Representing Asteroidal Sets on Subdivisions of Stars

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Abstract

Consider a simple undirected graph $G = (V, E)$. A family of subtrees, $\{T_v\}_{v \in V}$, of a tree $\mathcal{T}$ is called a $(\mathcal{T}; t)$-representation of G provided $uv \in E$ if and only if $|T_u \cap T_v| \geq t$. In this paper we consider $(\mathcal{T}; t)$-representations for graphs containing large asteroidal sets, where $\mathcal{T}$ is a subdivision of the n-star $K_{1,n}$. An asteroidal set in a graph G is a subset A of the vertex set such that for all 3-element subsets of A, there exists a path in G between any two of these vertices which avoids the neighborhood of the third vertex. We construct a representation of an asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$ and show that no graph containing a larger asteroidal set can be represented.

1 Introduction

The study of graph representations is an active research area in graph theory. Given a graph $G = (V, E)$, a representation of G is the following collection of objects: (1) a set S, (2) a function $f : V \to \mathscr{P}(S)$ (the power set of S), and (3) a function $g : f(V) \times f(V) \to \{0, 1\}$ so that $g(f(v_1), f(v_2)) = 1$ iff $v_1 v_2 \in E$. We call S the *host set*, f the *assignment function*, and g the *conflict rule*. A graph G is representable under a given host set S and conflict rule g if there exists a suitable assignment function f, in which case we say that G is $(S; g)$-representable.

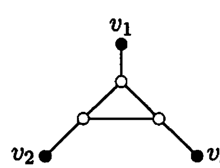
Much is known about graph representations when the conflict rule depends on the size of the intersection between assigned subsets. Such intersection representations have been studied extensively. A comprehensive list of authors can be found in [7].

Given conflict-tolerance $t = 1$, all graphs are $(S; 1)$-representable for large enough S. A central objective in the theory of graph representations is to find the smallest host set on which certain classes of graphs may be represented with respect to some given conflict rule. For example, the cycle C_n can be represented on a host set of size n. Indeed, assigning the set $\{i, i+1 \mod (n)\}$ to vertex v_i gives a set representation of C_n, however there is no possible assignment of subsets from a smaller host set that will induce C_n with $t = 1$.

In this paper we consider tree representations of graphs. Tree representations are a variation on the traditional graph representation. The host is a tree, giving more structure than merely a set of elements; objects assigned to vertices of a represented graph are subtrees of the host tree; and an edge exists between two vertices if and only if their assigned subtrees intersect in t or more nodes, where t is a prescribed conflict-tolerance. An important distinction between set representations and tree representations is that given a host tree, not all graphs have a representation where the conflict-tolerance is $t = 1$, even if we allow subdivision of the tree. For example, cycles of length four or greater are always forbidden.

The classes of graphs representable using different host trees differ significantly (see [1,2,3,4,6,8]), and are thus interesting to study. A well known and interesting example is the result of Lekkerkerker and Boland [9]. We first give a definition and then state their result.

Definition 1.1 *An asteroidal set in a graph G is $A \subset V(G)$ so that $\forall v_1, v_2, v_3 \in A$, and $\forall i, j, k \in \{1, 2, 3\}$ with i, j, and k distinct, there exists a path between v_i and v_j which does not intersect $N(v_k)$. If $|A| = m$, then A is said to be an m-asteroid of G. Furthermore, if A is the maximum size asteroidal set contained in G, then G is said to be m-asteroidal.*



$\{v_1, v_2, v_3\}$ is an asteroidal set in the graph

Theorem 1.2 *A graph is representable on an interval if and only if it is chordal and non-asteroidal.*

For convenience of notation, consider $\mathcal{K}_{1,n}$ to be a subdivision of the tree $K_{1,n}$. In 1972 James Walter wrote his dissertation [10] on graphs representable on $\mathcal{K}_{1,3}$ with tolerance $t = 1$. This can be thought of as a generalization of the result of Lekkerkerker and Boland, since $\mathcal{K}_{1,3}$ is a logical extension from studying paths. Walter discovered that graphs containing certain induced cycles and certain asteroidal configurations were not representable under this host tree and conflict rule. Walter's result provides motivation for the work done in this paper, and is stated as follows.

Theorem 1.3 *A graph G is $(\mathcal{K}_{1,3};1)$-representable iff G is chordal, at most 3-asteroidal, and for any two pairs v_1, v_2 and u_1, u_2 of vertices contained in asteroidal triples of G, any path connecting v_1 and v_2 must be adjacent to any path connecting u_1 and u_2.*

Given this result, it seems that cycles and asteroidal sets are interesting structures to study while considering tree representations of graphs. It was Jamison who stated that it would be interesting to explore what happens if the conflict-tolerance is greater than one. In 2001, Eaton and Barbato [1] studied representations of cycles on $\mathcal{K}_{1,3}$ with arbitrary tolerance t. They described all cycles representable on $\mathcal{K}_{1,3}$ with conflict tolerance t, showing that arbitrarily large cycles can be represented on $\mathcal{K}_{1,3}$ at the cost of increasing the tolerance. Their theorem is restated below.

Theorem 1.4 *For $t = 3, 4,$ and 5 the maximum n such that C_n is $(\mathcal{K}_{1,3};t)$-representable is $3t-3$. For $t \geq 6$ the largest such n satisfies the following inequality*

$$\frac{1}{4}t^2 + t + \frac{3}{4} \leq n \leq \frac{1}{4}t^2 + \frac{3}{2}t - \frac{3}{4}$$

A related result on cycles is due to Eaton and Faubert [3]. The two considered tree representations where the host tree is a caterpillar. That is, a tree in which every node is either on, or adjacent to, its longest path. Again, we provide a definition and then state their result.

Definition 1.5 *We say that $G \in cat[h,t]$ if there exists a caterpillar with maximum degree h such that G is representable on this caterpillar with tolerance t.*

Theorem 1.6 *If $n \leq (h-2)(t-1)+2$ with $h \geq 3$ and $t \geq 2$, then $C_n \in cat[h,t]$.*

The two went on to show that $C_n \in cat[4,3]$ for all values of n. They also completely classified the graphs which are in $cat[2,t]$ and $cat[3,1]$ [2].

Going back to the result of Eaton and Barbato, note that for $k \geq 6$, the cycle C_k contains an asteroidal set of size $\lfloor \frac{k}{2} \rfloor$. That is, their result on cycles implies that graphs containing arbitrarily large asteroidal sets are representable on $\mathcal{K}_{1,3}$. We explore this result further, and construct the largest asteroidal set representable on $\mathcal{K}_{1,n}$ with arbitrary conflict tolerance $t > 1$ and $n \geq 3$. The remainder of this paper will be devoted to proving the following main theorem and discussing a few open problems.

Theorem 1.7 *For $t > 1$ and $n \geq 3$, an asteroidal configuration of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$ which is $(\mathcal{K}_{1,n}; t)$-representable exists. Furthermore, any graph containing a larger asteroidal set is not $(\mathcal{K}_{1,n}; t)$-representable.*

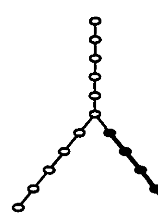
Note that any terms not defined in this introduction, but used throughout the paper, can be found in [11].

2 Construction of the Graph Containing the Asteroidal Set

2.1 Vertices and their Assigned Subtrees

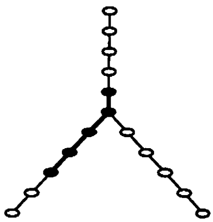
For convenience, we will refer to the graph we are constructing as the *target graph*. Recall that our host tree is $\mathcal{K}_{1,n}$. It shall be sufficient to assume that each branch of the host tree contains t nodes. For clarity, the word *node* will be used to indicate a vertex of the host tree, distinguishing these from vertices of the target graph. We call the unique node of degree n the branching node of the host tree, and the branches of the host tree are labeled with the integers $0, 1, ..., n-1$. Furthermore, within this paper, any reference to the size of a subtree or of an intersection should be interpreted in terms of number of nodes. It should be noted that each subtree defined below will correspond to a vertex in the target graph. The vertex set of the target graph will be described in terms of four disjoint subsets $\mathcal{V}, \mathcal{W}, \mathcal{P}$, and $\mathcal{Q}$. Note that the subscripts of the vertices in $\mathcal{V}$ and $\mathcal{P}$ will give information as to which branches their corresponding subtrees exist on, and should be interpreted modulo n. The desired asteroidal set will be a subset of the vertex set of the target graph. The description of each set of vertices below is accompanied by an image, giving an example of the subtree representative of one of the vertices from that set. The sample host tree $\mathcal{K}_{1,3}$ is shown; the tolerance used in the examples is $t = 5$; and the selected subtrees are denoted by thick edges and filled nodes.

1. Let $\mathcal{V}$ be a collection of vertices represented by distinct subtrees, each of size exactly t, and contain the leaf of a branch of $\mathcal{K}_{1,n}$. These subtrees are paths on the exterior portion of each branch of $\mathcal{K}_{1,n}$. Since there are n branches of $\mathcal{K}_{1,n}$, we will have n vertices in $\mathcal{V}$. We will refer to these vertices later as $v_0, ..., v_{n-1}$, where the subscripts denote the branch on which the corresponding assigned subtree exists.

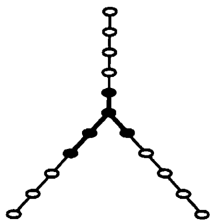


The subtree assigned to one of the vertices in $\mathcal{V}$

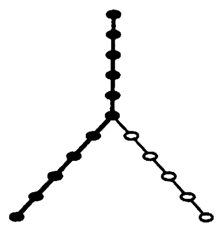
2. For $2 \leq k \leq n$, we define $\mathcal{W}_k$ to be the set of vertices represented by distinct subtrees which are of size exactly t and exist non-trivially on exactly k branches of $\mathcal{K}_{1,n}$. There are $\binom{n}{k}\binom{t-2}{k-1}$ such subtrees, and therefore the same number of corresponding vertices. We take $\mathcal{W}$ to be the union, over all values of k, of the sets $\mathcal{W}_k$. Clearly $\{\mathcal{W}_k\}_{k=2}^{n}$ forms a partition of $\mathcal{W}$, and therefore $|\mathcal{W}| = \sum_{k=2}^{n}\binom{n}{k}\binom{t-2}{k-1}$. Note that we assume the convention that $\binom{a}{b} = 0$ for $b > a$.



The subtree assigned to one of the vertices in $\mathcal{W}_2$



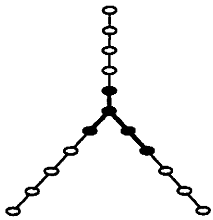
The subtree assigned to one of the vertices in $\mathcal{W}_3$



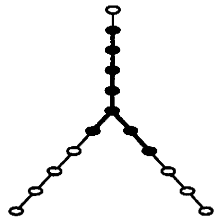
The subtree assigned to one of the vertices in $\mathcal{P}$

3. Construct $\mathcal{P}$, the collection of all vertices representable by subtrees existing non-trivially on exactly two consecutive branches of $\mathcal{K}_{1,n}$, and which extend out to the leaf node on each of these branches. We define vertex p_i, whose assigned subtree we denote by T_{p_i} and contains the entirety of branches i and $i+1$ for $0 \leq i \leq n-1$.

4. Let $\mathcal{Q}$ be a collection of vertices, each assigned a subtree which is an extension of a subtree assigned to a vertex in $\mathcal{W}$. That is, for each vertex in $\mathcal{W}$ we create $k(w)$ (recall that k is the number of legs of $\mathcal{K}_{1,n}$ on which T_w, the subtree assigned to w, lives non-trivially) new vertices in $\mathcal{Q}$. Each of these vertices is assigned a subtree which has been created by elongating, out to the second to last node, exactly one of the pre-existing legs of the subtree assigned to the corresponding vertex w. Consider the following example where we show T_w, the subtree assigned to one of the vertices from $\mathcal{W}$ and one of its extensions, a subtree assigned to a vertex in $\mathcal{Q}$.



T_w, the subtree assigned to the vertex $w \in \mathcal{W}$



One of the three extensions of T_w, giving the subtree assigned to one of vertices in $\mathcal{Q}$

2.2 Basic Claims About Adjacencies

Consider the graph $G = (\mathcal{V} \cup \mathcal{W} \cup \mathcal{P} \cup \mathcal{Q}\ ,\ E)$, where E is the edge set defined by the conflict tolerance relationship. Given the assignment of subtrees of $\mathcal{K}_{1,n}$ from the previous section, we have the following claims and observations about E. Note that we will use the notation T_x to refer to the subtree assigned to vertex x of G.

Observation 2.1 *$\mathcal{V}$ is an independent set.*

Observation 2.2 *There are no edges between the vertices of $\mathcal{Q}$ and $\mathcal{V}$.*

Observation 2.3 $N(v_i) = \{p_{i-1}, p_i\}$

Claim 2.4 *$\mathcal{W}$ is an independent set.*

Proof. Note that for each $w \in \mathcal{W}$, we have that $|T_w| = t$. Let w_i, w_j be distinct vertices in $\mathcal{W}$. Then their assigned subtrees are distinct. That is, there exists at least one node of $\mathcal{K}_{1,n}$ which is a node of $T_{w_i} \cup T_{w_j}$, but which is not a node of $T_{w_i} \cap T_{w_j}$. This gives that $|T_{w_i} \cap T_{w_j}| < |T_{w_i}| = t$. Therefore, given any two vertices from $\mathcal{W}$, their assigned subtrees have an intersection of fewer than t nodes, and thus the vertices cannot be adjacent. In short, $\mathcal{W}$ is an independent set. ■

Observation 2.5 *$\mathcal{V} \cup \mathcal{W}$ is an independent set.*

Claim 2.6 *Vertices p_i and p_j are adjacent if and only if $j \in \{i-1, i+1\}$.*

Proof. If $j \neq i-1$ or $i+1$ then p_i and p_j are assigned subtrees which exist on distinct pairs of branches of $\mathcal{K}_{1,n}$. That is, their assigned subtrees intersect in exactly a single node (the branching node). Since $t > 1$, we have that p_i and p_j are not adjacent. ■

Claim 2.7 *Let $w \in \mathcal{W}$, then $N(w) \cap \mathcal{P} \neq \emptyset$ iff T_w exists non-trivially only on two consecutive branches of $\mathcal{K}_{1,n}$. Furthermore, if $|N(w) \cap \mathcal{P}| \neq 0$, then $|N(w) \cap \mathcal{P}| = 1$.*

Proof. Consider $p \in \mathcal{P}$ and $w \in \mathcal{W}$ with $pw \in E$. Then $|T_p \cap T_w| \geq t$. However, since $|T_w| = t$, this implies that $T_p \cap T_w = T_w$. Now, since T_w exists on at least two branches of $\mathcal{K}_{1,n}$, and we know that $T_w \subset T_p$, then T_w exists non-trivially on the same two branches of $\mathcal{K}_{1,n}$ as T_p (which are indeed consecutive).

If T_w exists non-trivially only on two consecutive branches of $\mathcal{K}_{1,n}$ (call them branch i and $i+1$), then $T_w \subset T_{p_i}$. Therefore, $|T_w \cap T_{p_i}| = t$, implying that $p_i \in N(w)$. ■

Claim 2.8 *If q is a vertex from $\mathcal{Q}$, then q and p_j are adjacent iff T_q was obtained from some T_w by elongating branch j or branch $j+1$.*

Proof. If T_q exists non-trivially on exactly two branches of $\mathcal{K}_{1,n}$, then the result follows directly from the previous claim. We therefore assume that T_q exists non-trivially on at least 3 branches of $\mathcal{K}_{1,n}$. Let $qp_j \in E$. Assume that T_q was obtained from T_w by an elongation of branch i, where $i \notin \{j, j+1\}$. Now, since T_{p_j} exists non-trivially only on branches j and $j+1$ of $\mathcal{K}_{1,n}$, we have $a_j + a_{j+1} \geq t-1$, where a_j and a_{j+1} are the lengths of the legs of T_q on branch j and $j+1$ of $\mathcal{K}_{1,n}$. Furthermore, since $i \neq j$ and $i \neq j+1$, we must have that the previous statement is also true about T_w. This, however, gives a contradiction since $|T_w| = t$ and T_w lives non-trivially on at least 3 branches of $\mathcal{K}_{1,n}$. ■

Claim 2.9 *For $w_1, w_2 \in \mathcal{W}$, there exists $q_1 \in \mathcal{Q}$, with the property that $q_1 \in N(w_1) \setminus N(w_2)$.*

Proof. Let $w_1, w_2 \in \mathcal{W}$ with $w_1 \neq w_2$. Assume that $N(w_1) \subset N(w_2)$. Now, consider $\mathcal{Q}_{w_1}$ the set of all $q \in \mathcal{Q}$ so that T_q can be obtained via an elongation of one of the legs of T_{w_1}. Then, since $N(w_1) \subset N(w_2)$, we know $\mathcal{Q}_{w_1} \subset N(w_2)$. That is, $\forall q \in \mathcal{Q}_{w_1}, |T_{w_2} \cap T_q| \geq t$. However, $|T_{w_2}| = t$, so this implies $T_{w_2} \cap T_q = T_{w_2}, \forall q \in \mathcal{Q}_{w_1}$. This gives us that $T_{w_2} \subset \bigcap_{q \in \mathcal{Q}_{w_1}} T_q$. However, we know that $\bigcap_{q \in \mathcal{Q}_{w_1}} T_q$ is exactly T_{w_1}. Therefore, we have that $T_{w_2} \subset T_{w_1}$; but, since $|T_{w_1}| = |T_{w_2}|$, we know $T_{w_2} = T_{w_1}$. Thus $w_1 = w_2$, a contradiction. This gives that there must exist a $q_1 \in N(w_1) \setminus N(w_2)$. ■

2.3 Verification of the Asteroidal Properties of $\mathcal{V} \cup \mathcal{W}$

Theorem 2.10 *$\mathcal{V} \cup \mathcal{W}$ is an m-asteroidal set of the graph G which has vertex set $\mathcal{V} \cup \mathcal{W} \cup \mathcal{P} \cup \mathcal{Q}$ and edge set E, where $m = n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$.*

Proof. We recall that an asteroidal set A is an independent set which has the property that for any selection of 3 vertices, $a_1, a_2, a_3 \in A$ there is a path connecting any two of them, which avoids the neighborhood of the third.

Firstly, notice that $C = v_0p_0v_1..., v_{n-1}p_{n-1}v_0$ is a cycle in G. This cycle will be used extensively in the verification of the properties of the asteroidal set. Also worth noting is that, for convenience of notation we will use $P(u, v)$ to denote a path in G connecting the vertices u and v instead of the more commonly used (uv)-path. The following claims 2.11 through 2.16 verify that this definition is satisfied on $\mathcal{V} \cup \mathcal{W}$.

Claim 2.11 *For $v_1, v_2, v_3 \in \mathcal{V}$, $\forall i, j, k \in \{1, 2, 3\}$, with i, j, and k distinct, there exists a path, $P(v_i, v_j)$, so that $P(v_i, v_j) \cap N(v_k) = \emptyset$.*

Proof. Let $v_1, v_2, v_3 \in \mathcal{V}$. Now, we construct a path between v_i and v_j which avoids $N(v_k) = \{p_{k-1}, p_k\}$. Then the cycle C provides two paths from v_i to v_j, one of which must avoid the sequence $p_kv_kp_{k+1}$. ▼

Claim 2.12 *For $v_1, v_2 \in \mathcal{V}$, $w \in \mathcal{W}$, there exists a path, $P(v_1, v_2)$, so that $P(v_1, v_2) \cap N(w) = \emptyset$.*

Proof. The cycle C connects the vertices v_1 and v_2 via two paths. Now, recall that $N(w)$ contains at most one of the vertices on this cycle. If $N(w) \cap \mathcal{P} = \emptyset$, then we take either portion of the cycle as our path. If $N(w) \cap \mathcal{P}$ is non-empty, then there must be exactly one vertex $p \in N(w) \cap \mathcal{P}$, so we traverse the portion of the cycle from v_1 to v_2 in the direction which avoids p. This gives us the desired path. ▼

Claim 2.13 *For $v_1, v_2 \in \mathcal{V}$, $w \in \mathcal{W}$, $\forall i, j \in \{1, 2\}$ with i and j distinct, there exists a path, $P(v_i, w)$, so that $P(v_i, w) \cap N(v_j) = \emptyset$.*

Proof. We begin from w. Let $\mathcal{Q}_w$ denote the collection of all $q \in \mathcal{Q}$, so that T_q was obtained from T_w by elongating one of its legs. Now, notice that $|N(w) \cap \mathcal{Q}_w| \geq 2$, so there exists $q_1, q_2 \in N(w) \cap \mathcal{Q}_w$. Then, recall $\forall q \in \mathcal{Q}$, there exists two vertices $p_a, p_b \in N(q) \cap \mathcal{P}$. Since q_1 and q_2 are distinct, there are at least three distinct vertices $p_a, p_b, p_c \in N(q_1) \cup N(q_2)$. Also, recall that $N(v_j) \cap \mathcal{P} = \{p_{j-1}, p_j\}$. Therefore, there exists $p \in \{p_a, p_b, p_c\}$ with $p \notin \{p_{j-1}, p_j\}$. We take the path from w to this vertex p. We again traverse the path from vertex p to v_i which uses the portion of the cycle C avoiding the sequence $p_{j-1}v_jp_j$. ▼

Claim 2.14 *For $w_1, w_2, w_3 \in \mathcal{W}$, $\forall i, j, k \in \{1, 2, 3\}$, with i, j, and k distinct, there exists a path, $P(w_i, w_j)$, so that $P(w_i, w_j) \cap N(w_k) = \emptyset$.*

Proof. Recall the existence of $q_i \in N(w_i)$ so that $q_i \notin N(w_k)$, and $q_j \in N(w_j)$ so that $q_j \notin N(w_k)$. We traverse from w_i to q_i and from w_j to q_j. If $q_i = q_j$, then we have the desired path already, so assume they are not equal. Now, q_i and q_j may be adjacent, however, we are unsure, so we travel on. Recall that $N(w_k)$ may contain at most one vertex from $\mathcal{P}$. Also, recall that each vertex from $\mathcal{Q}$ is adjacent to exactly two vertices from $\mathcal{P}$. Therefore we can extend from q_i to at least one of its neighbors p_i and from q_j to at least one of its neighbors p_j without crossing into $N(w_k)$. Again, if $p_i = p_j$ we have the desired path, so we assume this is not the case. Now, we are on the cycle C. If $N(w_k) \cap \mathcal{P} = \emptyset$, then we connect p_i to p_j via either portion of the cycle. Otherwise $N(w) \cap \mathcal{P}$ is a single vertex, p, and we connect p_i to p_j by a path along the cycle in the direction which avoids p. In either case we have completed a path connecting the vertices w_i and w_j which avoids $N(w_k)$. ▼

Claim 2.15 *For $w_1, w_2 \in \mathcal{W}$, $v \in \mathcal{V}$, there exists a path, $P(w_1, w_2)$, so that $P(w_1, w_2) \cap N(v) = \emptyset$.*

Proof. If $N(w_1) \cap N(w_2) \neq \emptyset$ then we can draw a path from w_1 to w_2 via their common neighbor. Recall that there are no edges between the partitions $\mathcal{V}$ and $\mathcal{Q}$, so this path satisfies the requirements. Otherwise, $N(w_1) \cap N(w_2) = \emptyset$. Then, similar to the proof of claim 2.13, there exist paths from w_1 to p_a and from w_2 to p_b where $p_a, p_b \notin N(v)$. Now, we can use the cycle C to connect the vertices p_a and p_b with a path that does not intersect $N(v)$. ▼

Claim 2.16 *For $w_1, w_2 \in \mathcal{W}$, $v \in \mathcal{V}$, $\forall i, j \in \{1, 2\}$ with i and j distinct, there exists a path, $P(w_i, v)$, so that $P(w_i, v) \cap N(w_j) = \emptyset$.*

Proof. We begin from w_i. Recall that there exists $q \in N(w_i)$, so that $q \notin N(w_j)$. We move from w_i to q. Also, recall that $N(w_j)$ contains at most one vertex in the partition $\mathcal{P}$, and that q is adjacent to exactly two vertices from $\mathcal{P}$. That is, we can extend from q to one of its neighbors in $\mathcal{P}$ without crossing into $N(w_j)$. Now we find ourselves again on the cycle C. If $N(w_j) \cap \mathcal{P} = \emptyset$, we travel either half of the cycle from p to v. Otherwise, $N(w_j) \cap \mathcal{P}$ consists of a single vertex, and we travel the half of the cycle connecting p and v which avoids this vertex. The union of the two selected paths provides a single path from w_i to v with the desired property. ▼

We have shown that $\mathcal{V} \cup \mathcal{W}$ satisfies the definition of an asteroidal set. Recall that $|\mathcal{V}| = n$ and $|\mathcal{W}| = \sum_{k=2}^{n} \binom{n}{k} \binom{t-2}{k-1}$, and that $\mathcal{V} \cap \mathcal{W} = \emptyset$. Therefore we have exhibited an asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k} \binom{t-2}{k-1}$. ■

3 No Larger Asteroid is Representable Under the Restrictions of n and t

In the current section we show that the construction of a larger asteroidal set under the current restrictions of n and t is impossible. Note that showing this will prove the following theorem:

Theorem 3.1 *If G is a $(\mathcal{K}_{1,n};t)$-representable graph then G is at most $\left(n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}\right)$-asteroidal.*

Proof. Firstly, observe that an asteroidal set must be an independent set. Furthermore, it is easily seen that at most $\sum_{k=1}^{n} \binom{n}{k}\binom{t-2}{k-1}$ vertices may be assigned subtrees which contain the branching node of $\mathcal{K}_{1,n}$ such that no two subtrees intersect in t or more nodes. Now, note that we have almost 'saturated' the branching node of the host tree with subtrees while constructing the asteroidal set of size $n + \sum_{k=2}^{n} \binom{n}{k}\binom{t-2}{k-1}$.

Let $\mathcal{M}$ be the set of vertices which are assigned subtrees containing the branching node and extending non-trivially only on a single branch of $\mathcal{K}_{1,n}$. We need only to consider whether the vertices of $\mathcal{M}$ and $\mathcal{V}$ can coexist in the same asteroidal set.

Claim 3.2 *Let m_i be a vertex with assigned subtree, T_{m_i}, existing non-trivially only on branch i of $\mathcal{K}_{1,n}$ and which contains the branching node of $\mathcal{K}_{1,n}$. Then, any path from v_i to another asteroidal vertex a, must be adjacent to m_i.*

Proof. Note that by Lekkerkerker and Boland's result, a must have a non-trivial intersection with a branch of $\mathcal{K}_{1,n}$ other than branch i. If not, then $\{v_i, m_i, a\}$ is an asteroidal triple which is represented on an interval, a direct violation of their theorem.

We consider any path beginning at v_i and ending at a. Along this path there must be a vertex u_ℓ, the last vertex along the path whose assigned subtree does not contain the branching node of $\mathcal{K}_{1,n}$. Now, we must have that $T_{u_{\ell+1}}$ contains the branching node of the host tree, and also has an intersection of size at least t with T_{u_ℓ}. Therefore, either $T_{m_i} \subset T_{u_{\ell+1}}$ or $T_{u_{\ell+1}} \cap T_{u_\ell} \subset T_{m_i}$. Recalling that $|T_{m_i}| \geq t$ gives that either $|T_{m_i} \cap T_{u_{\ell+1}}| \geq t$ or $|T_{m_i} \cap T_{u_\ell}| \geq t$, inducing either the edge $u_{\ell+1}m_i$ or $u_\ell m_i$. That is, any path connecting v_i with another asteroidal vertex must be adjacent to the vertex m_i. ▼

The previous claim directly implies that the vertices m_i and v_i cannot be part of the same asteroidal set. A similar agument can be used to show the same result about v_i and a vertex whose subtree is a proper sub-path of the i^{th} branch of the host tree. This implies that per branch we may only have one vertex whose assigned subtree exists non-trivially only on that branch, no matter the configuration of subtrees. Note that this property is satisfied in the configuration constructed in section 2. ■

Combining the result from section 2 with this result, we have shown the main result from the introduction, and have rediscovered the surprising corollary originally seen by Eaton and Barbato:

Corollary 3.3 *Graphs containing arbitrarily large asteroidal sets are representable on $\mathcal{K}_{1,3}$ [1].*

Since the size of the largest cycle which is $(\mathcal{K}_{1,3}; t)$-representable is eventually quadratic in t, and the size of the largest representable asteroidal set grows exponentially, the following observation can be made directly from combining the main result here with Eaton and Barbato's result on cycles.

Observation 3.4 *If an asteroidal configuration of size m is $(\mathcal{K}_{1,3}; t)$-representable, it is not neccessary that every asteroidal configuration of size m or smaller has such a representation. In fact, the size gap can be made arbitrarily large.*

The following fairly obvious observation can be made, however.

Observation 3.5 *If an asteroidal configuration of size m is $(\mathcal{K}_{1,n}; t)$-representable, then there exists an asteroidal configuration of size $m - k$ for each $k \leq m - 3$ which is also $(\mathcal{K}_{1,n}; t)$-representable.*

4 Open Problems

There are still many interesting open problems in the theory of tree representations, as well as some questions stemming from the main result in this paper. Answers to the following questions would be interesting.

Question 4.1 *For small fixed values of n and t, which asteroidal configurations are $(\mathcal{K}_{1,n}; t)$-representable?*

We can already see that the answer is non-trivial. The cycle gives an example showing that even though some large asteroid may be representable, not all smaller asteroidal configurations can be represented. The number of m-asteroidal configurations grows quickly, so given the current tools, it seems that analyzing relatively small cases is in order.

Conjecture 4.2 *A graph G is $(\mathcal{K}_{1,n}; 1)$-representable iff G is chordal, at most n asteroidal, and G satisfies the condition that given any two pairs of vertices from asteroidal sets in G, any path connecting the first pair must be adjacent to any path connecting the second pair (Walter [10]).*

The argument for necessity in Walter's conjecture is straight forward. The argument towards sufficiency, however, seems to require surgical detail, and has not yet been resolved.

Question 4.3 *Exactly which graphs are $(\mathcal{K}_{1,n}; t)$-representable?*

A complete characterization of the class of $(\mathcal{K}_{1,n}; t)$-representable graphs is the ultimate goal here. The fact that Walter's 1972 conjecture still remains undecided may be a good indicator that a full solution is still out of reach, however.

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