On the Second Neighborhood Conjecture

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Abstract

Seymour's Second Neighborhood Conjecture claims that every simple digraph has a vertex whose first neighborhood is at most as large as its second neighborhood. We confirm this conjecture for neighbor-connection free simple digraphs and distance-two simple digraphs. As a consequence, the conjecture is true for triangle free digraphs and 4-cycle free digraphs.

Keywords: first neighborhood; second neighborhood; triangle free digraph.

1 Introduction

A digraph is simple if it has no loops, parallel edges or directed 2-cycle. Let D=(V,A) be a digraph with vertex set V and arc set A. For any $v \in V$, we define $N_D^+(v) = \{u|(v,u) \in A\}$ and call it the first neighborhood of v and $N_D^{++}(v) = \{u|u$ is distance 2 from $v\}$ and call it the second neighborhood of v. We denote $d_D^+(v) = |N_D^+(v)|$, $d_D^{++}(v) = |N_D^{++}(v)|$. We call a digraph D *neighbor-connection free* if for any vertex v of D, there is no arc joining two vertices of $N_D^+(v)$. A digraph D is *distance-two simple* if for any two

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distance-two vertices u and v, say the distance from u to v is 2, there exists exactly one directed 2-path from u to v. The *underlying graph* of a digraph D is a graph G obtained from D by removing the direction of each arc of D and change it to an undirected edge. Note that it is possible that the underlying graph of a neighbor-connection free (resp. distance-two simple) digraph contains 3-cycles (resp. 4-cycles). A digraph is *triangle free* if its underlying graph contains no 3-cycle. A digraph is *4-cycle free* if its underlying graph contains no 4-cycle. Clearly a triangle free digraph is neighbor-connection free and any 4-cycle free digraph is distance-two simple.

In 1990, Seymour made the following conjecture:

Conjecture 1.1 *([2]) (The Second Neighborhood Conjecture). Every simple digraph D has a vertex v such that $d_D^+(v) \leq d_D^{++}(v)$.*

This conjecture restricted to tournaments is known as Dean's conjecture [2] which was confirmed by Fisher [4] in 1996. In 2000, Havet and Thomassé [6] proved Dean's conjecture using a new method and this method is later applied by Fidler and Yuster [3] in 2007 to prove the Second Neighborhood Conjecture for tournaments missing the edges of a complete graph and tournaments missing a matching. Most recently, Ghazal [5] proved the weighted version of the Second Neighborhood Conjecture for any tournament missing a generalized star.

For general digraphs, Chen et al. [1] proved that every digraph contains a vertex v such that $d_D^+(v) \leq \gamma d_D^{++}(v)$, where $\gamma = 0.657298\ldots$ is the unique real root of the equation $2x^3+x^2-1=0$. In addition, Kaneko and Locke [7] proved the Second Neighborhood Conjecture for digraphs with minimum outdegree at most 6.

In this paper, we focus on neighbor-connection free simple digraphs and distance-two simple digraphs and confirm the Second Neighborhood Conjecture for these two families of graphs.

Theorem 1.2 *The Second Neighborhood Conjecture is true for neighbor-connection free simple digraphs and distance-two simple digraphs.*

Since each triangle free digraph is neighbor-connection free and each 4-cycle free digraph is distance-two simple, then we have the following:

Corollary 1.3 *The Second Neighborhood Conjecture is true for triangle free simple digraphs and 4-cycle free simple digraphs.*

2 Proof of Theorem 1.2

Let D be a neighbor-connection free digraph or a distance-two simple digraph. Pick a vertex v of D such that $d_D^+(v)$ is minimum. We are to show that $d_D^+(v) \leq d_D^{++}(v)$ as follows:

If $d_D^+(v) = 0$, then $d_D^+(v) = 0 \leq d_D^{++}(v)$. So we may assume that $d_D^+(v) \geq 1$. For convenience, let $N_D^+(v) = \{u_1, u_2, \cdots, u_k\}$ with $k \geq 1$. By the choice of v, we have $d_D^+(u_i) \geq d_D^+(v) = k$ for $1 \leq i \leq k$.

If D is a neighbor-connection free graph, then for any $i \neq j$, $(u_i, u_j) \notin A$. Therefore, for any $(u_i, x) \in A$, we have $x \in N_D^{++}(v)$. But $d_D^+(u_i) \geq d_D^+(v) = k$, then $d_D^+(v) = k \leq d_D^+(u_i) = |N_D^+(u_i)| \leq d_D^{++}(v)$.

So we may assume that G is distance-two simple. Note that $|N_D^+(v)\backslash\{u_i\}| = k-1$ and $d_D^+(u_i) \geq d_D^+(v) = k$. Then for each i with $1 \leq i \leq k$, there is at least one $w_i \in N_D^+(u_i) \setminus N_D^+(v)$. Clearly, the distance from v to each w_i is 2, where $1 \leq i \leq k$. Since D is distance-two simple, then $w_i \neq w_j$ if $i \neq j$. Therefore $d_D^+(v) = k \leq d_D^{++}(v)$. ■

By Corrollary 1.3 the Second Neighborhood Conjecture holds for triangle free simple digraphs as well as 4-cycle free simple digraphs. It would be interesting to prove the conjecture for digraphs with many triangles such as directed line graphs, or more generally, claw-free digraphs. So we post here two weak versions of the Second Neighborhood Conjecture.

Problem 2.1 *The Second Neighborhood Conjecture is true for directed line graphs.*

Problem 2.2 *The Second Neighborhood Conjecture is true for claw-free digraphs.*

Note that any line graph is a claw-free graph, so Problem 2.2 is stronger than Problem 2.1. Since any complete graph is claw-free, Problem 2.2 implies Dean's Conjecture.

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