The genus of edge amalgamations of a type of graph

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Abstract. If G_1 and G_2 are two graphs, then the edge amalgamation $G_1 *_e G_2$ is defined to be the graph obtained by identifying some given edge of G_1 with some given edge of G_2. In this paper, it is shown that $\gamma(G_1 *_e G_2 *_e \cdots *_e G_n) = \lceil \frac{n}{2} \rceil$, where $G_i (1 \le i \le n)$ is a critical graph of minimum genus 1.

Key words. genus, joint tree, edge amalgamation, orientable surface, embedding

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1 Introduction

Genus problem of a graph is a central problem of topological graph theory. And the problem is NP-complete. Much of the progress in this area has been in the computation of the genera of certain special classes of graphs with good symmetry, such as complete graph, complete bipartite graph etc. Further, for certain graphs by computations of certain special classes of graphs, Alpert[1, 2] has studied the minimum genera of edge amalgamations for complete graphs and complete bipartite graphs, respectively. In 2003, Liu set up the joint tree model[6], so that the corresponding relation was established between the joint trees and the embeddings. On the basis of joint trees, some works have been done[4, 5, 10, 11, 12, 13]. In this paper, on the basis of joint tree model, an embedding of a graph on a surface can be represented by an associated surface of it. By studying the properties of its associated surfaces, we obtain the genus of edge amalgamations of a type of graph.

Graphs considered in this paper are simple, connected and have no subdivision vertices. A *surface* is a compact 2-dimensional manifold without

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boundary. Since it can be obtained by identifying each of pairs of edges along a given direction on a polygon with even number of edges, an *orientable surface* is presented by a cyclic order P of letters satisfying the following conditions[8]:

Con.1 If $a \in P$, then $a^- \in P$.

Con.2 For each letter a on P, both a and a^- occur once on P.

Use the symbols $o(S)$ and $\mathcal{S}$ to denote the genus of surface S and set of orientable surfaces, respectively. On $\mathcal{S}$, an *elementary transformation*[8] is defined by the following three operations:

El.0 $\forall S \in \mathcal{S},\ S = Aaa^-B,\ A \neq 0$, or $B \neq 0 \Longleftrightarrow S = AB$.

El.1 $\forall S \in \mathcal{S},\ S = AabBb^-a^-C \Longleftrightarrow S = AaBa^-C$.

El.2 $\forall S \in \mathcal{S},\ S = AaBCa^-D \Longleftrightarrow S = BaADa^-C$.

If two surfaces S_1 and S_2 can be converted from one to another by finite sequences of elementary transformations, then we say that S_1 and S_2 are *equivalent*, denoted by $S_1 \sim S_2$. Note that S_1 and S_2 have same orientability and genus.

Lemma 1.1 [6] *$AaBbCa^-Db^-E \sim ADCBEaba^-b^-$, where $a, b \notin ABCDE \in \mathcal{S}$.*

Then by applying these operations above, each orientable surface is equivalent to only one of the following canonical forms:

$$O_i = \begin{cases} a_0a_0^-, & \text{if the genus of a surface is } 0; \\ \Pi_{k=1}^{i} a_kb_ka_k^-b_k^-, & \text{if the genus of a surface is } i. \end{cases}$$

Lemma 1.2 [6] *Let S_1 and S_2 be surfaces, $a, b \notin S_2$. If $S_1 \sim S_2aba^-b^-$, then $o(S_1) = o(S_2) + 1$.*

An *embedding* of a graph G into a surface S is a homeomorphism τ: $G \to S$, such that each component of $S - \tau(G)$ is a 2-cell. Two embeddings τ: $G \to S$ and h: $G \to S$ of G into a surface S are said to be *equivalent* if there is a homeomorphism f: $S \to S$ such that $f \circ \tau = h$. The *(minimum) genus* $\gamma(G)$ of a graph G is the minimum genus of the orientable surface into which G has an embedding.

Given a graph $G = (V, E)$, A *rotation* σ_v at a vertex v is a cyclic permutation of edges incident with v. And $\sigma = \Pi_{v \in V(G)}\sigma_v$ is called a rotation of G. For a spanning tree T of G, split the cotree edge (u_i, v_i) into two semi-edges $(u_i, \bar{u}_i)$ and $(v_i, \bar{v}_i)$, which are, respectively, incident with u_i and v_i for $1 \leq i \leq \beta$(Betti number), to obtain a new tree $\hat{T} = (V + V_1, E(T) + E_1)$, where $E_1 = \{(u_i, \bar{u}_i), (v_i, \bar{v}_i) | 1 \leq i \leq \beta\}$ and $V_1 = \{\bar{u}_i, \bar{v}_i | 1 \leq i \leq \beta\}$. Denote $\hat{T}$ by $\hat{T}^\dagger$, edges $(u_i, \bar{u}_i)$ and $(v_i, \bar{v}_i)$ are labeled by a_i with different indices: $+$(always omitted) and $-$ for $1 \leq i \leq \beta$(Betti number). The tree $\hat{T}^\dagger$ with a certain rotation σ of G is called a *joint tree*, denoted by $\hat{T}^\dagger_\sigma$.

On $\hat{T}^\dagger_\sigma$, according to a given orientation(clockwise or counterclockwise), write down such letters as a_i labelling the semi-edges to obtain a cyclic order of 2β letters,we call it an *associated surface*[7] of G. That two associated surfaces are *the same* is meant that they have the same cyclic order. Otherwise, distinct. Then an embedding of a graph on a surface can be represented by a joint tree, further by an associated surface of the graph.

Use the symbol $S_\sigma(T)$ to denote the associated surface of G corresponding to $\hat{T}^\dagger_\sigma$. Let $(S_\sigma(T),\ \Xi) = \{S_\sigma(T) \mid \sigma \in \Xi\}$, where Ξ is the set of all rotations of G.

Theorem 1.3 [7] *For any two distinct rotations σ_1 and σ_2 of G, $\hat{T}^\dagger_{\sigma_1}$ and $\hat{T}^\dagger_{\sigma_2}$, as well as $S_{\sigma_1}(T)$ and $S_{\sigma_2}(T)$, are distinct.*

Theorem 1.4 [7] *Let T_1 and T_2 be two distinct spanning trees of graph G. Then $(S_\sigma(T_1),\ \Xi) = (S_\sigma(T_2),\ \Xi)$.*

Theorem 1.5 [7] *For a fixed spanning tree T of graph G, there is a 1-to-1 correspondence between the associated surfaces and the embeddings of G.*

For $K_{3,3}$, choose a spanning tree such that a, $a_i(1 \leq i \leq 3)$ are cotree edges as shown in Fig.1.1. A joint tree of it is given in Fig.1.2.

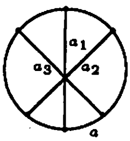

Fig.1.1 $K_{3,3}$

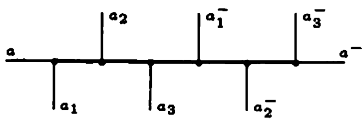

Fig.1.2 A joint tree of $K_{3,3}$

Then the associated surface of $K_{3,3}$ corresponding to the joint tree shown in Fig.1.2 can be shown as

$$S_3^3 = aa_2a_1^-a_3^-a^-a_2^-a_3a_1.$$

According to the rotation of $K_{3,3}$, an associated surface of minimum genus can be obtained.

2 Main results

A graph G is said to be *critical* if, for each edge e of G, $\gamma(G-e) = \gamma(G)-1$. An *amalgamation* $G_1 *_e G_2$ of graph G_1 and graph G_2 is obtained by identifying some given edge e_1 of G_1 with some given edge e_2 of G_2. And $e_1(e_2)$ is called an *amalgamated edge.* Identify some given edge e_2' of $G_1 *_e G_2$ with some given edge e_3 of G_3 to obtain a graph, denoted by $G_1 *_e G_2 *_e G_3$, where $e_2' \in G_2$ and is not adjacent to e_2. A surface S is called *maximal,* if $o(S) = 0$ and $o(S') > 0$ for $S \subset S'$. If vertices v_1, v_2, v_3, v_4 are on a cycle in this order, then edge v_1v_3 is said to cross edge v_2v_4.

Note that in this paper, all amalgamated edges are contained in the hamiltonian cycle of G or some cycle when the graph is not hamiltonian.

Theorem 2.1 [3] $\gamma(G) = \sum_{i=1}^{n} \gamma(G_i)$,

where $G_1, G_2, \cdots, G_n$ *are* n *blocks of the connected* G.

Lemma 2.2 [2] $\gamma(G_1 *_e G_2) \leq \gamma(G_1) + \gamma(G_2)$.

Lemma 2.3 *Let* G *be a critical hamiltonian graph of minimum genus* 1. *Then an associated surface of minimum genus of* G *can be shown as*

$$S = A_0\underline{A}_1 = A_0a_1A_1a_2A_2a_1^{-}A_3a_2^{-},$$

where A_0 *is maximal and* $A_i = a_{i+2}a_{i+2}^{-}$ *or* $\emptyset$ $(1 \leq i \leq 3)$.

Proof Firstly, choose all edges but one from hamiltonian cycle to obtain a spanning tree. Denote cotree edges by $a_i (1 \leq i \leq \beta)$. Since $\gamma(G) = 1$, there must be two edges, say a_1 and a_2, such that a_1 crosses a_2 and one vertex incident with a_1 is adjacent to one vertex incident with a_2. According to the rotation of G, S can be shown as $A_0a_1A_1a_2A_2a_1^{-}A_3a_2^{-}$. And $A_i = a_{i+2}a_{i+2}^{-}$ or $\emptyset$ $(1 \leq i \leq 3)$ provided that A_0 is maximal.

By contradiction. Without loss of generality, suppose that $\underline{A}_1$ contains a_i and a_i^{-} for $1 \leq i \leq 3$.

Case1. $\underline{A}_1 = a_1a_2a_3a_1^{-}a_2^{-}a_3^{-}$.

Remove all edges, denoted by the letters in A_0, from G, one can get a graph G_0, namely $K_{3,3}$. It is a critical graph of minimum genus 1. This contradicts that G is critical.

Case2. $\underline{A}_1 = a_1a_2a_3a_1^{-}a_3^{-}a_2^{-}$.

Since G is a critical graph of minimum genus 1, $G - a_2$ is a planar graph, where $G - a_2$ means removing the edge, denoted by a_2, from G. Then a_1 and a_3 can not belong to $\underline{A}_1$ at the same time.

Subcase2.1. $a_1 \in A_0$.

Since A_0 is maximal, there must be one, say a_4, in A_0 such that $a_4 \in \underline{A}_1$. Since $G - a_2$ is planar, a_4 can not cross a_3. Then a_4 must cross a_2. At this time, $\underline{A}_1 = a_2a_4a_2^{-}a_3a_3^{-}a_4^{-} \sim a_1a_2a_1^{-}a_3a_3^{-}a_2^{-}$.

Subcase2.2. $a_3 \in A_0$.

Since A_0 is maximal, there must be one, say a_4, in A_0 such that $a_4 \in \underline{A}_1$. a_4 can not cross a_1. If a_4 does not cross a_2, then $\underline{A}_1 = a_1a_2a_4a_4^-a_1^-a_2^-$. If a_4 cross a_2, then there exists a planar subgraph G_{00} of G shown in Fig.2.1.

Since G is nonplanar, without loss of generality, suppose that a_5 cross a_1, a_2, a_4. At this time, $G - a_1 - a_3$ is a critical graph of minimum genus 1. This contradicts that G is critical.

On the other hand, clearly, $o(S) = o(A_0a_1A_1a_2A_2a_1^-A_3a_2^-) = 1$. This completes the proof. □

Lemma 2.4 *Let G be a critical hamiltonian graph of minimum genus* 1. *Then the associated surface of minimum genus can also be simply shown as*

$$S = a_2a_1a_2^-a_1^-.$$

To consider genus problem of graph G in Lemma 2.4, it is suffice to consider the graph H_1(Fig.2.2), where there is one possibility for the rotation of each vertex at this time.

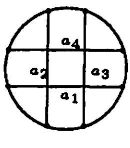

Fig.2.1 G_{00}

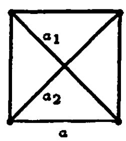

Fig.2.2 H_1

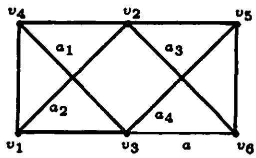

Fig.2.3 H_2

Theorem 2.5 *Let $G_i (1 \le i \le n)$ be a critical graph of minimum genus* 1. *Then*

$$\gamma(G_1 *_e G_2 *_e \cdots *_e G_n) = \lceil \frac{n}{2} \rceil.$$

Proof In order to simplify notation, $G_1 *_e G_2 *_e \cdots *_e G_n$ is written simply as $\underline{G}_n$.

Firstly, $\gamma(\underline{G}_n) \le \lceil \frac{n}{2} \rceil$.

Case1. G_i is hamiltonian for $1 \le i \le n$.

By induction on n. When $n = 1$, obviously, $\gamma(\underline{G}_1) = 1$. When $n = 2$, consider the graph H_2(Fig.2.3). Choose all edges from cycle C but a to obtain a spanning. And at this time, there are two possibilities for the rotation of vertex $v_i (1 \le i \le 2)$, one, for vertex $v_j (3 \le j \le 6)$. Its associated surface can be shown as

$$S_2 = aa_1b_2a_2a_1b_1a_2b_2^-b_1^-a \sim a_1a_2a_1^-a_2^-.$$

Then $\gamma(G_1 *_e G_2) \le 1$.

Suppose that $\gamma(\underline{G}_k) \le \lceil \frac{k}{2} \rceil$ for $1 \le k < n$. Choose a spanning tree of $\underline{G}_n$ to consider graph H_n with the same method as when $n = 2$.

When n is even, let

$$A_i^1 = \begin{cases} a_{2i+1}a_{2i}, & i \text{ is odd;} \\ a_{2i}a_{2i+1}, & i \text{ is even.} \end{cases} \qquad B_j^1 = \begin{cases} a_{\overline{2j}}a_{\overline{2j-3}}, & j \text{ is odd;} \\ a_{\overline{2j-3}}a_{\overline{2j}}, & j \text{ is even.} \end{cases}$$

Then an associated surface of $\underline{G}_n$ can be shown as

$$S_n^1 = a_1a_4a_2a_{\bar{1}}a_{\bar{2}}a_3a_{\bar{4}}a_5\Pi_{i=3}^{n-1}A_i^1a_{2n}a_{\overline{2n-1}}\Pi_{j=n}^3B_j^1$$

$$\sim a_1a_4a_2a_{\bar{1}}a_{\bar{2}}a_3a_{\bar{4}}a_5\Pi_{i=3}^{n-3}A_i^1a_{2n-4}a_{\overline{2n-5}}\Pi_{j=n-2}^3B_j^1a_{2n-1}a_{2n}a_{\overline{2n-1}}a_{\overline{2n}}$$

$$= S_{n-2}^1(a_{2n-1}a_{2n}a_{\overline{2n-1}}a_{\overline{2n}})$$

It follows from the induction hypothesis that $\gamma(\underline{G}_n) \leq \gamma(\underline{G}_{n-2}) + 1 \leq \lceil \frac{n}{2} \rceil$.

When n is odd, let

$$A_i^2 = \begin{cases} a_{2i}a_{2i+1}, & i \text{ is odd;} \\ a_{2i+1}a_{2i}, & i \text{ is even.} \end{cases} \qquad B_j^2 = \begin{cases} a_{\overline{2j-3}}a_{\overline{2j}}, & j \text{ is odd;} \\ a_{\overline{2j}}a_{\overline{2j-3}}, & j \text{ is even.} \end{cases}$$

Then an associated surface of $\underline{G}_n$ can be shown as

$$S_n^2 = a_1a_4a_2a_{\bar{1}}a_{\bar{2}}a_3a_5a_{\bar{4}}\Pi_{i=3}^{n-1}A_i^2a_{2n}a_{\overline{2n-1}}\Pi_{j=n}^3B_j^2$$

$$\sim a_1a_4a_2a_{\bar{1}}a_{\bar{2}}a_3a_5a_{\bar{4}}\Pi_{i=3}^{n-3}A_i^2a_{2n-4}a_{\overline{2n-5}}\Pi_{j=n-2}^3B_j^2a_{2n-1}a_{2n}a_{\overline{2n-1}}a_{\overline{2n}}$$

$$= S_{n-2}^2(a_{2n-1}a_{2n}a_{\overline{2n-1}}a_{\overline{2n}})$$

It follows from the induction hypothesis that $\gamma(\underline{G}_n) \leq \gamma(\underline{G}_{n-2}) + 1 \leq \lceil \frac{n}{2} \rceil$.

Case2. There exists some G_{i_0} containing no hamiltonian cycle. Since G_{i_0} is nonplanar, there must be some cycle. By considering the cycle in the same spirit as considering hamiltonian cycle to obtain that $\gamma(\underline{G}_n) \leq \lceil \frac{n}{2} \rceil$.

Conversely, by removing from $G_k(k = 2, 4, \cdots, 2\lfloor \frac{n}{2} \rfloor)$ all edges but the amalgamated edges and the one, which connects G_{k-1} to G_{k+1}, we obtain a subgraph of $\underline{G}_n$, denoted by G^0(Fig.2.4).

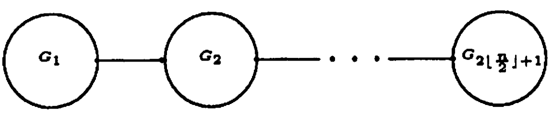

Fig.2.4 G^0

Then

$$G^0 \subseteq \underline{G}_n.$$

Therefore by applying Theorem 2.1,

$$\gamma(\underline{G}_n) \geq \gamma(G^0) = \lceil \frac{n}{2} \rceil.$$

This completes the proof. □

From the proof of Theorem 2.5, apparently, Corollary 2.6 is derived.

Corollary 2.6 $\gamma(H_n + v_0) = \lceil \frac{n}{2} \rceil$, *where* $v_0 \notin H_n$ *and* $n \geq 1$.

Conjecture 2.7 *Let* G_i *be a critical graph of minimum genus* k_i. *Then*

$$\gamma(G_1 *_e G_2 *_e \cdots *_e G_n) = \sum_{i=1}^{n} k_i - \lfloor \frac{n}{2} \rfloor.$$

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