Interlace Polynomials of n-Claw Graphs

Sarita Nemani
Department of Mathematics and Computer Science
Georgian Court University, Lakewood, NJ 08701, USA
nemanis@georgian.edu

Aihua Li
Department of Mathematical Science, Montclair State University
1 Normal Avenue, New Jersey 07043, USA
lia@mail.montclair.edu

Abstract

In this paper, we present the study of the interlace polynomials for n-claw graphs. For a positive integer $n > 1$, an n-claw graph W_n is a tree that has one center vertex and n claws. The center vertex is connected to one vertex of each of the n claws using one edge of the claw. We present iterative formulas and explicit formulas for the interlace polynomial of W_n. Furthermore, some interesting properties of the polynomial are discussed.

1 Introduction

In this paper, the set of vertices of a graph G is denoted by $V(G)$ and the set of edges by $E(G)$. For $a \in V(G)$, $G \setminus \{a\}$ is the resulting graph after removing the vertex a and all edges of G connected to a.

Consider an undirected graph G and $a, b \in V(G)$ with $ab \in E(G)$. The edge ab divides the set of vertices, $V(G) \setminus \{a, b\}$, into four sets:

$$\begin{aligned} V_a(G) &= \{c \in V(G) \mid ac \in E(G), bc \notin E(G)\}, \\ V_b(G) &= \{c \in V(G) \mid bc \in E(G), ac \notin E(G)\}, \\ V_{ab}(G) &= \{c \in V(G) \mid ac, bc \in E(G)\}, \text{ and} \\ V'_{ab}(G) &= \{c \in V(G) \mid ac, bc \notin E(G)\}. \end{aligned}$$

Note that $V(G) \setminus \{a, b\} = V_a(G) \cup V_b(G) \cup V_{ab}(G) \cup V'_{ab}(G)$, where the unions are disjoint. Now, let us recall the toggling process and then pivoting a graph [3].

Definition 1.1 *(Toggle process)*
Toggling the pair u, v in $V(G)$ means obtaining a new graph G' such that $V(G') = V(G)$ and $uv \in E(G')$ if and only if $uv \notin E(G)$, keeping the rest of the graph unchanged.

Definition 1.2 *(Pivot process)*
Pivoting G on an edge ab simply means obtaining a new graph G^{ab} from G by toggling every pair u, v such that the vertices u and v are from two different sets $V_a(G), V_b(G)$ and $V_{ab}(G)$, keeping the rest of the graph unchanged.

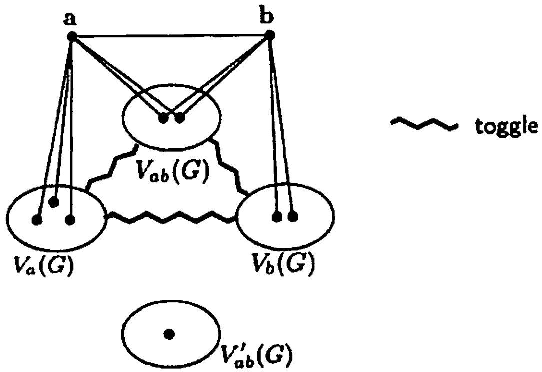

Figure 1: Pivot Process

The interlace polynomial of a graph G is defined iteratively:

Definition 1.3 *(Interlace Polynomial)*
For any undirected graph G with n vertices, the interlace polynomial $q(G, x)$ *of G is defined by*

$$q(G,x) = \begin{cases} x^n & \text{if } E(G) = \emptyset; \\ q(G \setminus \{a\}, x) + q(G^{ab} \setminus \{b\}, x) & \text{if } ab \in E(G). \end{cases}$$

Some basic known results are given below. Proof can be found in [3].

Proposition 1.4 *[3]*

1. *The map defined above gives a well defined polynomial on all simple graphs.*
2. *The interlace polynomial of any simple graph has zero constant term.*
3. *For any two disjoint graphs G_1 and G_2,*

$$q(G_1 \cup G_2, x) = q(G_1, x)\, q(G_2, x).$$

4. For any path P_n on n edges, the interlace polynomial is given by

$$q(P_1,x)=2x, \quad q(P_2,x)=x^2+2x, \quad \text{and}$$

$$q(P_n,x)=q(P_{n-1},x)+xq(P_{n-2},x) \text{ for } n\geq 3.$$

In this paper we will be developing the interlace polynomial of a special graph called n-claw graph, which is defined as follows:

Definition 1.5 *(n-Claw Graph)*
An n-claw graph is a graph, denoted by W_n, with the set of vertices and set of edges as follows:

$$V(W_n)=\{c,a_i,b_{i,j}\,|\,1\leq i\leq n, j=1,2\} \qquad \text{and}$$

$$E(W_n)=\{ca_i,a_ib_{i,j}\,|\,1\leq i\leq n, j=1,2\}.$$

Clearly, W_n is a tree with $|V(W_n)|=3n+1$ and $|E(W_n)||=3n$. Figure 2 shows the 4−claw graph.

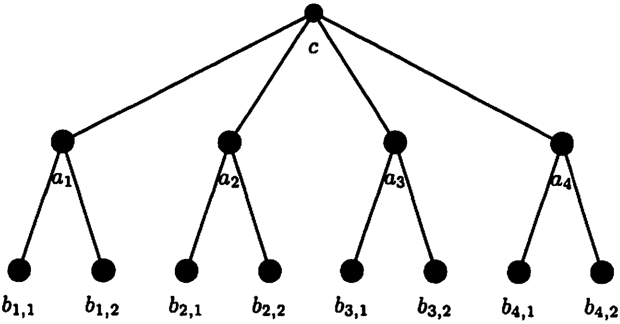

Figure 2: 4-Claw Graph

2 Iterative Formulas for the Interlace Polynomial of n-claw graphs

First, let us develop the interlace polynomial of W_n for small values of n.

Proposition 2.1 *Let $q_n(x)$ be the interlace polynomial of the n-claw graph W_n. (That is, $q(W_n,x)=q_n(x)$.)*

1. $q_0(x)=x$.
2. $q_1(x)=x^3+x^2+2x$.
3. $q_2(x)=x^5+2x^4+5x^3+5x^2+2x$.

Proof.

1. For $n=0$, we get $V(W_0)=\{c\}$ and $E(W_0)=\phi$. Hence we get the result.

2. For $n = 1$,

$$V(W_1) = \{c, a_1, b_{1,1}, b_{1,2}\} \text{ and } E(W_1) = \{ca_1, a_1b_{1,1}, a_1b_{1,2}\}.$$

When we pivot the graph W_1 on the edge ca_1 we get $W_1^{ca_1} = W_1$, since $V_c(W_1) = V_{a_1}(W_1) = \phi$. Thus $W_1^{ca_1} \setminus \{a_1\}$ is the graph with 3 vertices without edges. This means $q(W_1^{ca_1}, x) = x^3$. Next, $W_1 \setminus \{c\}$ is the path P_2, therefore, $q(W_1 \setminus \{c\}, x)$ is $x^2 + 2x$. The result follows from Proposition (1.4) and Definition (1.5).

Next, we find the iterative formula for interlace polynomial of W_n in general.

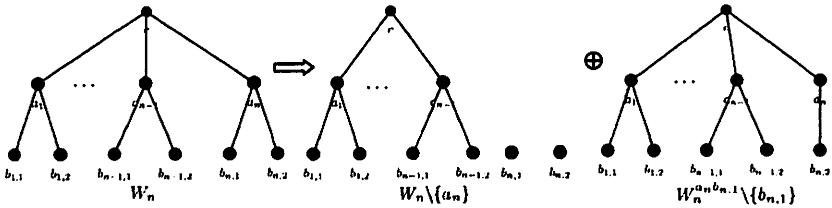

Figure 3: Decomposition of W_n with respect to $a_nb_{n,1}$

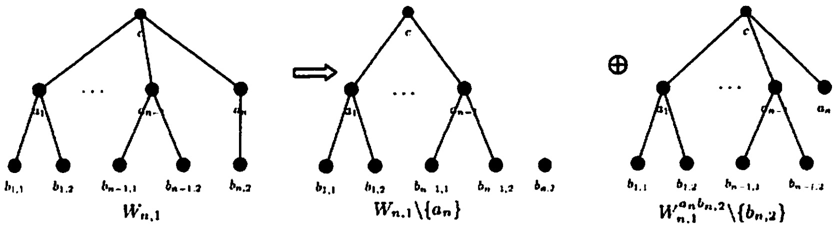

Figure 4: Decomposition of $W_{n,1}$ with respect to $a_nb_{n,2}$

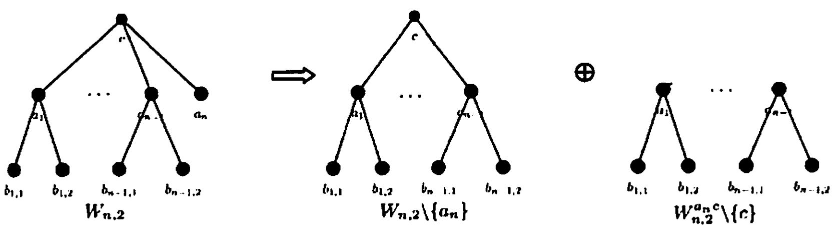

Figure 5: Decomposition of $W_{n,2}$ with respect to a_nc

Theorem 2.2 *For $n \geq 1$,*

$$q_n(x) = (x^2 + x + 1)q_{n-1}(x) + x^n(x+2)^{n-1}.$$

Proof.

To find the interlace polynomial of W_n, we pivot the graph W_n on the edge $a_n b_{n,1}$. See Figure 3. Then by Definition (1),

$$q_n(x) = q\left(W_n \backslash \{a_n\}, x\right) + q\left(W_{n,1}, x\right),$$

where $W_{n,1} = W_n^{a_n b_{n,1}} \backslash \{b_{n,1}\}$.

Here $W_n \backslash \{a_n\}$ is the disjoint union of W_{n-1} and $\{b_{n,1}, b_{n,2}\}$ (no edge). Therefore,

$$q(W_n \backslash \{a_n\}, x) = x^2 q_{n-1}(x).$$

To find $q\left(W_{n,1}, x\right)$, we pivot the graph $W_{n,1}$ on the edge $a_n b_{n2}$. See Figure 4. Then again by Definition (1.5), we get

$$q\left(W_{n,1}, x\right) = q\left(W_{n,1} \backslash \{a_n\}, x\right) + q\left(W_{n,2}, x\right)$$

where $W_{n,2} = W_{n,1}^{a_n b_{n,2}} \backslash \{b_{n,2}\}$.

Since $W_{n,1} \backslash \{a_n\}$ is the disjoint union of W_{n-1} and $\{b_{n,2}\}$,

$$q\left(W_{n,1} \backslash \{a_n\}\right) = x q_{n-1}(x).$$

To find $q\left(W_{n,2}, x\right)$, we pivot the graph $W_{n,2}$ on the edge $a_n c$. See Figure 5. Then

$$q\left(W_{n,2}, x\right) = q\left(W_{n,2} \backslash \{a_n\}, x\right) + q\left(W_{n,2}^{a_n c} \backslash \{c\}, x\right).$$

Now $W_{n,2} \backslash \{a_n\}$ is nothing but W_{n-1}. Therefore,

$$q\left(W_{n,2} \backslash \{a_n\}, x\right) = q_{n-1}(x).$$

Note that $W_{n,2}^{a_n c} \backslash \{c\}$ consists of n disjoint components obtained by

$$\{c\}, \{a_1 b_{1,1}, a_1 b_{1,2}\}, \ldots, \text{ and } \{a_{n-1} b_{n-1,1}, a_{n-1} b_{n-1,2}\},$$

which are isomorphic to P_2. Using Proposition (1.4),

$$q\left(W_{n,2}^{a_n c} \backslash \{c\}\right) = x q(P_2, x)^{n-1} = x(x^2 + 2x)^{n-1}.$$

Corollary 2.3 *Let $q_n(x)$ be as defined earlier.*

1. $q_3(x) = x^7 + 3x^6 + 9x^5 + 16x^4 + 16x^3 + 7x^2 + 2x$.
2. *The degree of $q_n(x)$ is $2n+1$.*
3. *The constant term of $q_n(x)$ is always* 0.

Proof.

1. Using the iterative formula obtained in Theorem (2.2) for $n = 3$,

$$\begin{aligned} q_3(x) &= (x^2 + x + 1) q_2(x) + x^3 (x+2)^2 \\ &= (x^2 + x + 1)(x^5 + 2x^4 + 5x^3 + 5x^2 + 2x) + x^3(x+2)^2 \\ &= x^7 + 3x^6 + 9x^5 + 16x^4 + 16x^3 + 7x^2 + 2x. \end{aligned}$$

2. We use induction on n. From Proposition (2.1) the result is true for $n = 0, 1, 2$. Assume the result for $n-1$, that is, the degree of $q_{n-1}(x)$ is $2n-1$. Then Theorem (2.2) shows that the leading coefficient of $q_n(x)$ is the leading coefficient of $x^2 q_{n-1}(x)$. Therefore, the degree of $q_n(x)$ is $2n+1$.

3. Once again using induction on n and with the help of Proposition (2.1) and Theorem (2.2), $q_n(0) = 0$ for all $n \geq 0$. Thus the constant term of the polynomial is always 0. Note that this confirms the known result for all simple graphs. (See statement (2) of Proposition (1.4)).

3 An Explicit Formula for $q_n(x)$

Since the degree of $q_n(x)$ is $2n+1$, let us rewrite the interlace polynomial of the n-claw graph W_n as

$$q_n(x) = a_{n,(2n+1)} x^{2n+1} + \cdots + a_{n,1} x \text{ for } n \geq 1.$$

Obviously, by Proposition (2.1), $q_0(x) = x$ means $a_{0,1} = 1$. Also we have $a_{1,1} = 2$, $a_{1,2} = a_{1,3} = 1$, $a_{2,1} = 2$, $a_{2,2} = a_{2,3} = 5$, $a_{2,4} = 2$, and $a_{2,5} = 1$. Using the iterative formula obtained in Theorem (2.2) and comparing the coefficients of the corresponding terms, we obtain the following result.

Theorem 3.1 *Let $n > 1$. The sequence $\{a_{n,k} \mid n > 1, 1 \leq k \leq 2n+1\}$, as denoted above, satisfies the following recursive relations:*

(1) $a_{n,1} = a_{(n-1),1}$

(2) For $n > 2$, $a_{n,2} = a_{(n-1),1} + a_{(n-1),2}$.

(3) For $3 \leq k \leq n-1$, $a_{n,k} = a_{(n-1),(k-2)} + a_{(n-1),(k-1)} + a_{(n-1),k}$.

(4) $a_{n,(2n)} = a_{(n-1),(2n-2)} + a_{(n-1),2(n-1)}$ *and* $a_{n,(2n+1)} = a_{(n-1),2(n-1)}$.

Proof.
From the iterative formula $q_n(x) = (x^2+x+1)q_{n-1}(x) + x^n(x+2)^{n-1}$, the x^{2n+1}-term and x^{2n}-term of $q_n(x)$ are those of $(x^2+x+1)q_{n-1}$, thus (1) and (2) are true. Similarly, for $3 \leq k \leq n-1$, $k = 2n$, or $k = 2n+1$, $(x^2+x+1)q_{n-1}$ is the only part in $q_n(x)$ contributing to the x^k-term. Thus (3) and (4) are satisfied.

Let us use these recursive relations to describe some coefficients of the interlace polynomial.

Theorem 3.2 *Let $n > 1$. The following coefficients of the polynomial $q_n(x)$ are determined as follows:*

1. *The leading coefficient is* $a_{n,(2n+1)} = 1$.
2. *The coefficient of x is always 2, that is,* $a_{n,1} = 2$.
3. *The coefficient of x^{2n} is* $a_{n,(2n)} = n$.

4. *The coefficient of x^2 is $a_{n,2} = 2n+1$.*
5. *The coefficient of x^3 is $a_{n,3} = (n+1)^2$ for $n \geq 3$.*

Proof.

1. Using the recursive relation given in Theorem 3.1(4),we get

$$a_{n,(2n+1)} = a_{(n-1),(2n-1)} = a_{(n-2),(2n-3)} = \cdots = a_{1,3} = 1.$$

2. By the recursive relation $a_{n,1} = a_{(n-1),1}$ and the fact from Proposition (2.1) that $a_{2,1} = 2$, we get the result.
3. To find the coefficient of x^{2n}, we use the recursive relation for $a_{n,(2n)}$ and the facts that the leading coefficient of $q_{n-1}(x)$ is $a_{(n-1),(2n-1)}$, and $q_{1,2} = 1$. By Mathematics Induction, Thus $a_{n,(2n)} = a_{(n-1),(2n-2)} + 1 = n$.
4. By Theorem 3.1(2), $a_{n,2} = a_{(n-1),2}+2$ since $a_{(n-1),1} = 2$. Also, we know from $q_2(x)$ that $a_{2,2} = 5$. Thus By induction, $a_{n,2} = 2(n-2) + a_{2,2} = 2n-4+5 = 2n+1$.
5. By Theorem 3.1(3) and the known formulas $a_{(n-1),2} = 2(n-1)+1 = 2n-1$ and $a_{(n-1),1} = 2$, we obtain $a_{n,3} = 2n+1+a_{(n-1),3}$. Now we apply induction on n to prove the result. The result is true for $n = 3$ since $a_3 = 16$ by Proposition (2.1). Let us assume that the result is true for $n-1$ $(n \geq 4)$, that is, $a_{n-1,3} = n^2$. Then $a_{n,3} = 2n+1+n^2 = (n+1)^2$.

Now let us find the interlace polynomial explicitly.

Theorem 3.3 *Let $q_n(x)$ be as before, $n \geq 1$. Then*

$$q_n(x) = \begin{cases} 3^{n-1}(n+3) & \text{for } x = 1; \\ (x^2+x+1)^{n-1}(x^3+x^2+2x) + \\ \frac{x^2(x+2)\left[(x^2+2x)^{n-1} - (x^2+x+1)^{n-1}\right]}{x-1} & \text{for } x \neq 1. \end{cases}$$

Proof.

1. Substituting $x = 1$ in the iterative formula obtained in Theorem (2.2), we get

$$\begin{aligned} q_n(1) &= 3q_{n-1}(1) + 3^{n-1} \\ &= 3[3q_{n-2}(1) + 3^{n-2}] + 3^{n-1} \\ &= 3^2 q_{n-2}(1) + 2 \times 3^{n-1} \\ &\ \ \vdots \\ &= 3^{n-1}q_1(1) + (n-1) \times 3^{n-1} \\ &= 3^{n-1} \times 4 + (n-1) \times 3^{n-1} \text{ (by Proposition (2.1))} \\ &= 3^{n-1}(n+3). \end{aligned}$$

2. To show this result, we use mathematical induction on n. Let $u = x^2 + x + 1$ (for simplicity, ignore the variable x) and $v_n(x) = u^{n-1}(x^3 + x^2 + 2x) + x^2(x+2)[(x^2+2x)^{n-1} - u^{n-1}]/(x-1)$, the right hand side expression of the formula we are proving. We want to show that $q_n(x) = v_n(x)$ for all positive integers. Obviously, $v_1(x) = x^3 + x^2 + 2x = q_1(x)$. Assume the formula is true for n, that is, $q_n(x) = v_n(x)$. By the iterative formula in Theorem (2.2)

$$\begin{aligned} q_{n+1}(x) &= uq_n(x) + x^{n+1}(x+2)^n \\ &= u\left[u^{n-1}(x^3 + x^2 + 2x) + \frac{(x^3 + 2x^2)\left[(x^2 + 2x)^{n-1} - u^{n-1}\right]}{x-1}\right. \\ &\quad +x(x^2 + 2x)^n \\ &= u^n(x^3 + x^2 + 2x) - \frac{u^n(x^3 + 2x^2)}{x-1} \\ &\quad + \frac{u(x^3 + 2x^2)(x^2 + 2x)^{n-1}}{x-1} + x(x^2 + 2x)^n. \end{aligned}$$

Note that $u = x^2 + x + 1 = x^2 + 2x - (x-1)$, we have

$$\begin{aligned} &\frac{u(x^3 + 2x^2)(x^2 + 2x)^{n-1}}{x-1} + x(x^2 + 2x)^n \\ = &\frac{(x^2 + 2x - (x-1))(x^3 + 2x^2)(x^2 + 2x)^{n-1}}{x-1} + x(x^2 + 2x)^n \\ = &\frac{(x^3 + 2x^2)(x^2 + 2x)^n - (x-1)(x^3 + 2x^2)(x^2 + 2x)^{n-1}}{x-1} + x(x^2 + 2x)^n \\ = &\frac{(x^3 + 2x^2)(x^2 + 2x)^n}{x-1} - x(x^2 + 2x)(x^2 + 2x)^{n-1} + x(x^2 + 2x)^n \\ = &\frac{(x^3 + 2x^2)(x^2 + 2x)^n}{x-1}. \end{aligned}$$

It gives

$$q_{n+1}(x) = u^n(x^3 + x^2 + 2x) + \frac{x^2(x+2)\left[(x^2 + 2x)^n - u^n\right]}{x-1} = v_{n+1}(x).$$

Thus, $q_{n+1}(x) = v_{n+1}(x)$ is true for all positive integers n and all real numbers $x \neq 1$.

From the formula above, which is in the rational form, we can develop a formula for $q_n(x)$ in the polynomial form.

Theorem 3.4 *For $n \geq 1$ and $x \in \mathbb{R}$,*

$$q_n(x) = x(x^2 + x + 1)^n + x\sum_{k=0}^{n-1}\binom{n}{k}(x^2 + x + 1)^k(x-1)^{n-k-1}.$$

Proof.
Refer to formula obtained in Theorem (3.3). Set $u = x^2 + x + 1$. Then,

$$
\begin{aligned}
q_n(x) &= \frac{u^{n-1}(x^3+x^2+2x)(x-1)+x^2(x+1)\left[(x^2+2x)^{n-1}-u^{n-1}\right]}{x-1} \\
&= \frac{x}{x-1}\left[u^{n-1}(x^3+x-2)+(x^2+2x)\left((x^2+2x)^{n-1}-u^{n-1}\right)\right] \\
&= \frac{x}{x-1}\left[u^{n-1}\left((x-1)(x^2-1)-3\right)+(x^2+2x)^n\right] \\
&= u^{n-1}(x^3-x)+\frac{x}{x-1}\left[(x^2+2x)^n-3u^{n-1}\right] \\
&= u^{n-1}(x^3-x)+\frac{x}{x-1}\left[\left(\sum_{k=0}^{n}\binom{n}{k}u^k(x-1)^{n-k}\right)-3u^{n-1}\right] \\
&= u^{n-1}(x^3-x)+x\sum_{k=0}^{n-1}\binom{n}{k}u^k(x-1)^{n-k-1}+\frac{xu^{n-1}}{x-1}(u-3) \\
&= u^{n-1}(x^3-x)+u^{n-1}(x^2+2x)+x\sum_{k=0}^{n-1}\binom{n}{k}u^k(x-1)^{n-k-1} \\
&= xu^n+x\sum_{k=0}^{n-1}\binom{n}{k}u^k(x-1)^{n-k-1} \\
&= x(x^2+x+1)^n+x\sum_{k=0}^{n-1}\binom{n}{k}(x^2+x+1)^k(x-1)^{n-k-1}.
\end{aligned}
$$

We now give explicit formulas for the coefficients of the important component $x(x^2+x+1)^n$ in $q_n(x)$. It can be shown by applying the binomial formula. We skip the proof.

Lemma 3.5 *For $n+1 \le m \le 2n+1$, the coefficient of the x^m-term of $x(x^2+x+1)^n$ is given by*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k}{m-2n+2k-1}.$$

By Theorem (3.4) and Lemma (3.5), we can confirm that $a_{n,2n+1}=1$ and $a_{n,2n}=n$ shown by other methods before. Also, it is easy to obtain $a_{n,2n-1}$:

Corollary 3.6 *For $n>1$, $a_{n,(2n-1)}=n(n+3)/2$.*

Proof.

In the formula given in Lemma (3.5), for $m=2n-1$, the only nonzero terms are from $k=1$ or $k=2$. It gives the term:

$$\left[\binom{n}{1}\binom{1}{0}+\binom{n}{2}\binom{2}{2}\right]x^{2n-1}=\frac{n(n+1)}{2}x^{2n-1}.$$

By Theorem (3.4), the summation part has one term of x^{2n-1}, which is the leading term, when $k=n-1$:

$$\binom{n}{n-1}x^{2n-1}.$$

Thus, the x^{2n-1}-term of $q_n(x)$ is given by:

$$a_{n,(2n-1)} = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}.$$

Using the formula given by Theorem (3.4) one can find $q_n(x)$ for any values of x. Some of them are listed in the following corollary:

Corollary 3.7 *Let $q_n(x)$ be as before, $n \geq 1$. Then*

1. $q_n(-1) = \frac{1}{2}\left((-1)^n - 3\right)$.
2. $q_n(-2) = -8 \cdot 3^{n-1}$.
3. $q_n(2) = 2^{3n+1}$.

4 A Matrix Application

In [4], it is shown that the interlace polynomial value of a graph at -1 is the rank of a matrix derived from the adjacent matrix of the graph.

Theorem 4.1 *[4] Let A_n be the $n \times n$ adjacent matrix of a graph G with n vertices and $r = rank(I_n + A_n)$ over $\mathbb{F}_2$ of the field of characteristic 2, where I_n is the $n \times n$ identity matrix. Suppose $q(G, x)$ is the interlace polynomial of G. Then*

$$q(G, -1) = (-1)^n(-2)^{n-r}.$$

Theorem 4.2 *Let A_{3n+1} be the $(3n+1) \times (3n+1)$ adjacent matrix of W_n and $r_n = rank(A_{3n+1} + I_{3n+1})$ over $\mathbb{F}_2$. Then r_n is odd. In fact,*

$$r_n = \begin{cases} 3n & \text{if } n \text{ is odd} \\ 3n+1 & \text{if } n \text{ is even.} \end{cases}$$

Proof.
Note that $|V(W_n)| = 3n + 1$. By Corollary (3.7) and Theorem (4.1),

$$\begin{aligned} q_n(-1) &= [(-1)^n - 3]/2 \\ &= (-1)^{3n+1}(-2)^{3n+1-r_n} \end{aligned}$$

Thus,

$$(-1)^n - 3 = (-1)^{r_n} 2^{3n-r_n+2}.$$

Furthermore, $(-1)^n - 3$ is always negative, whose value is -2 when n is even and -4 when n is odd. Thus r_n has to be odd. Comparing both sides, we obtain that $r_n = 3n + 1$ when n is even and $r_n = 3n$ when n is odd.

Corollary 4.3 *Let M_{3n+1} be the symmetric matrix below:*

$$M_{3n+1} = \begin{bmatrix} B & 0 & 0 & \cdots & \cdots & 0 & C \\ 0 & B & 0 & \cdots & \cdots & 0 & C \\ 0 & 0 & B & 0 & \cdots & 0 & C \\ & & \ddots & \ddots & & & \vdots \\ 0 & \cdots & & \cdots & 0 & B & C \\ C^T & C^T & C^T & \cdots & \cdots & C^T & D \end{bmatrix}_{(3n+1)\times(3n+1)},$$

where

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad D = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then over the field $\mathbb{F}_2$,

$$rank(M_{3n+1}) = \begin{cases} 3n & \text{if } n \text{ is odd} \\ 3n+1 & \text{if } n \text{ is even.} \end{cases}$$

Proof.
It is straightforward to see that the matrix $A_{3n+1} + I_{3n+1}$ in Theorem (4.2) is equal to M_{3n+1}. Then the result follows.

It is interesting to compare this simple proof, which uses a graph theory result, with a proof by linear algebra techniques.

Proof. (linear algebra proof)
Let

$$U_{3n+1} = \begin{bmatrix} I & 0 & \cdots & \cdots & 0 & 0 \\ 0 & I & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I & \cdots & 0 & 0 \\ & \ddots & \ddots & & & \vdots \\ 0 & \cdots & & 0 & I & 0 \\ -C^TB^{-1} & -C^TB^{-1} & \cdots & \cdots & -C^TB^{-1} & I \end{bmatrix}_{(3n+1)\times(3n+1)}.$$

We have

$$U_{3n+1}M_{3n+1} = \begin{bmatrix} B & 0 & 0 & \cdots & \cdots & 0 & C \\ 0 & B & 0 & \cdots & \cdots & 0 & C \\ 0 & 0 & B & 0 & \cdots & 0 & C \\ & & \ddots & \ddots & & & \vdots \\ 0 & \cdots & & \cdots & 0 & B & C \\ 0 & 0 & 0 & \cdots & \cdots & 0 & D-(n-1)C^TB^{-1}C \end{bmatrix}.$$

Then

$$\text{rank}(M_{3n+1}) = 3n + \text{rank}(D-(n-1)C^TB^{-1}C).$$

Note that , $D-(n-1)C^TB^{-1}C \equiv D-C^TB^{-1}C$ over $\mathbb{F}_2$ and is a 4×4 invertible matrix (of full rank). Thus when n is odd, it implies that over $\mathbb{F}_2$,

$$\text{rank}(M_{3n+1}) = 3(n-1) + \text{rank}(D) = 3(n-1) + 3 = 3n.$$

On the other hand, when n is even, we get,

$$\text{rank}(M_{3n+1}) = 3(n-1) + \text{rank}\left(D - C^TB^{-1}C\right) = 3(n-1) + 4 = 3n+1.$$

One advantage of the linear algebra method is that the determinant of the matrix M_{3n+1} can be calculated, in addition to the computation of the rank.

Corollary 4.4 *Let M_{3n+1} be as above. Then*

$$|M_{3n+1}| = \begin{cases} 0 & \textit{if } n \textit{ is odd} \\ (-1)^{n-1} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof.
Note that in the proof of the last corollary the matrix U is introduced and $|U|=1$. By calculation, $|B|=-1$ and $|D-C^TB^{-1}C|=1$. Therefore, when n is even, over $\mathbb{F}_2$,

$$|U|\cdot|M_{3n+1}| = |B|^{n-1}\cdot|D-C^TB^{-1}C| = (-1)^{n-1}.$$

When n is odd,

$$|U|\cdot|M_{3n+1}| = |B|^{n-1}|D| = 0.$$

References

[1] M. Aigner, H. Holst,Interlace polynomials, *Linear Algebra and its Applications,* 377 (2004) 11-30.

[2] R. Arratia, B. Bollobas, D. Coppersmith, G.B. Sorkin, Euler circuits and DNA sequencing by hybridization, *Discrete Appl. Math.,* 104 (1-3) (2000) 63-96 (MR 2001i:92032).

[3] R Arratia, B. Bollobas and G. B. Sorkin, The interlace polynomial of a graph, *Journal of Combinatorial Theory,* Series B 92 (2004) 199-233.

[4] P. N. Balister, B. Bollobas, J. Cutler and L. Pebody, The Interlace Polynomial of Graphs at -1, *Europ. J. Combinatorics,* (2002) 23, 761-767.

[5] A. Bouchet, Graph polynomials derived from Tutte's Mrtin polynomials, *Discrete Mathematics,* 302 (2005) 32-38.

[6] J. A. Ellis-Monaghan, Identities for circuit partition polynomials, with applications to the Tutte polynomial, *Advances in Applied Mathematics,* 32 (2004) 188-197.

[7] R. Glantz, M. Pelillo, Graph polynomials from principal pivoting, *Discrete Mathematics,* 306 (2006) 3253-3266.