Minimum Number of Vertices of Graphs without Perfect Matching, with Given Edge Connectivity and Minimum and Maximum Degrees

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Abstract. A *matching* M in a graph G is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex v is *unsaturated* by M if there is no edge of M is incident with v. A matching M is called a *perfect matching* if there is no vertex of the graph is unsaturated by M. Let G be a k-edge-connected graph, $k \geq 1$, on even n vertices, have minimum degree r and maximum degree $r+e$, $e \geq 1$. In this paper we find a lower bound for n when G has no perfect matchings.

1. Introduction

For our purposes, all graphs are finite, loopless and have no multiple edges. For most part of our notation and terminology we follow that of Diestel [1]. Thus G is a graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of a set A is denoted by $|A|$. Hence, the number of vertices and number of edges are $|V(G)|$ and $|E(G)|$ respectively.

A *matching* M in G is a subset of $E(G)$ in which no two edges have a vertex in common. A vertex v is *unsaturated* by M if there is no edge of M is incident with v. A matching M is called a *perfect matching* (or 1-*factor*) if there is no vertex of the graph is unsaturated by M.

Many problems concerning matching have been studied in the literature, see [3]. The perfect matchings of almost regular graph have been studied by Caccetta and Mardiono [2]. Volkmann and Zingsem [4] studied the number of vertices of connected bipartite graphs G with partite sets X and Y, $|X|=|Y|$, each vertex has degree r or $r+1$, and G has no perfect matching. Zhao [5] studied the number of vertices of graphs G, with each vertex has degree r or $r+1$, and G has no odd component and no perfect matching. The result is as follows.

Theorem 1.1 (Zhao). *A graph G on $2n$ vertices, each vertex has degree r or $r+1$, with no perfect matching and no odd component, satisfies $|V(G)| \geq 3r+4$.* []

Let G be a connected graph. An *edge cut set* E_1 of G is a minimal subset of $E(G)$ such that $G-E_1$ disconnected. The *edge-connectivity* of G is the minimum cardinality of edge cut sets, and is denoted by $k'(G)$. A graph G is *k-edge-connected* if $k \leq k'(G)$. Thus every connected graph is 1-edge-connected.

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In this paper we generalize Theorem 1.1. Let G be a k-edge-connected graph, $k\geq 1$, on even n vertices, have minimum degree r and maximum degree $r+e$, $e\geq 1$, and G has no perfect matching. We obtain the lower bound of n and show that the bound is sharp.

2. The Bounds

Let G be a graph. If S is a subset of $V(G)$, $G-S$ denotes the graph formed from G by deleting all the vertices in S together with their incident edges. A component of G is called *odd* or *even* according as its number of vertices is odd or even. The number of odd components of a graph G is denoted by $o(G)$. We need the following well-known Tutte's theorem ([1], p39) to establish our results.

Tutte's Theorem. A graph G has a perfect matching if an only if $o(G-S)\leq |S|$ for all $S\subseteq V(G)$. []

Let G be a k-edge-connected graph, $k\geq 1$, on even n vertices, G have minimum degree 1 and maximum degree $r+e$, $e\geq 1$. For the trivial case, when $r=e=1$, if a graph G has no odd component, and each vertex has degree 1 or $1+1$, then G has a perfect matching. Further, when G has minimum degree 1, has maximum degree $r+e$, and has no perfect matching, then $n\geq r+e+1$. Hence we only consider for the case $r\geq 2$.

Theorem 2.1. *Let G be a k-edge-connected graph, $k\geq 1$, on even n vertices, G have minimum degree r and maximum degree $r+e$, $r\geq 2$, $e\geq 1$. If G has no perfect matching, then*

a) $n\geq 3r+4$ *if* $e=1$, *r is even and* $k\leq \frac{r+1}{3}$,

b) $n\geq 3r+5$ *if* $e=1$, *r is odd, and* $k=1$,

c) $n\geq 3r+7$ *if* $e=1$, $r\geq 5$ *is odd and* $2\leq k\leq \frac{r+1}{3}$,

d) $n\geq 2\left\lceil \frac{3r}{2}\right\rceil+2k+2$ *if* $e=1$ *and* $\frac{r+2}{3}\leq k\leq \frac{r-1}{2}$,

e) $n\geq 4r+2$ *if* $e=1$ *and* $k\geq \frac{r}{2}$,

f) $n\geq 2r+2$ *if* $2\leq e\leq r$,

g) $n\geq 2\left\lceil \frac{r+e+1}{2}\right\rceil$ *if* $e\geq r+1$.

Proof. Suppose G has no perfect matching. By Tutte's Theorem, there exists a vertex set $S\subseteq V(G)$ such that $o(G-S)>|S|$. Since G is connected and n is even, then $|S|\neq\varnothing$ and $o(G-S)\geq |S|+2\geq 3$.

Let H be an odd component of $G-S$ joined to S by at most $r-1$ edges and having h vertices. By counting degree of vertices, we have

$$h(h-1)+(r-1)\geq rh,$$

$$h^2-(r+1)h+(r-1)\geq 0,$$

and hence

$$h\leq\frac{(r+1)-\sqrt{(r+1)^2-4(r-1)}}{2} \quad\text{or}\quad h\geq\frac{(r+1)+\sqrt{(r+1)^2-4(r-1)}}{2},$$

$$h\leq\frac{(r+1)-\sqrt{(r-1)^2+4}}{2} \quad\text{or}\quad h\geq\frac{(r+1)+\sqrt{(r-1)^2+4}}{2},$$

$$h<1 \quad\text{or}\quad h>r.$$

Since h is a positive integer, then $h\geq r+1$; and since h is odd, then $h\geq r+2$ when r is odd. Thus, when r is even (or odd) each odd component of $G-S$ has at least $r+1$ (or $r+2$) vertices, or each odd component of $G-S$ has at least $2\left\lceil\frac{r}{2}\right\rceil+1$ vertices.

Let m be the number of odd components of $G-S$ joined to S by at most $r-1$ edges. Since

$$(r+e)|S|\geq mk+r(|S|+2-m),$$

then

$$|S|\geq\frac{mk+(2-m)r}{e},$$

and

$$\begin{aligned} n &\geq m(2\left\lceil\frac{r}{2}\right\rceil+1)+2|S|+2-m \\ &\geq 2m\left\lceil\frac{r}{2}\right\rceil+2(\frac{mk+(2-m)r}{e})+2 \\ &= 2m\left\lceil\frac{r}{2}\right\rceil+\frac{2mk+2(2-m)r}{e}+2. \end{aligned}$$

Case 1: $e=1$.

Subcase 1.1: $k\leq\frac{r+1}{3}$.

Let $m=0$. Then $n\geq 4r+2$. So $n\geq 3r+4$ when r is even, and $n\geq 3r+7$ when $r\geq 5$ is odd.

Let $m=1$. Then $n\geq 2\left\lceil\frac{r}{2}\right\rceil+2r+2k+2$. Hence, $n\geq 3r+4$ when r is even, $n\geq 3r+5$ when r is odd and $k=1$, and $n\geq 3r+7$ when r is odd and $k\geq 2$.

Let m = 2. Then $|S|\geq 2k\geq 2$ and $o(G-S)\geq 4$. If each other odd components of $G-S$ has at least $\frac{r}{2}$ vertices, then

$$n \geq 2(2\left\lceil \frac{r}{2} \right\rceil + 1) + 2k + 2k\left\lceil \frac{r}{2} \right\rceil$$

$$\geq (2k+4)\left\lceil \frac{r}{2} \right\rceil + 2k + 2$$

$$\geq 6\left\lceil \frac{r}{2} \right\rceil + 4.$$

If there is an odd component of $G-S$ has less than $\frac{r}{2}$ vertices, then $|S| \geq \left\lceil \frac{r}{2} \right\rceil + 1$, and

$$n \geq 2(2\left\lceil \frac{r}{2} \right\rceil + 1) + 2|S| \geq 6\left\lceil \frac{r}{2} \right\rceil + 4.$$

We have $n \geq 3r+4$ when r is even and $n \geq 3r+7$ when r is odd.

Let $m \geq 3$. Then $n \geq 3(2\left\lceil \frac{r}{2} \right\rceil + 1) + 1 = 6\left\lceil \frac{r}{2} \right\rceil + 4$. Again, we have $n \geq 3r+4$ when r is even and $n \geq 3r+7$ when r is odd.

For all m we have:

$n \geq 3r+4$ if r is even,
$n \geq 3r+5$ if r is odd and $k=1$,
$n \geq 3r+7$ if $r \geq 5$ is odd and $k \geq 2$.

Subcase 1.2: $\frac{r+2}{3} \leq k \leq \frac{r-1}{2}$. Then $r \geq 5$.

If $m=0$, then $n \geq 4r+2$.

If $m=1$, then. $n \geq 2\left\lceil \frac{r}{2} \right\rceil + 2r + 2k + 2 \geq 2\left\lceil \frac{3r}{2} \right\rceil + 2k + 2$.

Let $m=2$. Then $|S| \geq 2k \geq 2$. If $|S| \geq r$, then

$$n \geq 2(2\left\lceil \frac{r}{2} \right\rceil + 1) + 2|S|$$

$$\geq 4\left\lceil \frac{r}{2} \right\rceil + 2r + 2.$$

Let $|S| \leq r-1$. Then every odd component of $G-S$ has at least three vertices. If we can show that every odd component of $G-S$ has at least $r-1$ vertices, then

$$n \geq 2(2\left\lceil \frac{r}{2} \right\rceil + 1) + 2(r-1) + 2$$

$$\geq 4\left\lceil \frac{r}{2} \right\rceil + 2r + 2.$$

Suppose there exists an odd component of $G-S$ has only x vertices, where $3 \le x \le r-2$. Then this component is joined to S by $t \ge xr - x(x-1)$ edges. Since $3 \le x \le r-2$, then $t \ge 3r-6$. Consequently,

$$(r+1)|S| \ge 2k + r(|S|-1) + (3r-6),$$

$$|S| \ge 2k+2r-6 \ge r,$$

this contradicts to $|S| \le r-1$.

Let $m=3$. Then $|S| \ge 2$. If $|S| \ge r$, then

$$n \ge 3(2\left\lceil \frac{r}{2} \right\rceil + 1) + 2|S| - 1$$

$$\ge 6\left\lceil \frac{r}{2} \right\rceil + 2r + 2.$$

Let $|S| \le r-1$. Then, as for the case $m=2$, every odd component of $G-S$ has at least $r-1$ vertices, and we have

$$n \ge 3(2\left\lceil \frac{r}{2} \right\rceil + 1) + (r-1) + 2$$

$$= 6\left\lceil \frac{r}{2} \right\rceil + r + 4$$

Let $m \ge 4$. Then $n \ge 4(2\left\lceil \frac{r}{2} \right\rceil + 1) + |S| \ge 8\left\lceil \frac{r}{2} \right\rceil + 5$.

Since $2k \le r-1$, then for every m we have $n \ge 2\left\lceil \frac{3r}{2} \right\rceil + 2k + 2$.

Subcase 1.3: $k \ge \frac{r}{2}$.

$$n \ge 2m\left\lceil \frac{r}{2} \right\rceil + 2mk + 2(2-m)r + 2$$

$$= (2k-r)m + 4r + 2$$

$$\ge 4r + 2.$$

Case 2: $e \ge 2$.

Let $2 \le e \le r$. If each odd component has at least r vertices, then

$$n \ge 1 + 3r > 2r + 2.$$

Let the minimum number of vertices of odd components is t, $1 \le t \le r-1$. Then $|S| \ge r-t+1$, and

$$n \ge |S| + (|S|+2)t$$

$$\ge (r-t+1) + (r-t+3)t$$

$$
\begin{aligned}
&= 2r+2-1+(t-1)r+2t-t^2 \\
&= 2r+2+(t-1)r-(t-1)^2 \\
&\geq 2r+2.
\end{aligned}
$$

If $e > r+1$, then we use $n \geq r+e+1$, and $n \geq 2\left\lceil \frac{r+e+1}{2} \right\rceil$ since n is even. []

3. Construction

In this section we show that the bound in Theorem 2.1 is sharp; we show that for every integer n, the same as the bound, there exists a k-edge-connected graph G, $k \geq 1$, on n vertices, G have minimum degree r and maximum degree $r+e$, $e \geq 1$, and G has no perfect matching.

We will use the following graph in our constructions. Let p be odd integer greater than 2. We construct a graph B_p as follows. Take a copy of complete graph K_p and delete $\frac{p-1}{2}$ of its edges which are disjoint. The resulting graph B_p has $p-1$ vertices of degree $p-2$ and one vertex of degree $p-1$.

Theorem 3.1. *Let k, r, e, and n be integers,* $1 \leq k \leq r$, $e \geq 1$, *and*

a) $n = 3r+4$ *if* $e=1$, *r is even and* $k \leq \frac{r+1}{3}$,

b) $n = 3r+5$ *if* $e=1$, *r is odd, and* $k=1$,

c) $n = 3r+7$ *if* $e=1$, $r \geq 5$ *is odd and* $2 \leq k \leq \frac{r+1}{3}$,

d) $n = 2\left\lceil \frac{3r}{2} \right\rceil + 2k + 2$ *if* $e=1$ *and* $\frac{r+2}{3} \leq k \leq \frac{r-1}{2}$,

e) $n = 4r+2$ *if* $e=1$ *and* $k \geq \frac{r}{2}$,

f) $n = 2r+2$ *if* $e \geq 2$ *and* $2 \leq e \leq r$,

g) $n = 2\left\lceil \frac{r+e+1}{2} \right\rceil$ *if* $e \geq r+1$.

Then there exists a k-edge-connected graph on n vertices, has minimum degree r and maximum degree $r+e$, *and has no perfect matching.*

Proof.

a) Let $e=1$, r be even and $k \leq \frac{r+1}{3}$.

We have $r+1-2k \geq k$. We construct a graph G_1 as follows. Take three copies, H_1, H_2, and H_3, of complete graph K_{r+1} and one other vertex v. Join $r+1-2k$, k, and k different vertices of H_1, H_2, and H_3, respectively, to v. The resulting G_1 is a k-edge-connected graph on $n = 3r+4$ vertices, has minimum degree r and maximum degree $r+1$, and has no perfect matching.

b) Let $e=1$, r is odd, and $k=1$.

We construct a graph G_2 as follows. Take a copy H_1 of graph B_{r+2}. Graph H_1 has $r+1$ vertices of degree r and one vertex of degree $r+1$. Take $2r+3$ vertices $u_1, u_2, \ldots, u_{r+1}$, $v_1, v_2, \ldots, v_{r+2}$, and for each $i = 1, 2, \ldots, r+2$, join v_i to every u_j, $i \le j \le i+r-1$, j is reduced to modulo $r+1$ if necessary. Then every u_j, $j = 1, 2, \ldots, r$, has degree $r+1$, every u_{r+1} and v_i, $i = 1, 2, \ldots, r+2$, has degree r. Then join u_{r+1} to one vertex of degree r in H_1. The resulting graph G_2 is a 1-edge-connected graph on $n=3r+5$ vertices, has minimum degree r and maximum degree $r+1$, and has no perfect matching.

c) Let $e=1$, $r \ge 5$ is odd and $2 \le k \le \dfrac{r+1}{3}$.

We have $r+1-2k \ge k$. We construct a graph G_2 as follows. Take three copies, H_1, H_2, and H_3, of graph B_{r+2} and one other vertex v. Each H_i has $r+1$ vertices of degree r and one vertex of degree $r+1$. Join $r+1-2k$, k, and k different vertices of degree r of H_1, H_2, and H_3, respectively, to v. The resulting graph G_3 is a k-edge-connected graph on $n=3r+7$ vertices, has minimum degree r and maximum degree $r+1$, and has no perfect matching.

d) Let $e=1$ and $\dfrac{r+2}{3} \le k \le \dfrac{r-1}{2}$.

We construct a graph G_4 as follows. Take $2r+2k+1$ vertices $u_1, u_2, \ldots, u_{r+k+1}$, $v_1, v_2, \ldots, v_{r+k}$. For each $i = 1, 2, \ldots, r+k+1$, join u_i to every v_j, $i \le j \le i+r-1$, j is reduced to modulo $r+k$ if necessary. The resulting graph H_1 has r vertices of degree $r+1$, and k vertices of degree r. When r is even, take a copy H_2 of graph K_{r+1}. When r is odd, take a copy H_2 of graph B_{r+2}. Graph H_2 has $r+1$ vertices of degree r (and one vertex of degree $r+1$ when r is odd). Then join each vertex of degree r in H_1 to one vertex of degree r in H_2, different vertices of H_1 are joined to different vertices of H_2. The resulting graph G_4 is a k-edge-connected graph on $n=2\left\lceil \dfrac{3r}{2} \right\rceil+2k+2$ vertices, has minimum degree r and maximum degree $r+1$, and has no perfect matching.

e) Let $e=1$ and $k \ge \dfrac{r}{2}$.

We construct a graph G_5 as follows. Take $4r+2$ vertices $u_1, u_2, \ldots, u_{2r}$, $v_1, v_2, \ldots, v_{2r+2}$, and for each $i = 1, 2, \ldots, 2r$, join u_i to every v_j, $i \le j \le 2r-1$, j is reduced to modulo $2r$ if necessary. Join v_{2r+1} to every u_i, $1 \le i \le r$, and join v_{2r+2} to every u_i, $r+1 \le i \le 2r$. The resulting graph G_5 is a k-edge-connected graph on

$n=4r+2$ vertices, has minimum degree r and maximum degree $r+1$, and has no perfect matching.

f) Let $2 \le e \le r$.

We construct a graph G_6 as follows. Take a complete bipartite graph $K_{r,r+2}$ with partition sets, say, X and Y, where $|X| = r$. Join one vertex of X to other $e-2$ vertices of X. The resulting graph G_6 is a k-edge-connected graph on $n=2r+2$ vertices, has minimum degree r and maximum degree $r+e$, and has no perfect matching.

g) Let $e \ge r+1$.

We have $e \ge r+2$ or $e \ge r+1$ depend on whether e and r have the same parity or different parity. We construct a graph G_7 as follows. When e and r have the same parity, take a complete graph $K_{r,e+1}$ and one other vertex v; then join v to all vertices, except one vertex of degree r, of $K_{r,e+1}$. When e and r have the different parity, take a complete graph $K_{r-1,e+1}$ and one other vertex v; then join v to all vertices of $K_{r-1,e+1}$. The resulting graph G_7 is a k-edge-connected graph on $n=2\left\lceil \frac{r+e+1}{2} \right\rceil$ vertices, has minimum degree r and maximum degree $r+e$, and has no perfect matching. □

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