and, if $j \geq 2$, then T_{i+1} can be obtained recursively from T_i by the operation $\mathcal{O}$. Since $\gamma(T) > 1$, T is not a star, and so $j \geq 2$. Let v be a vertex of T_{j-1} such that v does not belong to a $\gamma(T_{j-1})$-set. Then the tree T_j is obtained from T_{j-1} by joining v to a leaf of an star $K_{1,m}$ for some $m \geq 2$. Let o be the central vertex of $K_{1,m}$ and let S be a $\gamma(T_j)$-set. If D is a $\gamma(T_{j-1})$-set, then $D \cup \{o\}$ is a DS for T_j, implying that $\gamma(T_j) \leq \gamma(T_{j-1}) + 1$.

We now show that $S = S^* \cup \{o\}$, where S^* is a $\gamma(T_{j-1})$-set. Assume that $o \notin S$. Since v is not in any $\gamma(T_{j-1})$-set of T_{j-1}, it follows from [1] that $|(V(T_{j-1}) - \{v\}) \cap S| = \gamma(T_{j-1})$. Furthermore, $|(V(K_{1,m}) \cup \{v\}) \cap S| \geq 2$. This shows that $\gamma(T_j) = |S| \geq \gamma(T_{j-1}) + 2$, a contradiction. Hence $o \in S$ and $S^* = S - \{o\}$ is a DS of T_{j-1}, moreover S^* is a $\gamma(T_{j-1})$-set. This means that for any $\gamma(T_j)$-set S, $S = S^* \cup \{o\}$, where S^* is a $\gamma(T_{j-1})$-set.

Since $\gamma(T_{j-1}) < \gamma(T = T_j)$, by the inductive hypothesis T_{j-1} is γ_{sd}-critical. By Theorem 2.6, S^* is a 2-packing. Now it is obvious that S is 2-packing. By Theorem 2.6, T_j is γ_{sd}-critical. □

Acknowledgements

I would like to thank the referee for his/her careful review of the manuscript and some helpful comments which improved the presentation of the paper. I would like to note that the sec-

ond paragraph of part $\Leftarrow$ of the proof of Theorem 3.1 has been corrected by the referee.

References

[1] R.C. Brigham, P.Z. Chinn, and R.D. Dutton, *Vertex domination-critical graphs*, Networks 18 (1988), 173-179.

[2] T. Burton, and D. P. Sumner, *Domination dot-critical graphs*, Discrete Math. 306 (2006), 11-18.

[3] T.W. Haynes, S. T. Hedetniemi and P. J. Slater, editors. *Fundamentals of Domination in graphs* Marcel Dekker, Inc, New York, NY, 1998.

[4] N. Jafari Rad, *On the diameter of a domination dot critical graph*, Discrete Applied Mathematics 157 (2009), 1647-1649.

[5] D. Sumner and P. Blitch, *Domination critical graphs*, Journal of Combinatorial Theory Ser. B 34(1983), 65-76.

[6] D. P. Sumner, *Critical concept in domination*, Discrete Mathematics, 86 (1990), 33-46.