Total Efficient Domination and Cayley Graphs

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Abstract

A set $S \subseteq V$ is a dominating set of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. A set $S \subseteq V$ is a *total dominating set* of a graph $G = (V, E)$ if each vertex in V is adjacent to a vertex in S. In total domination a vertex no longer dominates itself. These two types of domination can be thought of as representing the vertex set of a graph as the union of the closed (domination) and open (total domination) neighborhoods of the vertices in the set S. A set $S \subseteq V$ is a *total, efficient dominating set* (also known as an *efficient open dominating set*) of a graph $G = (V, E)$ if each vertex in V is adjacent to exactly one vertex in S. In 2002 Gavlas and Schultz completely classified all cycle graphs that admit a total, efficient dominating set. This paper extends their result to two classes of Cayley graphs.

Keywords: Domination, Cayely Graph

1 Introduction

A set $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if each vertex in V is either in S or is adjacent to a vertex in S. A vertex is said to dominate itself and all its neighbors. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. A set $S \subseteq V$ is a *total dominating set* of a graph $G = (V, E)$ if each vertex in V is adjacent to a vertex in S. In total domination a vertex no longer dominates itself. These two types of domination can be thought of as representing the vertex set of a graph as the union of the closed (domination) and open (total domination) neighborhoods of the vertices in the set S. The *total domination number*, $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G. Since a total dominating set is a dominating set, $\gamma(G) \leq \gamma_t(G)$ for all graphs G. A set $S \subseteq V$ is an *efficient dominating set* of a graph $G = (V, E)$ if each vertex in V is adjacent to exactly one vertex in S [1].

Unlike many domination problems where a vertex dominates itself (closed domination), efficient domination is a question of existence rather than a question of optimization. The set of all vertices V always forms a dominating set for every graph G. In a graph with no isolated vertices, the set of all vertices V also forms a trivial total dominating set. For both domination and total domination, the existence question is handled quite easily. In contrast, we can no longer blithely use the set of all vertices in a graph to form a trivial efficient dominating set. Worse still is the realization that not every graph admits an efficient dominating set. The graph C_5 does not admit an efficient dominating set since each vertex dominates three vertices. When a graph admits at least one efficient dominating set, the size of each efficient dominating set is the same and, in fact, is $\gamma(G)$. [1]

A set $S \subseteq V$ is a *total, efficient dominating set* (or *TEDS*) of a graph $G = (V, E)$ if each vertex in V is adjacent to exactly one vertex in S where a vertex no longer dominates itself (open domination). Gavlas and Schultz define this concept as an efficient open dominating set in [3]. Given the vast quantity of literature on total domination, use of the term *total* rather than *open* seems more evocative. In [3] Gavlas and Schultz show that the cardinality of every TEDS in a graph G is the same. In [4] Gavlas, Slater and Schultz extend this result by showing that if S is a TEDS then $|S| = \gamma_t(G)$. This fact lends great support for the change in terminology from open, efficient domination to total, efficient domination.

Total efficient domination also forces another type of domination. A set S is a *paired dominating set* if S is a dominating set and the subgraph induced by S contains a perfect matching. The *paired-domination number*, $\gamma_{pr}(G)$, is defined to be the minimum cardinality of a paired-dominating set S in G. In a TEDS S, vertices dominate each other and no single vertex

v can dominate two or more other vertices in S since v would then be dominated more than once. Gavlas and Schultz show that a total, efficient dominating set is also a paired dominating set.

Figure 1 shows the contrast between an Efficient Dominating Set and a Total Efficient Dominating Set (TEDS).

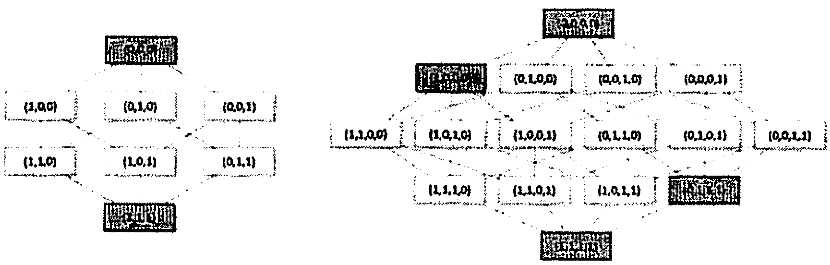

Figure 1: An Efficient Dominating Set in Q_3 and a Total Efficient Dominating Set in Q_4

As with efficient dominating sets, the question for a TEDS is one of existence and not of optimization. By using both vertices of K_2, a total, efficient dominating set is constructed. The complete graph K_3 does not admit a total, efficient dominating set since each vertex dominates two vertices. Thus, a single vertex is insufficient to dominate every vertex and two vertices will dominate a single vertex twice. Gavlas and Schultz also show that for a *TEDS* S in a graph G with n vertices $\sum_{v \in S} \deg(v) = n$. Such counting indicates that if the Grötzsch Graph in Figure 2 admits a *TEDS* S then $\sum_{v \in S} \deg(v) = 11$. With vertex degrees 3, 4 and 5, S must have the degree sequence 5,3,3 or 4,4,3. In either case paired domination cannot be satisfied. Furthermore, let G be a k-regular graph with n vertices. If a *TEDS* S exists in G then $2k \mid n$. Thus, the Petersen graph in Figure 3 does not admit a *TEDS* S.

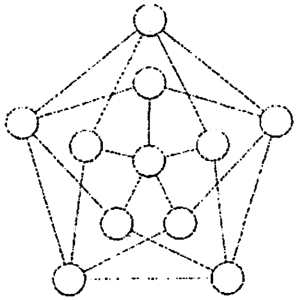

Figure 2: The Grötzsch Graph

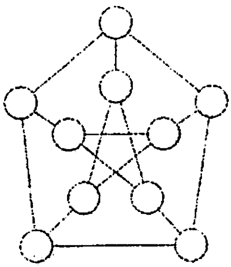

Figure 3: The Petersen Graph

While $2k \mid n$ is a necessary condition, it is not sufficient to guarantee the existence of a *TEDS* S as demonstrated by the connected graph in Figure 4.

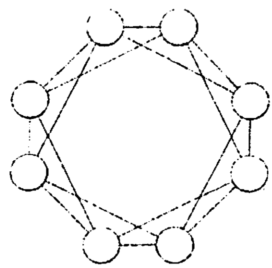

Figure 4: No TEDS exists yet $(2 \cdot 4) \mid 8$

We say that G is *triangle complete* if for every pair of adjacent vertices i and j there exists a vertex k adjacent to both. Simply put, every K_2 is a subgraph of some K_3. The connected graph in Figure 4 is triangle complete.

Theorem 1 *If a graph G is triangle complete then G does not admit a TEDS S.*

Proof. Since any TEDS S is also a paired dominating set then should a TEDS S exist, a pair of adjacent vertices i and j exist in the TEDS S Since G is triangle complete there exists a vertex k adjacent to both i and j. But now k is dominated at least twice which contradicts the existence of a TEDS S.
■

Gavlas and Schultz completely classified all path and cycle graphs that admit a TEDS.

Theorem 2 *A TEDS S exists in P_n if and only if $n \not\equiv 1 \mod 4$ [3].*

Theorem 3 *A TEDS S exists in C_n if and only if $n \equiv 0 \mod 4$ [3].*

We offer the following necessary and sufficient result for k-regular graphs that admit a TEDS:

Theorem 4 *A k-regular graph G of order n has a TEDS if and only if $\gamma_t(G) = \frac{n}{k}$.*

Proof. Let G be any k-regular graph and assume that G has a TEDS. Let S be a TEDS of G. We know $\sum_{v \in S} \deg(v) = n$, and since G is k-regular, we have $|S| = \gamma_t(G) = \frac{n}{k}$.

On the other hand, suppose $\gamma_t(G) = \frac{n}{k}$ and let D be a $\gamma_t(G)$ set. Since G is k-regular, we have $\sum_{v \in D} \deg(v) = k\left(\frac{n}{k}\right) = n$. Since D is an open dominating set of G, no vertex of D dominates any vertex of G more than once. If this were not the case, then there would be at least one vertex of G not dominated by D. Thus D is a TEDS of G. ■

2 Circulant Graphs

The cycle graph is one particular type of a more general class of graphs. The circulant digraph $G(\mathbb{Z}_n, C)$, where $n \geq 3$ and connection set $C \subseteq \mathbb{Z}_n \setminus \{0\}$, has vertex set $V = \{1, 2, ..., n\}$ and the $i \rightarrow j$ arc exists if and only if $j - i \in C$. If C is closed under additive inverses, then $G(\mathbb{Z}_n, C)$ is a graph, rather than a digraph. The identity element is excluded from C to prevent loops. Note that $C_n = G(\mathbb{Z}_n, \{\pm 1\})$. Here we extend the classification of the existence of a TEDS for cycle graphs to circulant graphs.

The graph in Figure 4is $G(\mathbb{Z}_8, \{\pm 1, \pm 2\})$. The graph $G(\mathbb{Z}_n, C)$ is regular of degree $|C|$. When closed under additive inverses, the order of C will be even if and only if $\frac{n}{2} \notin C$. Furthermore the circulant graph is one particular type of an even more general class of graphs. The Cayley digraph $G(H, C)$ for any group H and $C \subseteq H \setminus \{e\}$has as its vertex set the group elements of H and the $i \rightarrow j$ arc exists if and only if $ji^{-1} \in C$. Again, if C is closed under inverses, then $G(H, C)$ is a graph, rather than a digraph.

Theorem 5 *The graph $G(\mathbb{Z}_n, C)$, for some connection set C, admits a TEDS if and only if $2k \mid n$ where $|C| = k$.*

Proof. Clearly $2k \mid n$ where $|C| = k$ is a necessary condition for the existence of a TEDS S in $G(\mathbb{Z}_n, C)$. Is it also sufficient? Based on the parity of k two cases exist. If k is even then let $C = \{\pm 1, \pm 3, \cdots \pm (k-1)\}$. Now $S = \{1, 2, 2k+1, 2k+2, 4k+1, 4k+2, \ldots, 2\left(\frac{n}{2k} - 1\right)k+1, 2\left(\frac{n}{2k} - 1\right)k+2\}$ is a TEDS for $G(\mathbb{Z}_n, C)$. Each pair of adjacent vertices, $2jk+1$ and $2jk+2$, dominate exactly once, each of the $2k$ vertices from $2jk+1-(k-1) = 2jk - k + 2$ to $2jk + 2 + (k-1) = k + 2jk + 1$ and only those vertices. The next set of vertices $2(j+1)k + 1$ and $2(j+1)k + 2$ start at vertex number $2(j+1)k + 1 - (k-1) = 2k(j+1) - k + 2 = 2kj + 2k - k + 2 = k + 2jk + 2$. For odd k, Let $C = \{\pm 1, \pm 3, ..., \pm (k-2), \frac{n}{2}\}$. Now, $S = \{1, 1 + \frac{n}{2}, 1 + k, 1 + k + \frac{n}{2}, ..., 1 + \left(\frac{n}{2k} - 1\right)k, 1 + \left(\frac{n}{2k} - 1\right)k + \frac{n}{2}\}$ is a TEDS for $G(\mathbb{Z}_n, C)$. Clearly, S admits a perfect matching. ■

So, while the graph in Figure 4 does not admit a TEDS, $G(\mathbb{Z}_8, \{\pm 1, \pm 3\})$, a Cayley graph with a connection set of order 4, does admit a TEDS. The graph in Figure 5 is an example of the construction given in the proof of Theorem 5.

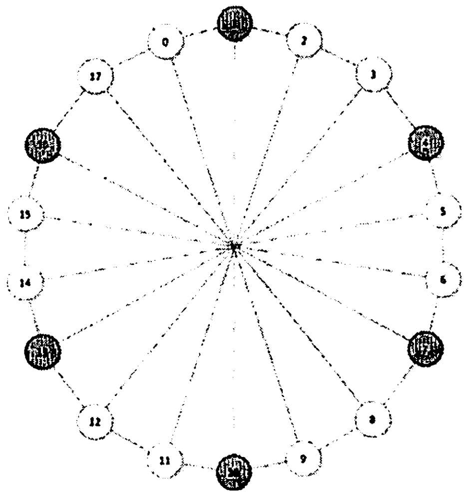

Figure 5: $G(\mathbb{Z}_{18}, \{\pm 1, 9\})$

Note the proof of Theorem 5 always generates a connected graph. It is necessary that the order of a group H is even in order to admit a TEDS. This is also a sufficient condition since every group H of even order contains an element, h, of order 2. The Cayley graph $G(H, \{h\})$ is a collection of $\frac{n}{2}$ K_2's and the entire vertex set forms a TEDS.

3 Dihedral Groups

As previously noted, the Circulant graph **Circ** $(\mathbb{Z}_n, X)$ is a special case of a larger class of graphs, Cayley graphs: graphical representations of groups. For a group G with a binary operation, we can define the Cayley graph on G in the following way:

Definition 6 *Let H be a finite group with identity e. Let C be a subset of H satisfying $e \notin C$ and $C = C^{-1}$, that is, $a \in C$ if and only if $a^{-1} \in C$. The **Cayley graph** on H with **connection set** C, denoted $G(H, C)$, satisfies: the vertices of $G(H, C)$ are the elements of H; there is an edge joining $a, b \in$*

$G(H,C)$ if and only if $a^{-1}b \in C$. We do not require that the connection set C generate the group H. We do not include the identity element e to avoid loops and require C closed under inverses to avoid digraphs.

The elements of the Dihedral group, D_n, are all the possible positions of a regular n-gon. Note that there are $2n$ different positions found by the n rotations and n flips across each horizontal, vertical, and diagonal axis. As noted earlier, every group of even order contains a TEDS for some connection set C. Since D_n has order $2n$, every dihedral group with connection set $C = \{f\}$ yields a Cayley graph that admits a TEDS containing every vertex in the graph.

Theorem 7 *The graph $G(D_n, \{f, rf\})$ admits a TEDS if and only if $2 \mid n$.*

Proof. On the one hand suppose $G(D_n, \{f, rf\})$ admits a TEDS. This graph is a $2n$-cycle by definition and since G admits a TEDS, we know that $2n \equiv 0 \mod 4$. Thus $4 \mid 2n$ which implies $2 \mid n$.
On the other hand suppose $2 \mid n$. Since our graph is regular of degree 2, we see that any two adjacent vertices in our TEDS S totally and efficiently dominate 4 vertices. To construct our TEDS S, we must choose our first pair of adjacent vertices. Choose the adjacent vertices e and f. We see that e dominates the set $\{f, rf\}$ and f dominates the set $\{e, r^{n-1}\}$.
If this is all of D_n stop, and we see that S is a TEDS. If this fails to be all of D_n we must choose our next pair of vertices to be added to S. The next set of vertices should be r^2 and $r^2 f$. We see that r^2 dominates the set $\{r^2 f, r^3 f\}$ and $r^2 f$ dominates the set $\{r, r^2\}$. If this dominates all of D_n stop. If not continue in this manner. The final pair of vertices included in S will be r^{n-2} and $r^{n-2}f$. Where r^{n-2} will dominate $\{r^{n-2}f, r^{n-1}f\}$ and $r^{n-2}f$ will dominate $\{r^{n-2}, r^{n-3}\}$. Thus our TEDS is

$$S = \{e, f, r^2, r^2 f, ..., r^{n-2}, r^{n-2} f\}.$$

■

Despite the fact that $\mathbb{Z}_{12} \ncong D_6$, it is easy to select connection sets such that the resulting Cayley graphs are isomorphic. Figure 6 displays an example of a TEDS in $G(D_6, \{f, rf\})$, a graph isomorphic to $G(\mathbb{Z}_{12}, \{\pm 1\})$.

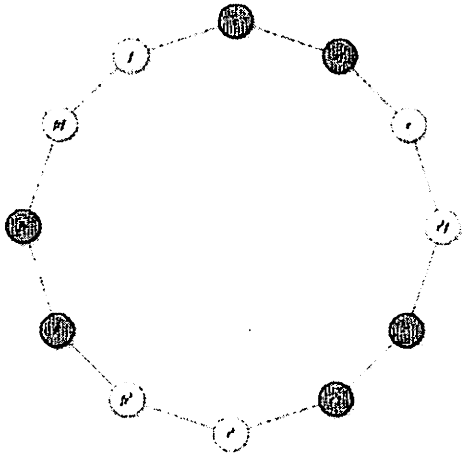

Figure 6: $G(D_6, \{f, rf\})$

Theorem 8 *The graph $G(D_n, C)$, for some connection set C, admits a TEDS S if and only if $k \mid n$ where $|C| = k$.*

Proof. On the one hand suppose $G(D_n, C)$ admits a TEDS. Since $|C| = k$ we know that our graph is k-regular. Choose a TEDS S. Since our graph G is k-regular, we see that each member $v \in S$ must dominate k number of other vertices. Therefore each adjacent pair of vertices in S dominates exactly $2k$ vertices. Since our set S must efficiently dominate all $2n$ vertices in G we see that $2k \mid 2n$ which implies $k \mid n$.

On the other hand suppose $k \mid n$, which implies $2k \mid 2n$. Note that $2k$ is the number of vertices that each adjacent pair of vertices in our TEDS S dominates, and that $2n$ is the number of vertices in G. This relationship shows us that we must have $\frac{n}{k}$ pairs of vertices in our TEDS. Define

$$C = \{f, rf, r^2f, r^3f, ..., r^{k-1}f\}$$

and we see that $|C| = k$. Now we must choose pairs of vertices for our TEDS. We start with set of adjacent vertices $\{e, r^{k-1}f\}$ and see that e dominates the set of vertices $\{f, rf, r^2f, ..., r^{k-1}f\}$ and $r^{k-1}f$ dominates the set of vertices $\{e, r, r^2, ..., r^{k-1}\}$ this is $2k$ vertices. Since e and $r^{k-1}f$ dominate each other and none of the same vertices, we see that $\{e, r^{k-1}f\}$ totally and efficiently dominates the set

$$\{e, r, r^2, ..., r^{k-1}f, rf, r^2f, ..., r^{k-1}f\},$$

which is $2k$ vertices. If this set is all of D_n, we are done and define $S = \{e, r^{k-1}f\}$ as our TEDS. If not, we need to select another pair of adjacent vertices. Consider the vertices $\{r^k, r^{2k-1}f\}$. We see that r^k dominates the set of vertices $\{r^k f, r^{k+1}f, r^{k+2}f, ..., r^{2k-1}f\}$ and $r^{2k-1}f$ dominates $\{r^k, r^{k+1}, r^{k+2}, ..., r^{2k-1}\}$. Thus the set $\{e, r^k, r^{k-1}f, r^{2k-1}f\}$ totally and efficiently dominates the set

$$\{e, r, r^2, ..., r^{2k-1}, f, rf, r^2 f, ..., r^{2k-1}f\}.$$

If this set is the all of D_n, we are done and can define $S = \{e, r^k, r^{k-1}f, r^{2k-1}f\}$. If not, we continue in this manner until we select the set of vertices in our TEDS: $\{r^{(n-1)-k}, r^{n-1}f\}$ which will totally and efficiently dominate the set of vertices

$$\left\{r^{(n-1)-k}, r^{(n-1)-k+1}, ..., r^{(n-1)}, r^{(n-1)-k}f, r^{(n-1)-k+1}f, ..., r^{(n-1)}f\right\}.$$

We see that a TEDS for $G(D_n, C)$ is

$$S = \left\{e, r^k, r^{2k}, ..., r^{(n-1)-k}, r^{k-1}f, r^{2k-1}f, r^{3k-1}f, ..., r^{n-1}f\right\}.$$

■

Figure 7 displays an example of a TEDS in D_9. Note that this figure is an example of the construction provided in Theorem 8.

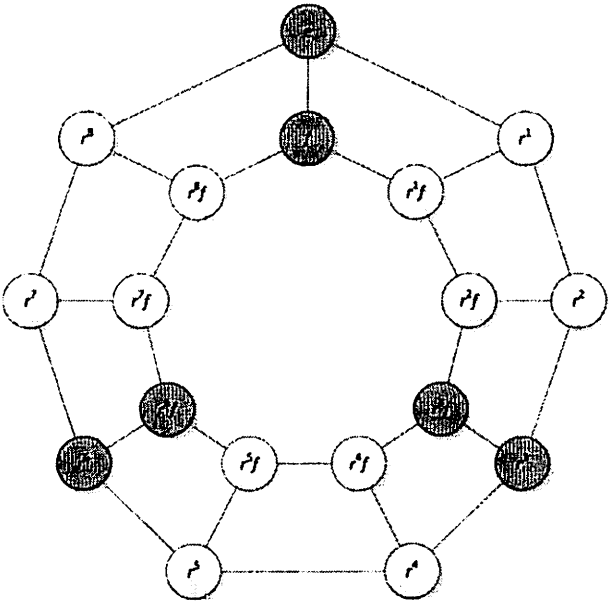

Figure 7: $G(D_9, \{f, r, r^8\})$

4 Future Work

All groups with an even number of elements contain an element of order 2. Using this element of order 2 as a connection set generates a disconnected Cayley graph that yields a TEDS. In other words, we can always choose our connection set to be the element of order 2 to get a graph of $n/2$ copies of K_2. For a connection set with at least two elements, whose order satisfied the necessary divisibility requirement, one can always find a connection set which generates a connected Cayley graph that admits a TEDS for $\mathbb{Z}_n$ and D_n. Two very different paths for future work exist. Perhaps this property does not hold for all groups. Thus, we want to find a group of even order n with divisor k such that $2k$ divides n but no connection set of order k yields a Cayley graph that admits a TEDS. One might want to consider A_4 which provides a counterexample to the converse of LaGrange's Theorem. Or perhaps this is a property that holds for all groups and we want to prove it. A smaller next step might attempt to prove a theorem similar to Theorem 8 for all finite abelian groups, such as:

Theorem 9 *The graph* $G(\mathbb{Z}_2 \times \mathbb{Z}_{2n}, \{(1,0),(0,1),(0,-1)\})$ *admits a TEDS if and only if* $3 \mid n$.

One might also exploit graph isomorphisms to prove the existence of this property in certain groups. For example, Figure 7 above and Figures 8 and 9 below demonstrate the existence of TEDS in Cayley graphs for isomorphic graphs (of non-isomorphic groups).

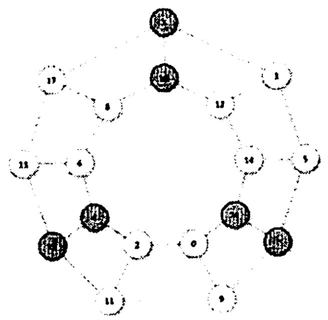

Figure 8: $G(\mathbb{Z}_{18}, \{\pm 2, 9\})$

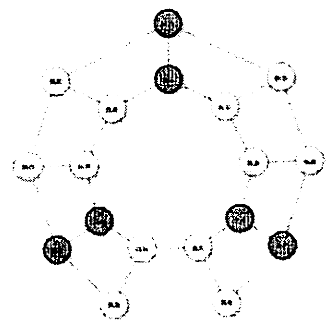

Figure 9: $G(\mathbb{Z}_2 \times \mathbb{Z}_9, \{(1,0),(0,\pm 1)\})$

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