Spectral Conditions for Some Stable Properties of Graphs

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Abstract

Using the spectral invariants of graphs, we present sufficient conditions for some stable properties of graphs.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [2]. A graph G is Hamiltonian if G has a Hamiltonian cycle, a cycle containing all the vertices of G. A graph G is traceable if G has a Hamiltonian path, a path containing all the vertices of G. A graph G is pancyclic if G has a cycle of length l for each l between three and the order of G. The concept of stability was introduced by Bondy and Chvátal in [1]. Let P be a property defined on all graphs of order n. Let k be a nonnegative integer. The P is said to be k - stable if whenever $G + uv$ has property P and $d_G(u) + d_G(v) \geq k(n, P)$, where $uv \notin E$, then G itself has property P. It is well known that the Hamiltonicity and traceability are respectively n - stable and $(n-1)$ - stable. The k - closure of a graph G, denoted $cl_k(G)$, is a graph obtained from G by recursively joining two nonadjacent vertices such that their degree sum is at least k.

For each k, where $k = 1$ or 2, Q_k^n $(n \geq 2k)$ is defined as a graph obtained by joining k distinct vertices of the complete graph K_{n-k} to each of k - independent vertices. We also use $K_{n-1} + e$ to denote Q_1^n. $K_{n-1} + v$ is defined as a graph that consists of a complete graph of order $n - 1$ together with an isolated vertex v. Let EP_n be the set of graphs of the following three types of graphs on n vertices: (a) a regular graph of degree $\frac{n}{2} - 1$, (b) a graph consisting of two complete components, or (c) the join of a regular graph of degree $\frac{n}{2} - 1 - r$ and a graph on r vertices, where $1 \leq r \leq \frac{n}{2} - 1$.

JCMCC **88** (2014), pp. 199-205

Let EC_n be the set of graphs of the following two types of graphs on n vertices: (a) the join of a trivial graph and a graph consisting of two complete components, or (b) the join of a regular graph of degree $\frac{n-1}{2} - r$ and a graph on r vertices, where $1 \leq r \leq \frac{n-1}{2} - 1$. We use $C(n, r)$ to denote the number of r - combinations of a set with n distinct elements.

Let $\mu_1(G) \leq \mu_2(G) \leq \ldots \leq \mu_n(G)$ denote the eigenvalues of the graph G of order n. The Laplacian of a graph G is defined as $L(G) = D(G) - A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $0 = \lambda_1(G) \leq \lambda_2(G) \leq \ldots \leq \lambda_n(G)$ of $L(G)$ are called the Laplacian eigenvalues of the graph G. Define $\Sigma_2(G) := \sum_{i=1}^{n} \lambda_i^2(G)$. Since $0 = \lambda_1^2(G) \leq \lambda_2^2(G) \leq \ldots \leq \lambda_n^2(G)$ are the eigenvalues of $L^2(G)$, we have that $\Sigma_2(G)$ is the sum of the diagonal entries in $L^2(G) = \sum_{i=1}^{n}(d_i^2(G) + d_i(G)) = \sum_{i=1}^{n} d_i^2(G) + 2e(G)$. By Lemma 13.1.3 in [4], we have that $\lambda_i(G^c) = n - \lambda_{n-i+2}(G)$ for each i with $2 \leq i \leq n$. Then $\Sigma_2(G^c)$, which will be used in our theorems below, is equal to $\sum_{i=2}^{n}(n - \lambda_i(G))^2$. The signless Laplacian of a graph G is defined as $L^+(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix of the degree sequence of G and $A(G)$ is the adjacency matrix of G. The eigenvalues $\gamma_1(G) \leq \gamma_2(G) \leq \ldots \leq \gamma_n(G)$ of $L^+(G)$ are called the signless Laplacian eigenvalues of the graph G.

The following interesting results were obtained by Fiedler and Nikiforov [3].

Theorem 1 Let G be a graph of order n.
(i) If $\mu_n(G^c) \leq \sqrt{n-1}$, then G contains a Hamiltonian path unless $G = K_{n-1} + v$.
(ii) If $\mu_n(G^c) \leq \sqrt{n-2}$, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.

The following results obtained by Zhou [8] are also interesting.

Theorem 2 Let G be a graph of order n.
(i) If $\gamma_n(G^c) \leq n$ and $G \notin EP_n$, then G contains a Hamiltonian path.
(ii) If $n \geq 3$, $\gamma_n(G^c) \leq n - 1$ and $G \notin EC_n$, then G contains a Hamiltonian cycle.

The following results were obtained by Li [7] .

Theorem 3 Let G be a graph of order n.
(i) If $\Sigma_2(G^c) \leq (n-1)(n+2)$, then G contains a Hamiltonian path unless $G = K_{n-1} + v$.
(ii) If $\Sigma_2(G^c) \leq (n-2)(n+1)$, then G contains a Hamiltonian cycle unless $G = K_{n-1} + e$.

Theorem 4 Let G be a 2 - connected graph of order $n \geq 12$.
(*i*) If $\mu_n(G^c) \leq \sqrt{\frac{(2n-7)(n-1)}{n}}$, then G contains a Hamiltonian cycle or $G = Q_2^n$.
(*ii*) If $\Sigma_2(G^c) \leq (2n-7)(n+1)$, then G contains a Hamiltonian cycle or $G = Q_2^n$.

Motivated by the theorems above, we will prove the following theorems on stable properties of graphs.

Theorem 5 Let G be a graph of order n. Suppose that P is a $r(n,P)$ - stable property and the complete graph K_n of order n has property P. Moreover, if $e(G) \geq l(n,P)$, then G has the property P. Then
(*i*) If $\mu_n(G^c) \leq \sqrt{\frac{(2n-r(n,P)-1)(C(n,2)-l(n,P))}{n}}$, then G has property P.
(*ii*) If $\Sigma_2(G^c) \leq (2n-r(n,P)+1)(C(n,2)-l(n,P))$, then G has property P.

Theorem 6 Let G be a graph of order n. Suppose that P is a $r(n,P)$ - stable property and the complete graph K_n of order n has property P. Then
(*i*) If $\mu_n(G^c) < \sqrt{\frac{(2n-r(n,P)-1)(2n-r(n,P)-2)}{n}}$, then G has property P.
(*ii*) If $\Sigma_2(G^c) < (2n-r(n,P)+1)(2n-r(n,P)-2)$, then G has property P.
(*iii*) If $\gamma_n(G^c) < (2n-r(n,P)-1)$, then G has property P.

2. Proofs

Proof of Theorem 5. Let G be a graph satisfying the conditions in Theorem 5 and G does not have property P. Then $H := cl_{r(n,P)}(G)$ does not have property P and therefore H is not K_n. Thus there exist two vertices x and y in $V(H)$ such that $xy \notin E(H)$ and for any pair of non-adjacent vertices u and v in $V(H)$ we have $d_H(u) + d_H(v) \leq r(n,P) - 1$. Hence for any pair of adjacent vertices u and v in $V(H^c)$ we have that $d_{H^c}(u) + d_{H^c}(v) = n - 1 - d_H(u) + n - 1 - d_H(v) \geq 2n - r(n,P) - 1$. So

$$\sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (2n - r(n,P) - 1)e(H^c).$$

Moreover, we have that

$$\sum_{v \in V(H^c)} d_{H^c}^2(v) = \sum_{uv \in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (2n - r(n,P) - 1)e(H^c).$$

(*i*) Suppose that $\mu_n(G^c) \leq \sqrt{\frac{(2n-r(n,P)-1)(C(n,2)-l(n,P))}{n}}$.

From the inequality of Hofmeister [6], we have that

$$n\mu_n^2(H^c) \geq \sum_{v\in V(H^c)} d_{H^c}^2(v) \geq (2n - r(n,P) - 1)e(H^c).$$

Since H^c is a subgraph of G^c,

$$e(H^c) \leq \frac{n\mu_n^2(H^c)}{2n - r(n,P) - 1} \leq \frac{n\mu_n^2(G^c)}{2n - r(n,P) - 1}.$$

Hence

$$e(H) = C(n,2) - e(H^c) \geq C(n,2) - \frac{n\mu_n^2(G^c)}{2n - r(n,P) - 1} \geq l(n,P).$$

Therefore H has the property P, a contradiction. Thus we complete the proof of (i) in Theorem 5.

(ii) Suppos that $\Sigma_2(G^c) \leq (2n - r(n,P) + 1)(C(n,2) - l(n,P))$.
From Theorem 13.6.2 in [4], we have that $\Sigma_2(G^c) \geq \Sigma_2(H^c) = \sum_{i=1}^{n} d_i^2(H^c) + 2e(H^c)$. Therefore

$$e(H^c) \leq \frac{\Sigma_2(G^c)}{2n - r(n,P) + 1}.$$

Hence

$$e(H) = C(n,2) - e(H^c) \geq C(n,2) - \frac{\Sigma_2(G^c)}{2n - r(n,P) + 1} \geq l(n,P).$$

Therefore H has the property P, a contradiction. Thus we complete the proof of (ii) in Theorem 5.

Proof of Theorem 6. Let G be a graph satisfying the conditions in Theorem 6 and G does not have property P. Then $H := cl_{r(n,P)}(G)$ does not have property P and therefore H is not K_n. Using the same proof as in the Proof of Theorem 5, we have that

$$\sum_{v\in V(H^c)} d_{H^c}^2(v) = \sum_{uv\in E(H^c)} d_{H^c}(u) + d_{H^c}(v) \geq (2n - r(n,P) - 1)e(H^c).$$

Since H is not complete and there exist two vertices, say x and y, in $V(H)$ such that $d_H(x)+d_H(y) \leq r(n,P)-1$. Then $e(H) \leq C(n-2,2)+r(n,P)-1$. Thus $e(H^c) \geq C(n,2) - C(n-2,2) - r(n,P) + 1 = 2n - r(n,P) - 2$.

(*i*) Suppse that $\mu_n(G^c) < \sqrt{\frac{(2n-r(n,P)-1)(2n-r(n,P)-2)}{n}}$.
From the inequality of Hofmeister [6], we have that

$$n\mu_n^2(H^c) \geq \sum_{v\in V(H^c)} d_{H^c}^2(v) \geq (2n - r(n,P) - 1)(2n - r(n,P) - 2).$$

Since H^c is a subgraph of G^c,

$$n\mu_n^2(G^c) \geq n\mu_n^2(H^c) \geq (2n - r(n,P) - 1)(2n - r(n,P) - 2),$$

a contradiction. Thus we complete the proof of (*i*) in Theorem 6.

(*ii*) Suppose that $\Sigma_2(G^c) < (2n - r(n,P) + 1)(2n - r(n,P) - 2)$.
From Theorem 13.6.2 in [4], we have that $\Sigma_2(G^c) \geq \Sigma_2(H^c) = \sum_{i=1}^{n} d_i^2(H^c) + 2e(H^c)$ again. Therefore

$$\Sigma_2(G^c) \geq (2n - r(n,P) + 1)e(H^c) \geq (2n - r(n,P) + 1)(2n - r(n,P) - 2),$$

a contradiction. Thus we complete the proof of (*ii*) in Theorem 6.

(*iii*) Suppse that $\gamma_n(G^c) < (2n - r(n,P) - 1)$.
From Lemma 3 in [8] and the fact that H^c is a subgraph of G^c, we have that

$$\gamma_n(G^c) \geq \gamma_n(H^c) \geq \frac{\sum_{v\in V(H^c)} d_{H^c}^2(v)}{e(H^c)} \geq (2n - r(n,P) - 1),$$

a contradiction. Thus we complete the proof of (*iii*) in Theorem 6.

3. Applications of Theorem 5 and Theorem 6

In this section, we will present some applications of Theorem 5. From Theorem 9.1 in [1], we have the following theorem.

Theorem 7 The property that a graph of order n has cylces of lengths between 4 and n is $(2n - 4)$ - stable.

Obviously, the property that a graph of order n has cylces of lengths between 4 and n is very close to the pancyclicity of graphs. The following thereom was proved by Häggkvist, Faudree, and Schelp [5].

Theorem 8 Every Hamiltonian graph G of order n and size $e(G) \geq \frac{(n-1)^2}{4} + 2$ is pancyclic or bipartite.

Theorem 9 Let G be a graph of order $n \geq 6$. If $\mu_n(G^c) \leq \sqrt{\frac{3(n^2-9)}{4n}}$, then G has cycles of lengths between 4 and n.

Proof of Theorem 9. Let P be the property that a graph of order n has cylces of lengths between 4 and n. Let G be a graph satisfying the conditions in Theorem 9. Recall that the largest eigenvalue of a bipartite graph $K_{s,t}$ is $\sqrt{st}$. Since $\mu_n((K_{n-1}+e)^C) = \sqrt{n-2}$ and $\mu_n(G^c) \leq \sqrt{\frac{3(n^2-9)}{4n}}$. Thus $G \neq K_{n-1}+e$. Notice that $\mu_n(G^c) \leq \sqrt{\frac{3(n^2-9)}{4n}} \leq \sqrt{n-2}$. Hence by (ii) in Theorem 1 we have that G is Hamiltonian. Now we can apply Theorem 8 to the graph G. We first observe that G cannot be bipartite. Suppose, to the contrary, that G is a bipartire. Then G is a subgraph of $K_{s,t}$, where $s \geq 1$, $t \geq 1$, and $s+t=n$. Then

$$\frac{n}{2} - 1 \leq \max\{s-1, t-1\} \leq \mu_n((K_{s,t})^C) \leq \mu_n(G^c) \leq \sqrt{\frac{3(n^2-9)}{4n}},$$

a contradiction. Let $r(n,P) = 2n-4$ and $l(n,P) = \frac{(n-1)^2}{4} + 2$ in Theorem 5. By (i) in Theorem 5, we complete the proof of Theorem 9.

Theorem 10 Let G be a 2 - connected graph of order $n \geq 12$. If $\Sigma_2(G^c) \leq \frac{5(n^2-9)}{4}$, then G has cycles of lengths between 4 and n or bipartite.

Proof of Theorem 10. Let P be the property that a graph of order n has cylces of lengths between 4 and n. Let G be a graph satisfying the conditions in Theorem 10. Since $\Sigma_2((Q_2^n)^c) = 4(n-4)+2(n-3)^2+2(n-4)+2(n-3) > \frac{5(n^2-9)}{4}$, $G \neq Q_2^n$. Notice that $\Sigma_2(G^c) \leq \frac{5(n^2-9)}{4} \leq (2n-7)(n+1)$. Thus, by (ii) in Theorem 4, G is Hamiltonian. Now we can apply Theorem 8 to the graph G. Let $r(n,P) = 2n-4$ and $l(n,P) = \frac{(n-1)^2}{4} + 2$ in Theorem 5. By (ii) in Theorem 5, we complete the proof of Theorem 10.

Obviously, we can use the theorems proved by Bondy and Chvátal in Appendix of [1] and Theorem 6 to obtain the spectral conditions for stable properties of graphs. The details of those conditions are omitted here.

4. Acknowledgments
The author would like to thank the referee for his or her suggestions and comments which improve this paper.

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