Dudeney's round table problem and neighbour-balanced Hamilton decompositions

Midori Kobayashi
University of Shizuoka, Shizuoka, 422-8526 Japan
midori@u-shizuoka-ken.ac.jp
Nobuaki Mutoh
University of Shizuoka, Shizuoka, 422-8526 Japan
muto@u-shizuoka-ken.ac.jp
Gisaku Nakamura
University of Shizuoka, Shizuoka, 422-8526 Japan

Abstract

Dudeney's round table problem asks for a set of Hamilton cycles in K_n having the property that each 2-path in K_n lies in exactly one of the cycles. In this paper, we show how to construct a solution of Dudeney's round table problem for even n from a semi-antipodal Hamilton decomposition of K_{n-1}.

1 Introduction

Dudeney's round table problem is the following problem.

> "Seat n people at a round table on $(n-1)(n-2)/2$ consecutive days so that each person sits between every pair of other people exactly once."

This problem is equivalent to asking for a set of Hamilton cycles in K_n having the property that each 2-path (a path of length 2) in K_n lies in exactly one of the cycles. We call this set a *Dudeney set* in K_n.

It has been conjectured that there is a Dudeney set for every complete graph. The conjecture is still unsettled in general (see [1, 3] for the details); however, when n is even it is known that the conjecture holds.

Theorem A [2] *There exists a Dudeney set in K_n when n is even.*

The proof of Theorem A is complicated and long, so a simple proof is desirable.

A Hamilton decomposition $\mathcal{H}$ of K_n (n is odd) is called *semi-antipodal* if the set of chords at distance $(n-1)/2$ of the cycles in $\mathcal{H}$ is the edge set of K_n. (A *chord* of a cycle C is an edge not in the edge set of C whose endvertices are in the vertex set of C. A *chord at distance k* of a cycle C is a chord of C whose endvertices lie at distance k in the cycle C.)

In this paper, we prove the following theorem which will be useful to have a simple proof of Theorem A.

Theorem 1.1 *Let $n \geq 5$ be an odd integer. A Dudeney set in K_{n+1} is constructed from a semi-antipodal Hamilton decomposition of K_n.*

2 A proof of Theorem 1.1

For two sequences $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ of length n, define a sequence $X \times Y$ of length $2n$ as

$$X \times Y = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n),$$

where x_i and y_i are variables. Define Y^{rev} and $s^j Y^{\text{rev}}$ $(0 \leq j \leq n-1)$ as

$$\begin{aligned} Y^{\text{rev}} &= (y_n, y_{n-1}, \ldots, y_1), \\ s^j Y^{\text{rev}} &= (y_{n-j}, y_{n-j-1}, \ldots, y_{n-j+1}), \end{aligned}$$

where the subscripts of the x_i and y_i are calculated modulo n and $s^0 Y^{\text{rev}} = Y^{\text{rev}}$. Then we have

$$X \times s^j Y^{\text{rev}} = (x_1, y_{n-j}, x_2, y_{n-j-1}, x_3, y_{n-j-2}, \ldots, x_n, y_{n-j+1}).$$

Let $n \geq 5$ be odd and $\mathcal{H}$ a semi-antipodal Hamilton decomposition of K_n. Let $H = (a_1, a_2, \ldots, a_n)$ be a Hamilton cycle in $\mathcal{H}$. Consider $(a_1, a_2, \ldots, a_n)$ as a sequence of length n, then we obtain

$$H \times s^j H^{\text{rev}} = (a_1, a_{n-j}, a_2, a_{n-j-1}, a_3, a_{n-j-2}, \ldots, a_n, a_{n-j+1}) \quad (0 \leq j \leq n-1).$$

Put $H \times s^j H^{\text{rev}} = (c_1, c_2, \ldots, c_{2n})$, then we have $c_i = c_{i+1}$ and $c_{i+n} = c_{i+n+1}$ for some i $(1 \le i \le n)$, where the subscripts of the c_i are calculated modulo $2n$.

We define a Hamilton path $P(H \times s^j H^{\text{rev}})$ as

$$\begin{aligned} P(H \times s^j H^{\text{rev}}) &= (c_{i+1}, c_{i+2}, \ldots, c_{i+n}) \\ &= (c_{i+n+1}, c_{i+n+2}, \ldots, c_{2n}, c_1, c_2, \ldots, c_i), \end{aligned}$$

and put

$$\mathcal{P}(H) = \{P(H \times s^j H^{\text{rev}}) \mid 0 \le j \le n-1\}.$$

Then $\mathcal{P}(H)$ is a set of n Hamilton paths in K_n. This set is well-defined. In fact, a Hamilton cycle H has many representations, for example, $H = (a_1, a_2, \ldots, a_n) = (a_2, a_3, \ldots, a_n, a_1) = (a_n, a_{n-1}, \ldots, a_2, a_1)$, etc, but the set $\mathcal{P}(H)$ is uniquely determined.

Put

$$\mathcal{P} = \cup_{H \in \mathcal{H}} \mathcal{P}(H),$$

then $\mathcal{P}$ is a set of $n(n-1)/2$ Hamilton paths in K_n.

Lemma 2.1 *$\mathcal{P}$ has each 2-path in K_n exactly once.*

Proof. Let (a, b, c) be any 2-path in K_n. The edge $\{a, c\}$ is contained in some Hamilton cycle H in $\mathcal{H}$, then we have $H = (a, c, \ldots)$. There is an integer j with $0 \le j \le n-1$ such that

$$H \times s^j H^{\text{rev}} = (a, b, c, \ldots, c, b, a, \ldots),$$

that is, a Hamilton path $P(H \times s^j H^{\text{rev}})$ has the 2-path (a, b, c). Therefore the 2-path (a, b, c) belongs to $\mathcal{P}$ at least once.

Since the number of 2-paths in K_n is $n(n-1)(n-2)/2$ and the number of 2-paths in a Hamilton path in K_n is $(n-2)$, $\mathcal{P}$ has each 2-path in K_n exactly once. □

Lemma 2.2 *Each pair of distinct vertices of K_n is the pair of endvertices of exactly one of the Hamilton paths in $\mathcal{P}$.*

Proof. Let a, b be any distinct vertices of K_n. There is a Hamilton cycle H in $\mathcal{H}$ such that a, b are semi-antipodal vertices in H. Then a, b are endvertices of $P(H \times H^{\text{rev}})$ or $P(H \times s^{n-1} H^{\text{rev}})$.

Since the number of the pairs of distinct vertices of K_n is $n(n-1)/2$ and the number of the pairs of endvertices in $\mathcal{P}$ is $n(n-1)/2$, we obtain Lemma 2.2. □

Let K_{n+1} be the complete graph with the vertex set $V(K_n) \cup \{\infty\}$, where $V(K_n)$ is the vertex set of K_n and $\infty \notin V(K_n)$.

For a Hamilton path $P \in \mathcal{P}$, we define $P(\infty)$ to be a Hamilton cycle in K_{n+1} putting the vertex ∞ between the endvertices of P. Define

$$\mathcal{D} = \{P(\infty) \mid P \in \mathcal{P}\},$$

then $\mathcal{D}$ is a set of Hamilton cycles in K_{n+1} with $|\mathcal{D}| = nr$.

Proposition 2.1 *$\mathcal{D}$ is a Dudeney set in K_{n+1}.*

Proof. Let $Q = (a, b, c)$ be any 2-path in K_{n+1}. We only need to consider the cases (i) $a, b, c \notin \infty$, (ii) $b = \infty$ and (iii) $a = \infty$.

When $a, b, c \neq \infty$, Q is contained in $\mathcal{D}$ by Lemma 2.1. When $b = \infty$, Q is contained in $\mathcal{D}$ by Lemma 2.2. When $a = \infty$, there is a Hamilton cycle H in $\mathcal{H}$ satisfying that $\{b, c\} \in H$. Then we have $H = (b, c, \ldots)$ and

$$H \times s^{n-1}H^{\text{rev}} = (b, b, c, \ldots).$$

Hence we have

$$P(H \times s^{n-1}H^{\text{rev}}) = (b, c, \ldots).$$

Therefore the 2-path $Q = (\infty, b, c)$ is contained in $P(H \times s^{n-1}H^{\text{rev}})(\infty)$, so contained in $\mathcal{D}$.

Since the number of the 2-paths in K_{n+1} is $(n+1)n(n-1)/2$ and the number of the Hamilton cycles in $\mathcal{D}$ is $n(n-1)/2$, $\mathcal{D}$ has each 2-path in K_{n+1} exactly once. Therefore $\mathcal{D}$ is a Dudeney set in K_{n+1}. □

Thus we complete the proof of Theorem 1.1.

3 Neighbour-balanced Hamilton decompositions and an open problem

Neighbour-balanced Hamilton decompositions (NBHDs) of K_n are defined in [4] as follows. Let $n \geq 5$ be an odd integer and k an integer with $2 \leq k \leq (n-1)/2$. A *k-neighbour-balanced Hamilton decomposition* $\mathcal{H}$ of K_n is a Hamilton decomposition such that the set of all chords at distance k of the cycles in $\mathcal{H}$ is the edge set of K_n. In this terminology, a semi-antipodal Hamilton decomposition is an $(n-1)/2$-NBHD. It is known that the existence of an $(n-1)/2$-NBHD is equivalent to that of a 2-NBHD:

Theorem B [4] *Let $n \geq 5$ be an odd integer. There exists a semi-antipodal Hamilton decomposition of K_n if and only if there exists a 2-neighbour-balanced Hamilton decomposition of K_n.*

With the aid of a computer, we found that there exist semi-antipodal Hamilton decompositions of K_n for n with $5 \leq n \leq 29$ except $n = 9$. But nothing else is known, so we propose an open problem.

Problem 3.1 *Construct a semi-antipodal Hamilton decomposition or equivalently a 2-neighbour-balanced Hamilton decomposition of K_n for every odd $n \geq 11$.*

References

[1] K. Heinrich, D. Langdeau and H. Verrall, Covering 2-paths uniformly. *J. Combin. Des.* 8 (2000), 100–121.

[2] M. Kobayashi, Kiyasu-Z. and G. Nakamura, A solution of Dudeney's round table problem for an even number of people. *J. Combin. Theory. Ser. A* 62 (1993), 26–42.

[3] M. Kobayashi, B. D. McKay, N. Mutoh and G. Nakamura, Black 1-factors and Dudeney sets. *J. Combin. Math. Combin. Comput.* 75 (2010), 167–174.

[4] M. Kobayashi, N. Mutoh, G. Nakamura and C. Nara, Neighbour-balanced Hamilton decompositions of the complete graphs. Manuscript (2012).