Random Seidel Switching on Graphs

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Abstract

We consider the random process arising from a sequence of random Seidel switching operations on n vertices. We show that this process can be interpreted as a random walk on a Cayley graph of an abelian group, and use spectral methods to show that the random process converges to a stationary distribution in $O(n \log(n))$ steps. We then consider two generalizations: we allow multiple states for each edge, and restrict the process to a fixed host graph H. We then analyze the general case and obtain convergence results for any graph H.

1 Introduction

Let $G = G(V, E)$ be a finite, simple graph. For a vertex $v \in V$, the operation of *switching at* v transforms G to a new graph G_v by deleting all edges adjacent to v, and adding all potential edges from v to vertices not previously connected. This operation is known as *vertex switching*, *node switching*, or *Seidel Switching*. It was originally introduced by J.H. van Lint and J.J. Seidel [11] as tool to study equilateral point sets in elliptic spaces.

Two graphs G_0 and G_k are said to be *switching equivalent* if there is a sequence of vertices $v_1, v_2, ..., v_k$ such that G_i is obtained by switching v_i in G_{i-1} for $i = 1, ..., k$. It is easy to see that performing the sequence of operations in reverse order will transform G_k back to G_0, and so the relation is both reflexive and symmetric. Also note that this demonstrates each switching operation is invertible, a fact that will be needed later. Transitivity follows immediately from the definition, and so this is an equivalence

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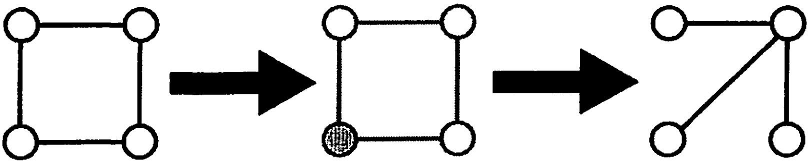

Figure 1: Seidel Switching on 4 vertices. We begin with a graph on 4 vertices, select one vertex, then switch the adjacency relations for that vertex.

relation among graphs on a fixed number of vertices. The equivalence classes are known as *switching classes.* The number of switching classes on n vertices is equal to the number of two-graphs with n vertices [10] as well as Euler graphs [7].

Seidel Switching has applications to spectral graph theory as illustrated by the following theorem, also due to Seidel [9].

Theorem 1.1. *Let G and G' be two regular graphs of degree d in the same switching class. Then G and G' are cospectral.*

In this paper, we consider a randomized switching process. At each step we randomly select a vertex, and apply a switching operation at that vertex. We analyze this random process, and obtain results about the times of convergence to the stationary distribution. Our method can be outlined as follows:

1. The set of compositions of switching operations is an abelian group, which we denote $\Gamma(n)$.
2. There is an isomorphism $\nu\colon \Gamma(n) \to \mathbb{Z}_2^{n-1}$
3. The random process can be understood as a lazy random walk on a Cayley graph of $\Gamma(n)$.
4. The eigenvalues of the transition matrix of this random walk can be found using the irreducible representations of $\Gamma(n)$, and can be used to bound the convergence time to the stationary distribution.

We remark that this same outline can be used to analyze other processes on graphs, and a paper using the same techniques to analyze a randomized version of the Lights Out process [1,5] on graphs is in preparation.

This paper is organized as follows. In Section 2 we state known results concerning Markov chains, random walks on graphs, and representation theory that we will use. In Section 3 we formally introduce randomized Seidel Switching, and show that the stationary distribution is uniform and give bounds on the convergence time. We omit proofs in this section as results are special cases of more general results stated in the following section. In Section 4 we introduce two generalizations of Seidel switching. We consider both switching with multiple colors and restricted to a fixed host graph. We then study a randomized version of this generalized switching, again showing how it can be viewed as a random walk on a graph and use spectral methods to bound the convergence times.

2 Preliminaries

We consider switching actions on a finite, simple graph $G = (V, E)$. We let $n = |V|$, the number of vertices of G. For each vertex $v \in V$, we let d_v be the degree of v. For two vertices v and w, we write $v \sim w$ if v and w are adjacent. Let D be the diagonal degree matrix with entries $D_{vv} = d_v$, and let A be the adjacency matrix with entries

$$A_{vw} = \begin{cases} 1 & \text{if } v \sim w \\ 0 & \text{if otherwise.} \end{cases}$$

While initially our graphs are undirected, in our analysis we will consider both directed graphs and graphs with weighted edges. If G has weighted edges, then A_{vw} will encode the weight on the edge between v and w. If G is directed, we will say $v \rightarrow w$ to denote there is an edge from v to w, but not necessarily from w to v.

A *walk* on a graph is a sequence of vertices $(v_0, v_1, ..., v_k)$ where $(v_i, v_{i+1}) \in E$. A *random walk* of length k is a sequence of random vertices $(x_0, ..., x_k)$ where the starting vertex x_0 is chosen according to an initial distribution, and

$$\Pr(x_{i+1} = v | x_i) = \begin{cases} 1/d_{x_i} & \text{if } x_i \sim v \\ 0 & \text{if } x_i \nsim v \end{cases}.$$

The transition probability matrix for a random walk on G is given by $W = D^{-1}A$. As the transition probabilities do not depend on i, this random process is a time-homogeneous Markov chain. For any initial distribution f (viewed as a row vector, which will be the convention throughout this paper) on the vertices of G, the distribution after t steps is given by fW^k.

The process is ergodic if there is a unique stationary distribution π such that for any vertex v,

$$\lim_{t \to \infty} fW^t(v) = \pi(v)$$

for any initial distribution f. The random walk on a strongly connected weighted graph G is ergodic if and only if

1. G is irreducible. That is, for any u, v there is a time t such that $W^t(u, v) > 0$

2. G is aperiodic. That is, the greatest common factor of the set $\{t\colon W^t(u, v)$ $0\}$ is 1.

[4, Section 1.5].

Aperiodicity can be artificially imposed by considering a *lazy random walk*, which has transition matrix $W' = \frac{1}{2}(W + I)$. This can be thought of as either the original random walk where half the time no action is taken (hence the name). Alternately, one can view it as a random walk on a modified graph where each vertex is given a self loop and weights are redistributed accordingly.

One can measure the "distance" from the stationary distribution in a variety of ways. Here we will use the χ-squared distance.

Definition 1. Let W be the transition matrix of an ergodic random walk on a graph G. Let π be the stationary distribution. The χ-squared distance after t steps, $\Delta'(t)$, is defined by

$$\Delta'(t) = \max_{x \in V(G)} \left(\sum_{y \in V(G)} \frac{(W^t(x, y) - \pi(y))^2}{\pi(y)} \right)^{1/2}.$$

We work with this metric because in cases where G has a large amount of symmetry, there is an elegant expression for $\Delta'(t)$ in terms of the spectrum of the transition matrix.

Definition 2. A graph automorphism is a function $f\colon V(G) \to V(G)$ such that for all vertices $x, y \in V(G)$, $A(x, y) = A(f(x), f(y))$.

Definition 3. A graph G is *vertex transitive* if for any two vertices u, v there is a graph automorphism $f\colon V(G) \to V(G)$ such that $f(u) = v$.

Informally, a vertex transitive graph is a graph where vertices are indistinguishable except for labels.

For such vertex transitive graphs, we can bound the Δ' distance, see for example [4, Theorem 1.18].

Theorem 2.1. *Let G be a vertex transitive graph on n vertices, and let W be the transition matrix for an ergodic random walk on G. Let λ_i be the eigenvalues of W, with $-1 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_n = 1$. Then the χ-squared distance after s steps is given by*

$$\Delta'(t) = \left(\sum_{i\neq n} \lambda_i^{2t}\right)^{1/2}.$$

Let Γ be a group, and ω a probability distribution on Γ. The Cayley graph with respect to ω, $Cay(G,\omega)$, is the weighted, directed graph with vertex set Γ and weighted adjacency matrix $A(g,h) = \omega(g^{-1}h)$. For $S \subset G$, $Cay(G,S)$ corresponds to $\omega(g) = \begin{cases} \frac{1}{|S|} & \text{if } g \in S \\ 0 & \text{if } g \notin S \end{cases}$. In this case, $g \to h$ if and only if $g^{-1}h \in S$, i.e. if there is an $s \in S$ such that $h = gs$. We say that S is symmetric in G if $h \in S \Rightarrow h^{-1} \in S$, and similarly that ω is symmetric if $\omega(h) = \omega(h^{-1})$ for all h. When ω is symmetric then $Cay(G,\omega)$ is undirected. We will use the following well known fact about presentations of abelian groups, see for example [6]:

Theorem 2.2. *Let Γ be a finite abelian group. Any irreducible representation of a finite abelian group is one dimensional. Let $\rho\colon \Gamma \to \mathbb{C}$ be a irreducible representation of Γ, and let W be the probability transition matrix of a lazy random walk on $Cay(G,\omega)$. If ρ is viewed as a row vector in $\mathbb{C}^{|\Gamma|}$, then ρ is an eigenvector with eigenvalue*

$$\lambda_\rho = \frac{1}{2} + \sum_{g\in\Gamma} \omega(g)\rho(-g).$$

Since ω is a probability distribution, the weighted degree of every vertex of $Cay(G,\omega)$ is one, and so $D = I$. Therefore $W = \frac{1}{2}(I+D^{-1}A) = \frac{1}{2}(I+A)$, and so the spectrum of W is simply the spectrum of A shifted by $\frac{1}{2}$. Thus the problem of determining the eigenvalues of a random walk on the Cayley graph of an abelian group comes down to understanding the irreducible characters. Thankfully, these are simple to compute due to the following theorem [6]

Theorem 2.3. *Let $\Gamma = \mathbb{Z}_{n_1} \times \ldots \times \mathbb{Z}_{n_k}$, and let $\theta_q = e^{\frac{2\pi i}{q}}$ for any positive integer q. For each $\vec{x} \in \Gamma$ define $\rho_{\vec{x}}\colon \Gamma \to \mathbb{C}$ be the homomorphism where $\rho_{\vec{x}}(e_i) = \theta_{n_i}$ where e_i is the cartesian product of the the additive generator 1 in the i-th group, and the identity 0 in all other groups. Then $\rho_{\vec{x}}$ is an irreducible character of Γ, and moreover every irreducible character of Γ is $\rho_{\vec{x}}$ for some $\vec{x} \in \Gamma$.*

3 Randomized Seidel Switching

We wish to consider the behavior of the random process arising from a random sequence of switchings on graphs with n vertices. It will be useful to think of the switching actions as functions on the state space of all graphs on n vertices. Let $G(n)$ denote the set of all graphs on n labeled vertices. Let $s_v\colon G(n) \to G(n)$ be the action of switching at vertex v, so that $s_v(G) = G_v$ for any graph G. We let $s_\emptyset$ denote the identity function, and let $\overline{K_n} \in G(n)$ denote the empty graph on n vertices.

Let $\{x_t\}_1^\infty$ be a sequence of vertices, independently chosen uniformly at random among the n vertices. Consider the random process $\hat{X}(t)$ where $\hat{X}(0) = \overline{K_n}$, and for each $t \geq 1$, $\hat{X}(t+1) = s_{x_t}(\hat{X}(t))$. As we will see later, this sequence may be periodic and so will not converge to a stationary distribution. To eliminate this concern, we consider a "lazy version" of this process where half the time no action is taken. We define the sequence $X(t)$ where $X(0) = \overline{K_n}$, and $X(t+1) = s_{y_t}(X(t))$ where $\Pr(y_t = \emptyset) = \frac{1}{2}$ and $\Pr(y_t = x_t) = \frac{1}{2}$.

Our goal is to analyze $X(t)$. In particular, we wish to consider the following questions.

1. How can we understand or interpret $X(t)$?
2. What is the stationary distribution of $X(t)$?
3. How fast does $X(t)$ converge to its stationary distribution?

We begin by examining the algebraic structure of compositions of switching functions s_v. Proofs in this section are omitted, as the results stated are special cases of results that appear in Section 4

Theorem 3.1. *Let $\Gamma(n)$ be the group of all compositions of the switching operators $\{s_{v_i}\}_{i=1}^n$ with respect to composition. Then $\Gamma(n) \cong \mathbb{Z}_2^{n-1}$*

For our analysis, we will need to explicitly understand the isomorphism between the groups, and in particular the image of the switching functions s_v.

For the groups $\mathbb{Z}_2 \times ... \times \mathbb{Z}_2$, let e_i correspond to the cartesian product of the the additive generator 1 in the i-th group, and the identity 0 in all other groups.

Corollary 3.2. *There exists an isomorphsim $\nu\colon \Gamma(n) \to \mathbb{Z}_2^{n-1}$ with*

$$\nu(s_{v_i}) = \begin{cases} e_i & \text{if } i = 1, ..., n-1 \\ \sum_{i=1}^{n-1} e_i & \text{if } i = n \end{cases}$$

The key idea in our analysis is recognizing that $X(t)$ can be viewed a lazy random walk on a state graph. Moreover, we will show that this state graph is isomorphic to a Cayley graph of $\Gamma(n)$.

We begin by defining the *Switching State Graph* as follows:

Definition 4. The *Switching (State) Graph* of n vertices, denoted SG_n, is a graph with vertex set the set of all graphs on n (labeled) vertices in the same switching class as the empty graph. For two graphs G and H, $G \sim H$ if there is a vertex v such that $s_v(G) = H$.

The notion of the Switching Graph is useful because it allows us to view our random process $X(t)$ as a lazy random walk on a graph. At each step in a random walk on SG_n one moves to a neighbor with probability $\frac{1}{2}$, and stays at the same vertex with probability $\frac{1}{2}$. Since moving to a random neighbor is equivalent to picking a random vertex v and applying the switching function s_v, we have the following proposition.

Proposition 3.3. *The random process $X(t)$ described above is identically distributed to a lazy random walk on the switching graph starting from the empty graph.*

Thus we have reduced the problem of analyzing $X(t)$ to that of understanding the lazy random walk on SG_n. We begin by recognizing that we can fully understand the structure of SG_n in terms of the switching group $\Gamma(n)$.

Proposition 3.4. *The Switching State Graph SG_n is isomorphic to $Cay(\mathbb{Z}_2^{n-1}, T)$, where $T = \{e_i\}_{i=1}^{n-1} \cup \{\sum_{i=1}^{n-1} e_i\}$ and e_i is the standard basis element of $\mathbb{Z}_2^{n-1}$. This is the hypercube of dimension $n-1$ with diagonal chords added.*

Using Propositions 3.3 and 3.4, along with the tools presented in Section 2, we can answer the questions posed at the beginning of this section.

Theorem 3.5. *Let $X(t)$ be the random graph at time t obtained from the randomized Seidel Switching switching process described above. Then*

1. *$X(t)$ converges to a uniform distribution on all graphs in the switching class of $\overline{K_n}$.*

2. *The χ-squared distance from the stationary distribution after t steps is bounded by*

$$\Delta'(t) \leq \left(\sum_{j=1}^{n-1} \binom{n-1}{j} \left(1 - \frac{j}{n}\right)^{2t} \right)^{\frac{1}{2}}$$

3. *$\Delta'(t) \leq e^{-c}$ if $t > \frac{1}{2} n \log(n) + c$ for any $c > 0$*

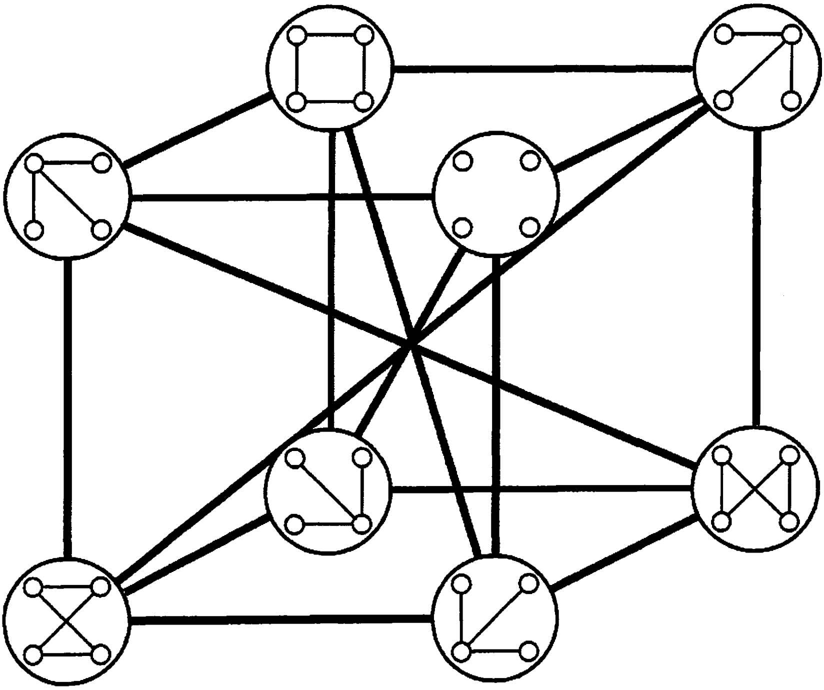

Figure 2: The Switching State Graph on 4 vertices. The "vertices" of the state graph are the graphs on 4 vertices that are in the same switching class as the empty graph. Two graphs are adjacent if they differ by a single switching operation. Note that the graph is bipartite, which demonstrates why we consider a lazy random walk to guarantee convergence.

4 Restricted and Multi-colored Switching

One can generalize the switching operation to colorings of graphs in a number of ways. Consider if instead of two states (off or on) for any of the edges, there are q states (for some fixed integer q). It is natural to think of these as different colors an edge. The switching operation is then some permutation π on the set of potential colors of an edge. Brewster and Graves [2] considered the action of an arbitrary, fixed permutation π and studied homomorphisms between colorings of graphs. Cameron and Tarzi [3] considered the action under all transpositions, as well as restricted cases where not all transpositions were allowed. We will consider the case where π is the cyclic operation "$+1 \mod q$", though many of the techniques generalize to arbitrary permutations.

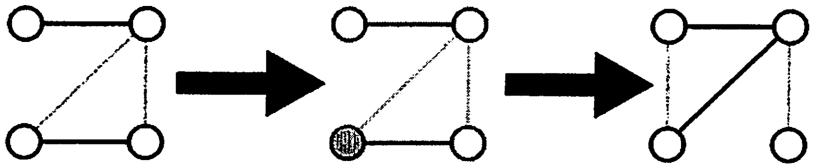

Figure 3: An example of multi-color switching with $q = 3$. No edge corresponds to state 0, a grey edge to state 1, and a black edge to state 2. After selecting bottom left vertex, we increment the state of each adjacent edge by 1 mod 3

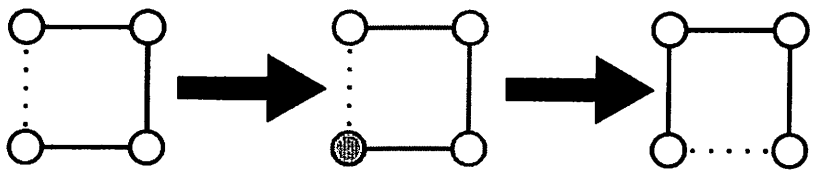

Figure 4: Switching restricted to the host graph $H = C_4$. Note that switching at the bottom left vertex only creates new edges that are in the host graph

We consider a further generalization simultaneously. Previously, any non-edge could become an edge. We consider instead when only certain edges can be created. For a fixed *host graph* H, we can define switching actions that are restricted to colorings of the edges of H.

We let $C_q(H)$ be the set of all edge-colorings of H using q colors, that is all functions $\tau\colon E(H) \to \{0, 1, ..., q-1\}$. We call τ a q-coloring of the edges of H.

Definition 5. We define the q-H-switching of a vertex v to be the operator $s_v\colon C_q(H) \to C_q(H)$ where $s_v(\tau)(e) = \tau(e) + 1 \mod q$. $s_\emptyset$ will refer to the identity map on $C_q(H)$. The operators s_v for $v \in V$ will be called the elementary q-H-switching operators.

Note that the Seidel Switching as explored in Section 3 is q-H-switching with $q = 2$, and $H = K_n$.

4.1 The q-H Switching Group

It is clear from the definition that the elementary switching operators commute and that $s_v^q = s_\emptyset$ for every v. Thus each elementary operator s_v is invertible with $s_v^{-1} = s_v^{q-1}$. Letting $\Gamma_q(H)$ be the set of all compositions of elementary q-H-switching operators, we see the following.

Proposition 4.1. *$\Gamma_q(H)$ is an abelian group under composition.*

We call $\Gamma_q(H)$ the *q-H-switching group*, or just *switching group* if q and H are clear from context. We begin with a somewhat surprising result; we can view $\Gamma_q(H)$ as the vector space over $\mathbb{Z}_q$ spanned by $\{s_v\}_{v\in V}$, where here s_v is viewed as a vector in $\mathbb{Z}_q^E$ with 1 for all edges with endpoint v. Thus, $\Gamma_q(H)$ is the column space over $\mathbb{Z}_q$ of the edge-vertex adjacency matrix. It seems that such an object should vary depending on the structure of the graph H, but it turns out only to depend on whether H is bipartite or not, and on the parity of q.

Theorem 4.2. *Let H be a connected graph on n vertices.*

$$\Gamma_q(H) \cong \begin{cases} \mathbb{Z}_q^{n-1} & \textit{if } H \textit{ is bipartite} \\ \mathbb{Z}_q^n & \textit{if } H \textit{ is not bipartite and } q \textit{ is odd} \\ \mathbb{Z}_q^{n-1} \times Z_r & \textit{if } H \textit{ is not bipartite and } q = 2r \end{cases}$$

Proof. Let us index the n generators of $\mathbb{Z}_q^n$ by the vertices of H, and denote them by vectors $\{f_v\}_{v\in V}$, where f_v corresponds to a 1 in the v coordinate and 0 elsewhere. We define the map $\phi\colon \mathbb{Z}_q^n \to \Gamma_q(H)$ by $\phi(f_v) = s_v$, extended linearly so that ϕ is a homomorphism.

We first note that ϕ is surjective, as any element in $\Gamma_q(H)$ occurs from switching vertices of H some number of times less than q. By the First Isomorphism Theorem, $\Gamma_q(H) \cong \mathbb{Z}_q^n / \ker\phi$.

It remains to analyze $\ker\phi$. We will view $\Gamma_q(H)$ as a module over $\mathbb{Z}_q$. Suppose that $g = \sum_v \alpha_v f_v \in \ker\phi$, where $\alpha_i \in \mathbb{Z}_q$. The value on e is only influenced by s_u and s_w, so if $(u, w) \in E(H)$, then

$$\alpha_u = -\alpha_w. \tag{1}$$

So if $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ is a walk on the vertices of H, then

$$\alpha_{i_l} = (-1)^{l+1}\alpha_{i_1} \tag{2}$$

That is, values alternate between c and $q-c \in \mathbb{Z}_q$ for a fixed c as one moves along the path. We now split into cases based on the structure of H.

Case 1. H is bipartite

Let A and B be the independent sets with respect to which H is bipartite, and without loss of generality suppose that $v_1 \in A$. Let $c = \alpha_1$. Since H is connected, for any vertex v_j, there exists a walk $v_1 = v_{i_1}, v_{i_2}, ..., v_{i_{k_j}} = v_j$ for some $k_j \in \mathbb{N}$. Thus $\alpha_j \equiv (-1)^{k_j-1}\alpha_1$. Because H is bipartite, k_j is even precisely when $v_j \in A$. Therefore $\alpha_v = \begin{cases} c & \text{if } v \in A \\ -c & \text{if } v \in B \end{cases}$. It follows that $\ker\phi = \{\sum_{v\in A} cf_v + \sum_{w\in B}(q-c)f_w\}_{c=0}^{q-1}$, and thus $\ker\phi \cong \mathbb{Z}_q$ and $\Gamma_q(H) \cong \mathbb{Z}_q^n/\mathbb{Z}_q \cong \mathbb{Z}_q^{n-1}$.

Case 2. H contains an odd cycle

Suppose H contains a cycle of length j for some odd integer j. Let v_1 be a vertex in that cycle, and let $v_1 = v_{i_1}, v_{i_2}, ..., v_{i_{j+1}} = v_1$ be a walk around the cycle, starting and ending at v_1. Then $\alpha_1 = (-1)^j\alpha_k = -\alpha_1$. Therefore $2\alpha_1 = 0$.

Subcase 1. q is odd

If q is odd, $2\alpha_1 = 0$ implies $\alpha_1 = 0$. Since H is connected, for every vertex v_i there is a walk from v_1 to v_i. Thus by Equation 2, $\alpha_i = 0$ for all i. Thus $\ker\phi = 0$, so $\Gamma_q(H) \cong \mathbb{Z}_q^n$.

Subcase 2. q is even

If q is even, $q = 2r$, and $2\alpha_1 = 0$ implies $\alpha_1 = 0$, or $\alpha_1 = r$. As above, if $\alpha_1 = 0$, then $\alpha_i = 0$ for all i. Similarly, if $\alpha_1 = r$, $\alpha_i = \pm r$ for all i. But since $q = 2r$, $-r = r$ in $\mathbb{Z}_q$, so $\alpha_i = r$ for all i. Thus $\ker\phi = \{0, \sum_v rf_v\} \cong \mathbb{Z}_2$. Thus $\ker\phi \cong \mathbb{Z}_2$, and therefore $\Gamma_q(H) \cong \mathbb{Z}_q^n/\mathbb{Z}_2 \cong \mathbb{Z}_q^{n-1} \times \mathbb{Z}_r$.

□

For calculations later, it will be necessary to explicitly construct the isomorphism. In particular, we will need to determine the image of the elementary switching operators s_v.

Corollary 4.3.

1. *When H is bipartite with respect to two disjoint subsets $A, B \subset V$, there exists an isomorphsim $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^{n-1}$ with $\nu(s_{v_i}) = e_i$ for $i \leq n-1$, and $\nu(s_{v_n}) = \sum_{v_i\in A, i\neq n} e_i - \sum_{v_j\in B, j\neq n} e_j$.*

2. *When H is not bipartite, and q is odd, there exists an isomorphsim $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^n$ with $\nu(s_{v_i}) = e_i$ for all i.*

3. *When H is not bipartite, and q is even, there exists an isomorphsim $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^{n-1} \times \mathbb{Z}_r$ with $\nu(s_{v_i}) = e_i$ for all i. Note the final generator $e_n = (0, ...0, 1)$ is of order r, whereas all the other e_i are order $q = 2r$.*

Proof. As in the proof of Theorem 4.2, we let $\{f_v\}_{v \in V}$ denote the standard basis of $\mathbb{Z}_q^n$, indexed by the vertices of H. We let $\{e_i\}$ denote the standard generator of $\mathbb{Z}_{n_1} \times ...\mathbb{Z}_{n_k}$ with a 1 in the i-th spot and 0 elsewhere.

To construct ν, we simply follow the standard proof of the First Isomorphism Theorem.

Case 1. H is bipartite

The use of the First Isomorphism Theorem in the proof of Theorem 3.1 above yields the isomorphism $\widetilde{\phi}\colon Z_q^n/\ker\phi \to \Gamma_{n,q}$ given by $\widetilde{\phi}(a + \ker\phi) = \phi(a)$. Thus $\Gamma_q(H)$ corresponds to the cosets of $\ker\phi = \{\sum_{v \in A} cf_v + \sum_{w \in B} (q - c)f_v\}_{c=0}^{q-1}$. Without loss of generality, suppose that $v_n \in B$. There are q elements in each coset, and we define the map $\psi\colon Z_q^n/\ker\phi \to Z_q^{n-1}$ to be the map sending $a + \ker\phi$ to its representative with the coefficient of $e_n = 0$. This is an isomorphism, and so we define $\nu\colon \Gamma_{n,q} \to \mathbb{Z}_q^{n-1} \times Z_r$ to be the composition of $\widetilde{\phi}^{-1}$ and ψ. Tracing back through the composition of maps we see that $\nu(s_{v_i}) = e_i$ for $i \leq n-1$, and $\nu(s_{v_n}) = \sum_{v_i \in A, i \neq n} e_i - \sum_{v_j \in B, j \neq n} e_j$.

Case 2. H is not bipartite, q is odd

Let $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^n$ be the inverse of ϕ, that is $\nu(s_{v_i}) = e_i$ for all i.

Case 3. H is not bipartite, q is even, $q > 2$

As in the first case, the use of the First Isomorphism Theorem yields the isomorphism $\widetilde{\phi}\colon Z_q^n/\ker\phi \to \Gamma_q(H)$ given by $\widetilde{\phi}(a + \ker\phi) = \phi(a)$. Thus $\Gamma_q(H)$ corresponds to the cosets of $\ker\phi = \{0, \sum_i rf_{v_i}\}$. There are two elements in each coset, and we define the map $\tau\colon Z_q^n/\ker\phi \to Z_q^{n-1} \times Z_r$ to be the map sending $a + \ker\phi$ to its unique representative with the coefficient of e_n lying in the set $\{0, 1, ..., r-1\}$. This is an isomorphism, and so we define $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^{n-1} \times Z_r$ to be the composition of $\widetilde{\phi}^{-1}$ and τ. Then $\nu(s_{v_i}) = e_i$ for all i as required.

Case 4. H is not bipartite, $q = 2$ As above, the use of the First Isomorphism Theorem yields the isomorphism $\widetilde{\phi}\colon Z_q^n/\ker\phi \to \Gamma_q(H)$ given by $\widetilde{\phi}(a + \ker\phi) = \phi(a)$. Thus, $\Gamma_q(H)$ corresponds to the cosets of $\ker\phi = \{0, \sum_v f_v\}$. There are two elements in each coset, and we define the map $\tau\colon Z_q^n/\ker\phi \to Z_q^{n-1}$ to be the map sending $a + \ker\phi$ to its unique representative with the coefficient of e_n equal to 0. This is an isomorphism, and so we define $\nu\colon \Gamma_q(H) \to \mathbb{Z}_q^{n-1} \times Z_r$ to be the composition of $\widetilde{\phi}^{-1}$ and τ. Then $\nu(s_{v_i}) = e_i$ for all $i \leq n-1$, and $\nu(s_{v_n}) = \sum_i e_i$.

□

We have identified $\Gamma_q(H)$ for all connected H, but it remains to consider the case when H is disconnected. Fortunately, the switching group decomposes as a product of the switching groups of the connected components in the most natural way possible. Suppose that H is the disjoint union of H_1 and H_2. Then for $v_i \in H_j$, s_{v_i} only changes the color of edges in H_j. Therefore there is no interaction between the switching functions on H_1 and H_2. In other words, we have the following;

Proposition 4.4. *If H is the disjoint union of two subgraphs H_1 and H_2, then $\Gamma_q(H) \cong \Gamma_q(H_1) \times \Gamma_q(H_2)$.*

Proof. We note that if $v \in H_1$, $w \in H_2$, then $s_v \perp s_w$, when viewed as vectors in $\mathbb{Z}_q^{|E|}$. Thus $\text{span}\{s_v\}_{v\in H_1} \perp \text{span}\{s_v\}_{v\in H_2}$, and so

$$\Gamma_q(H) \cong \text{span}\{s_u\}_{u\in H} \cong \text{span}\{s_v\}_{v\in H_1} \oplus \text{span}\{s_v\}_{v\in H_2} \cong \Gamma_q(H_1) \times \Gamma_q(H_2)$$

□

We can now give a classification of the structure of the abelian groups that are isomorphic to $\Gamma_q(H)$ for some q and H.

Theorem 4.5.

1. *Let H be a graph on n vertices. Then there exist $b, c \geq 0$ such that $2b + 3c \leq n$ and*

$$\Gamma_q(H) \cong \begin{cases} \mathbb{Z}_q^{n-b-c} \times \mathbb{Z}_r^j & \text{if } q = 2r \\ \mathbb{Z}_q^{n-c} & \text{if } q \text{ is odd} \end{cases}$$

2. *If $q = 2r$, and $b, c \geq 0$ such that $2b + 3c \leq n$, then there exists a graph H on n vertices such that $\Gamma_q(H) \cong \mathbb{Z}_q^{n-b-c} \times \mathbb{Z}_r^j$. If q is odd and $2b \leq n$, then there exists a graph H on n vertices such that $\Gamma_q(H) \cong \mathbb{Z}_q^{n-b}$*

Proof. Let $H_1 \times ... \times H_k$ be the connected components of H, and let $n_1, ..., n_k$ be the number of vertices in each component. Then by Lemma 4.4, $\Gamma_q(H) \cong \Gamma_{H_1,q} \times ... \times \Gamma_{H_k,q}$. We first consider the case when q is even. By Theorem 3.1,

$$\Gamma_{H_i,q} \cong \begin{cases} Z_q^{n_i-1} \times \mathbb{Z}_r & \text{if } H_i \text{ is not bipartite} \\ \mathbb{Z}_q^{n_i-1} & \text{if } H \text{ is bipartite} \end{cases} .$$

Let b be the number of non-bipartite connected components and c the number of bipartite connected components. Then $\Gamma_q(H) \cong \mathbb{Z}_q^{n-b-c} \times \mathbb{Z}_r^j$. For a component H_i to be bipartite, $n_i \geq 2$, and to be non-bipartite, $n_i \geq 3$. Thus $2b + 3c \leq n$. The case when q is odd is simpler; let $b = 0$ and let c be the number of bipartite components. Then since

$$\Gamma_{H_i,q} \cong \begin{cases} \mathbb{Z}_q^{n_i} & \text{if } H_i \text{ is not bipartite} \\ \mathbb{Z}_q^{n_i-1} & \text{if } H \text{ is bipartite} \end{cases},$$

$\Gamma_q(H) \cong \mathbb{Z}_q^{n-c}$.

For the other half of the proof, suppose $q = 2r$ and $2b + 3c \leq n$. Let H be the disjoint union of c 3-cycles, $b - 1$ edges, and one path of length $n + 2 - 2b - 3c$. Then H is a graph on n vertices and $\Gamma_q(H) \cong \mathbb{Z}_q^{2j} \times \mathbb{Z}_r^j \times \mathbb{Z}_q^{i-1} \times \mathbb{Z}_q^{n+1-2b-3c} \cong \mathbb{Z}_q^{n-b-c} \times \mathbb{Z}_r^j$.

If q is odd, and $2b \leq n$, let H be a disjoint union of $b-1$ edges, and one path of length $n + 2 - 2b$. Then H is a graph on n vertices with switching group $\Gamma_q(H) \cong \mathbb{Z}_q^{b-1} \times \mathbb{Z}_q^{n+1-2b} \cong \mathbb{Z}_q^{n-b}$ □

4.2 Random q-H-Switching

We wish to consider the process generated by a sequence of randomly chosen q-H-switching operators. We introduce one further generalization. While in Section 3 the operators were chosen uniformly at random, we now allow an arbitrary probability distribution on the vertices. Formally, let $\omega\colon V \to (0, 1)$ be a probability distribution on V. Let H be a connected graph with vertex set V, and $q \geq 2$ a positive integer.

Let $\tau_\emptyset$ refer to the 0 coloring of the edges; that is, $\tau_\emptyset(e) = 0$ for all edges $e \in E$. For an elementary switching operation s_v, recall that its inverse $s_v^{-1} = s_v^{q-1}$. Also, recall that $s_\emptyset$ refers to the identity operator.

Let $\{x_t\}_1^\infty$ be a sequence of independent identically distributed random vertices, where $\Pr(x_t = v) = \omega(v)$. We consider the following random process. Starting from the empty coloring, we pick a vertex at random and apply either the q-H-switching operator or its inverse (with equal probability) at that vertex. To guarantee convergence, we consider a lazy random process where half the time no action is taken. Formally, this is the sequence $X(t)$ where $X(0) = \tau_\emptyset$, and $X(t+1) = \sigma_t(X(t))$ where $\{\sigma_t\}$ is a sequence of random independently chosen switching operators where for $q > 2$, $\Pr(\sigma_t = s_\emptyset) = \frac{1}{2}$, $\Pr(\sigma_t = s_v) = \frac{\omega(v)}{4}$, and $\Pr(\sigma_t = s_v^{-1}) = \frac{\omega(v)}{4}$. When $q = 2$ and $s_v^{-1} = s_v$, we let $\Pr(\sigma_t = s_\emptyset) = \frac{1}{2}$, $\Pr(\sigma_t = s_v) = \frac{\omega(v)}{2}$.

Our immediate goal is to interpret $X(t)$ as a random walk on a weighted graph, then use the myriad of tools available for studying such processes. For an arbitrary probability distribution ω on V, we define the symmetric distribution $\hat{\omega}$ on $\Gamma_q(H)$ by

$$\hat{\omega}(\gamma) = \begin{cases} \frac{\omega(v)}{2} & \text{if } \gamma = s_v, v \in V \\ \frac{\omega(v)}{2} & \text{if } \gamma = s_v^{q-1}, v \in V \\ 0 & \text{if otherwise} \end{cases}.$$

We define the *switching state graph* to be the directed, weighted graph

$$G_q(H) = Cay(\Gamma_q(H), \hat{\omega}).$$

Thus $G_q(H)$ has weighted adjacency matrix given by

$$A(\tau_i, \tau_j) = \hat{\omega}(\tau_i \circ \tau_j^{-1}) = \begin{cases} \frac{\omega(v)}{2} & \text{if } \tau_j = s_v(\tau_i), v \in V \\ \frac{\omega(v)}{2} & \text{if } \tau_j = s_v^{-1}(\tau_i), v \in V \\ 0 & \text{if otherwise} \end{cases}$$

That is, the weight on the edge from τ_i to τ_j corresponds to the elementary operator or its inverse that sends τ_i to τ_j.

We will show that $X(t)$ is equal in distribution to a lazy random walk on the weighed graph $G_q(H)$ starting at the empty coloring. In order to guarantee that $G_q(H)$ is connected we only consider distributions ω whose support on G is a generating set.

Let W be the transition matrix of the lazy random walk on $G_q(H)$.

Proposition 4.6. *For any coloring τ_i in the q-H switching class of $\tau_\emptyset$,* $\Pr(X(t) = \tau_i) = \mathbf{1}_{\{\tau_\emptyset\}} W^t(\tau_i)$, *where $\mathbf{1}_{\{\tau_\emptyset\}}$ a row vector with a 1 in the entry for $\tau_\emptyset$ and 0 elsewhere.*

Proof. We must show that $P(X(t+1) = \tau_i | X(t) = \tau_j) = W(\tau_j, \tau_i)$ for any t. W is the transition matrix of a lazy random walk on $G_q(H)$, so $W = \frac{1}{2}(I + D^{-1}A)$, where I is the identity matrix, D the weighted diagonal degree matrix, and A the weighted adjacency matrix of $G_q(H) = Cay(\Gamma_q(H), \omega)$. Since ω is a probability distribution, the weighted out-degree of each vertex is 1, and so $W = \frac{1}{2}(I + A)$. If $\tau_i = \tau_j$, then $W(\tau_i, \tau_i) = \frac{1}{2} = \Pr(X(t+1) = \tau_i | X(t) = \tau_i)$. For $\tau_i \neq \tau_j$,

$$W(\tau_i, \tau_j) = \frac{1}{2} A(\tau_i, \tau_j) = \begin{cases} \frac{\omega(v)}{4} & \text{if } s_v(\tau_i) = \tau_j \text{ for some } v \in V \\ \frac{\omega(v)}{4} & \text{if } s_v^{-1}(\tau_i) = \tau_j \text{ for some } v \in V \\ 0 & \text{if otherwise} \end{cases}.$$

But this is precisely equal to $\Pr(X(t+1) = \tau_j | X(t) = \tau_i)$. We have demonstrated that $X(t)$ is a Markov chain with transition matrix W, proving the proposition.

□

Since we can now understand $X(t)$ as the lazy random walk on a connected graph, it remains to analyze that graph. Since $G_q(H)$ is a Cayley graph with respect to the symmetric distribution ω, it is a connected, vertex transitive, undirected, weighted graph. Thus a lazy random walk converges to the uniform distribution π. It remains to understand how quickly the walk converges.

We remark that in the case where q is odd and H is not bipartite, this reduces to the case of a geometric random walk on $\mathbb{Z}_q^n$, see for example [8].

By Theorem 2.1, the χ-squared distance can be calculated as $\Delta'(t) = \left(\sum_{i\neq n} \lambda_i^{2t}\right)^{1/2}$, where $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = 1$ are the eigenvalues of the transition matrix. Thus to understand the rate of convergence of $X(t)$ to its stationary distribution it remains only to understand the eigenvalues of W.

Theorem 4.7. *Let H be a connected graph on n vertices, ω a probability distribution on V, and let $\theta_q = e^{\frac{2\pi i}{q}}$ denote a qth root of unity. Let W be the transition matrix of the random walk on the state graph $G_q(H)$. Then the spectrum of W depends on the parity of q and the structure of H as follows:*

1. *If $q = 2$, there is one eigenvalue of W corresponding to each vector $\vec{x} \in \mathbb{Z}_2^{n-1}$ with*

$$\lambda_{\vec{x}} = \frac{1}{2}\left(1 + \omega(v_n)\prod_{i=1}^{n-1}(-1)^{x_i} + \sum_{i=1}^{n-1}\omega(v_i)(-1)^{x_i}\right)$$

2. *If H is bipartite with respect to subsets $A, B \subset V$, then there is one eigenvalue of W corresponding to each vector $\vec{x} \in \mathbb{Z}_q^{n-1}$ with*

$$\lambda_{\vec{x}} = \frac{1}{2}\left(1 + \omega(v_n)\Re\left(\frac{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}{\prod_{v_j\in B, j\neq n}\theta_q^{x_i}}\right) + \sum_{i=1}^{n-1}\omega(v_i)\Re(\theta_q^{x_i})\right)$$

where $\Re(\cdot)$ denotes the real part.

3. *If H is not bipartite and q is odd, then there is one eigenvalue of W corresponding to each vector $\vec{x} \in \mathbb{Z}_q^{n-1}$ with*

$$\lambda_{\vec{x}} = \frac{1}{2}\left(1 + \sum_{i=1}^{n}\omega(v_i)\Re(\theta_q^{x_i})\right).$$

4. *If H is not bipartite and $q \geq 4$ is even, $q = 2r$, then there is one eigenvalue of W corresponding to each vector $\vec{x} \in \mathbb{Z}_q^{n-1} \times \mathbb{Z}_r$ with*

$$\lambda_{\vec{x}} = \frac{1}{2}\left(1 + \sum_{i=1}^{n-1} \omega(v_i)\Re(\theta_q^{x_i}) + \omega(v_n)\Re(\theta_r^{x_n})\right).$$

Proof. We prove the case that H is bipartite. The proofs of other cases are similar. By Theorem 2.2, the eigenvalues of W are $\lambda_\rho = \frac{1}{2} + \sum_{g\in\Gamma} \hat{\omega}(g)\rho(-g)$ where ρ is a one dimensional irreducible representation of $\Gamma_q(H)$. Theorem 2.3 states the irreducible representations of $\mathbb{Z}_q^n$ are the functions $\rho_{\vec{x}}$ for $\vec{x} \in \mathbb{Z}_q^{n-1}$ where

$$\rho_{\vec{x}}(\sum_{i=1}^{n-1} \alpha_i e_i) = \prod_{i=1}^{n-1} \theta_q^{x_i \alpha_i}.$$

Let $A, B \subset V$ be the subsets of vertices with respect to which H is bipartite. Thus the irreducible representations of $\Gamma_q(H)$ are $\rho_{\vec{x}} \circ \nu$ where $\nu \colon \Gamma_q(H) \to \mathbb{Z}_q^{n-1}$ is the isomorphism defined in Corollary 4.3 with $\nu(s_{v_i}) = e_i$ for $i \leq n-1$ and $\nu(s_{v_n}) = \sum_{v_i\in A, i\neq n} e_i - \sum_{v_j\in B, j\neq n} e_j$. Let $\tilde{\rho}_{\vec{x}} : \Gamma_q(H) \to \mathbb{C}$ be the composition for $\rho_{\vec{x}}$ and ν. Since the $\tilde{\rho}_{\vec{x}}$ are the irreducible characters of $\Gamma_q(H)$, we have that for each $\vec{x} \in R_q^n$ there is an eigenvalue

$$\begin{aligned}\lambda_{\vec{x}} = \frac{1}{2} + \frac{1}{2}\sum_{g\in\Gamma_q(H)} \hat{\omega}(g)\tilde{\rho}_{\vec{x}}(-g) &= \frac{1}{2} + \frac{1}{2}\sum_{v} \hat{\omega}(v)\tilde{\rho}_{\vec{x}}(s_v) + \hat{\omega}(v)\tilde{\rho}_{\vec{x}}(s_v^{-1}) \\ &= \frac{1}{2} + \frac{1}{4}\sum_{i=1}^{n} \omega(v_i)\rho_{\vec{x}}(\nu(s_{v_i})) + \omega(v_i)\rho_{\vec{x}}(\nu(s_{v_i}^{-1}))\end{aligned}$$

Let $\zeta_{\vec{x}}(i) = \omega(v_i)(\rho_{\vec{x}}(\nu(s_{v_i})) + \rho_{\vec{x}}(\nu(s_{v_i}^{-1})))$. Then $\lambda_{\vec{x}} = \frac{1}{2} + \frac{1}{4}\sum_{i=1}^{n} \zeta_{\vec{x}}(i)$. For $i \leq n-1$,

$$\begin{aligned}\zeta_{\vec{x}}(i) &= \omega(v_i)\left(\rho_{\vec{x}}(\nu(s_{v_i})) + \rho_{\vec{x}}(\nu(s_{v_i}^{-1}))\right) \\ &= \omega(v_i)\left(\rho_{\vec{x}}(e_i) + \rho_{\vec{x}}(-e_i)\right) \\ &= \omega(v_i)\left(\theta_q^{x_i} + \frac{1}{\theta_q^{x_i}}\right) \\ &= \omega(v_i)\left(\theta_q^{x_i} + \overline{\theta_q^{x_i}}\right) \\ &= \omega(v_i)2\Re(\theta_q^{x_i})\end{aligned}$$

It remains to calculate $\zeta_{\bar{x}}(n)$.

$$\zeta_{\bar{x}}(n) = \omega(v_n)\left(\rho_{\bar{x}}(\nu(s_{v_n})) + \rho_{\bar{x}}(\nu(s_{v_n}^{-1}))\right)$$

$$= \omega(v_n)\left(\rho_{\bar{x}}(\sum_{v_i\in A, i\neq n} e_i - \sum_{v_j\in B, j\neq n} e_j) + \rho_{\bar{x}}(\sum_{v_i\in A, i\neq n} -e_i + \sum_{v_j\in B, j\neq n} e_j)\right)$$

$$= \omega(v_n)\left(\frac{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}{\prod_{v_j\in B, j\neq n}\theta_q^{x_j}} + \frac{\prod_{v_j\in B, j\neq n}\theta_q^{x_j}}{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}\right)$$

$$= \omega(v_n)2\Re\left(\frac{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}{\prod_{v_j\in B, j\neq n}\theta_q^{x_j}}\right)$$

Thus

$$\lambda_{\bar{x}} = \frac{1}{2} + \frac{1}{2}\left(\omega(v_n)\Re\left(\frac{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}{\prod_{v_j\in B, j\neq n}\theta_q^{x_j}}\right) + \sum_{i=1}^{n-1}\omega(v_i)\Re(\theta_q^{x_i})\right)$$

□

Combining Theorems 2.1 and 4.7 gives our desired bounds on convergence times of the random generalized switching process.

Theorem 4.8. *Let H be a connected graph on n vertices, and let ω be a probability distribution on the vertices. Let $\theta_q = e^{\frac{2\pi i}{q}}$ denote a qth root of unity. The random walk on $G_q(H)$, and hence the random q-H-switching process after t steps has χ^2 distance from the stationary distribution bounded as follows:*

1. *If $q = 2$,*

$$\Delta'(t) \leq \frac{1}{2^t}\left[\sum_{\bar{x}\in \mathbf{Z}_2^{n-1}}\left(1 + \omega(v_n)\prod_{i=1}^{n-1}(-1)^{x_i} + \sum_{i=1}^{n-1}\omega(v_i)(-1)^{x_i}\right)^{2t}\right]^{\frac{1}{2}}$$

2. *If H is bipartite with respect to subsets $A, B \subset V$,*

$$\Delta'(s) \leq \frac{1}{2^t}\left[\sum_{\bar{x}\in \mathbf{Z}_q^{n-1}}\left(1 + \omega(v_n)\Re\left(\frac{\prod_{v_i\in A, i\neq n}\theta_q^{x_i}}{\prod_{v_j\in B, j\neq n}\theta_q^{x_j}}\right) + \sum_{i=1}^{n-1}\omega(v_i)\Re(\theta_q^{x_i})\right)^{2t}\right]$$

3. If H is not bipartite and q is odd,

$$\Delta'(t) \leq \frac{1}{2^t}\left[\sum_{\vec{x}\in \mathbf{Z}_q^n}\left(1+\sum_{i=1}^{n}\omega(v_i)\Re(\theta_q^{x_i})\right)^{2t}\right]^{\frac{1}{2}}$$

4. If H is not bipartite and $q \geq 4$ is even, $q = 2r$,

$$\Delta'(t) \leq \frac{1}{2^t}\left[\sum_{\vec{x}\in \mathbf{Z}_q^{n-1}\times \mathbf{Z}_r}\left(1+\sum_{i=1}^{n-1}\omega(v_i)\Re(\theta_q^{x_i})+\Re(\theta_r^{x_n})\right)^{2t}\right]^{\frac{1}{2}}$$

Corollary 4.9. *When $q = 2$ and $\omega(v_i) = \frac{1}{n}$, and $H = K_n$ then we are considering randomized Seidel Switching as discussed in Section 3. In this case, we obtain the bounds*

1. $\Delta'(t) \leq \left(\sum_{j=1}^{n-1}\binom{n-1}{j}(1-\frac{j}{n})^{2t}\right)^{\frac{1}{2}}$
2. $\Delta'(t) \leq e^{-c}$ *if* $t > \frac{1}{2}n\log(n)+c$.

Proof. There will be $\binom{n-1}{j}$ vectors $\vec{x} \in \mathbb{Z}_2^{n-1}$ with j 1's, and $(n-1-j)$ 0's. For each of these, $\lambda_{\vec{x}} = \frac{1}{2}(1+\frac{1}{n}(n-1-2j)+\frac{(-1)^j}{n} \leq (1-\frac{j}{n})$.

For the second fact, we note that

$$\begin{aligned}\left(\sum_{j=1}^{n-1}\binom{n-1}{j}\left(1-\frac{j}{n}\right)^{2t}\right)^{\frac{1}{2}} &\leq \left(\sum_{j=1}^{n-1}e^{j\log(n-1)-\frac{2jt}{n}}\right)^{\frac{1}{2}} \\ &\leq \left((n-1)e^{(n-1)\log(n-1)-\frac{2(n-1)t}{n}}\right)^{\frac{1}{2}} \\ &= e^{\frac{(n-1)\log^2(n-1)}{2}-\frac{(n-1)\log(n-1)t}{n}} \\ &\leq e^{-c}\end{aligned}$$

if

$$\frac{(n-1)\log^2(n-1)}{2}-\frac{(n-1)\log(n-1)t}{n} \leq -c. \qquad (3)$$

Solving for t and simplifying shows that (3) is satisfied when

$$t > \frac{n\log(n)}{2}+c.$$

□

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