Counting tilings by taking walks

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Abstract

Given a graph G we show how to compute the number of (perfect) matchings in the graphs $G \square P_n$ and $G \square C_n$ by looking at appropriate entries in a power of a particular matrix. We give some generalizations and extensions of this result, including showing how to compute tilings of $k \times n$ boards using monomers, dimers and 2×2 squares.

1 Introduction

One of the most popular and well known problems in enumerative combinatorics is counting the number of tilings of some board where the tiling pieces are drawn from some finite collection of types of tiles. Perhaps the most famous example of this is using dominoes to tile an $m\times n$ board. In this case an explicit formula for the number of distinct tilings is known (see [3]) and is given by

$$\prod_{i=1}^{\lceil \frac{m}{2} \rceil} \prod_{j=1}^{\lceil \frac{n}{2} \rceil} \left(4\cos^2 \frac{\pi i}{m+1} + 4\cos^2 \frac{\pi j}{n+1} \right).$$

This formula can be found by relating the problem of tilings to the problem of finding perfect matchings in a graph, which can be connected to finding permanents of a particular matrix. This permanent can then be computed explicitly. Another common technique in counting tilings is to derive recurrence relationships and then solve the recurrence.

In this note we will present a different way to count some simple tilings based on an idea introduced in Butler et al. [2] used to count domino tilings of $3\times 2n$ boards. We will look at "boards" that can be described as the Cartesian product of a graph with a long path or a graph with a long cycle. For these graphs we will count the tilings by first relating the

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construction of a tiling to a walk in an auxiliary graph. Since walks in a graph can be counted by taking powers of the graph's adjacency matrix, we can then count such tilings by reading off entries in some appropriate matrix to a power.

Notation

The Cartesian product of two graphs G and H, denoted $G \,\Box\, H$, is the graph with vertex set

$$V(G \,\Box\, H) = \{(u,v) : u \in V(G) \text{ and } v \in V(H)\}$$

and edges $(u_1,v_1) \sim (u_2,v_2)$ where either $u_1 = u_2$ and $v_1 \sim v_2$ in H; or $u_1 \sim u_2$ in G and $v_1 = v_2$. As an example, the $m \times n$ board can naturally be associated with the graph $P_m \,\Box\, P_n$.

A *matching* in a graph is a subset of edges where no vertex is used twice. The number of matchings of the graph G will be denoted $m(G)$. A *perfect matching* in a graph is a matching which uses all of the vertices. The number of perfect matchings of the graph G will be denoted $pm(G)$.

Given a subset U of the vertices of G, then we will let $G[U]$ be the *induced* subgraph of G on the vertices of U.

2 Counting matchings and perfect matchings

The most basic problem of counting tilings is related to using dominos (1×2 pieces). Such tilings are in one-to-one correspondence with perfect matchings. If we allow ourselves to tile with dominos and monomoes (1×1 pieces) then we are in a one-to-one correspondence with matchings. Our first result is how to count matchings and perfect matchings in $G \,\Box\, P_n$ or $G \,\Box\, C_n$ where G is a (small) fixed graph.

Theorem 1. *Let G be a graph on the vertex set $[k] = \{1, 2, \ldots, k\}$. Define $\mathcal{A}$ to be the $2^k \times 2^k$ matrix with rows/columns indexed by subsets of $[k]$ and with entries defined as follows: for any subsets S and T of $[k]$,*

$$\mathcal{A}_{S,T} = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset, \\ m(G[V \setminus (S \cup T)]) & \text{otherwise,} \end{cases}$$

then $m(G \,\Box\, P_n) = (\mathcal{A}^n)_{\emptyset,\emptyset}$ and $m(G \,\Box\, C_n) = \operatorname{trace}(\mathcal{A}^n)$.

Similarly, if we define $\mathcal{B}$ to be the $2^k \times 2^k$ matrix with rows/columns indexed by subsets of $[k]$ and with entries defined as follows: for any subsets S and T of $[k]$,

$$\mathcal{B}_{S,T} = \begin{cases} 0 & \text{if } S \cap T \neq \emptyset, \\ pm(G[V \setminus (S \cup T)]) & \text{otherwise,} \end{cases}$$

then $pm(G \square P_n) = (\mathcal{B}^n)_{\emptyset,\emptyset}$ *and* $pm(G \square C_n) = \text{trace}(\mathcal{B}^n)$.

Before giving the proof, we note that the construction of these matrices can be completely automated and Sage code is available to produce the matrices discussed in this note.

Proof. Consider the graph $G \square P_n$. This can be viewed as n copies of G placed sequentially with edges joining the corresponding vertices of G (see Figure 1 for an illustration).

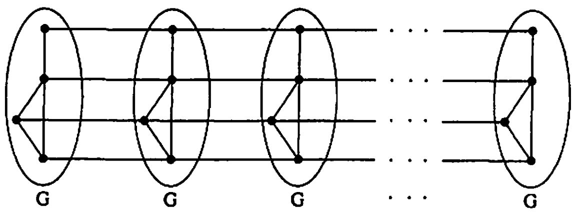


Figure 1: An illustration of the graph $G \square P_n$.

The edges which are used in a matching can now be placed into two groups. Namely (i) the edges which lie with both vertices in one of the copies of G and (ii) the edges which bridge between two copies of G.

We now construct an auxiliary multi-graph $\mathcal{G}$ whose vertices are all possible ways to place edges between two copies of G (i.e., the type (ii) edges in a matching). There will be 2^k such vertices, one for each subset of G and so we will index the vertices by the subsets of G.

The edges of $\mathcal{G}$ will be used to count the type (i) edges in the matching. Given two vertices of $\mathcal{G}$, we view them as a matching coming in on one side and going out on the other. If there is any overlap between these two vertices then we cannot have a matching, as a vertex of G would be used in more than one edge, so such vertices of $\mathcal{G}$ will not be connected. When there is no overlap then we will add an edge for each legal use of the unused vertices of G. Here, unused means that we will not use the vertex for edges in the matching running between copies of G. More particularly, if we are constructing a graph to count the matchings then we will add an edge for each matching using the unused vertices of G; if we are constructing a graph to count the perfect matchings then we will add an edge for each perfect matching using the unused vertices of G.

With $\mathcal{G}$ constructed we note that there is a one-to-one correspondence of tilings of $G \square P_n$ and walks of length n beginning and ending at the vertex labeled $\emptyset$ in $\mathcal{G}$. Namely, each tiling can be viewed as a sequence of how the vertices in each one of the copies of G are used, which corresponds to the vertices of $\mathcal{G}$, and how edges between copies of G are used, which correspond to the edges of $\mathcal{G}$, giving us the desired walk. We need to start and stop at $\emptyset$ so that no edges used in the matching are sticking out at the ends. Similarly there is a one-to-one correspondence of tilings of $G \square C_n$ and all walks of length n in $\mathcal{G}$. In other words, the only difference is that we need to be able to close up the two ends of the tilings.

Finally, to count walks in a multi-graph it suffices to take powers of the adjacency matrix, where in a multi-graph the entry between two vertices corresponds to the number of edges between those two vertices (see [1]). $\square$

Tiling boards and tori

As an example, suppose that we want to count the number of different domino tilings of $C_4 \square P_n$ and $C_4 \square C_n$. Then we would construct the following matrix (where the vertices of C_4 are labeled sequentially with $1, 2, 3, 4$ and on the left side of the matrix we indicate the labeling of the subset):

$$
\begin{array}{r}
\emptyset \\ \{1\} \\ \{2\} \\ \{3\} \\ \{4\} \\ \{1,2\} \\ \{1,3\} \\ \{1,4\} \\ \{2,3\} \\ \{2,4\} \\ \{3,4\} \\ \{1,2,3\} \\ \{1,2,4\} \\ \{1,3,4\} \\ \{2,3,4\} \\ \{1,2,3,4\}
\end{array}
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

The entry of 2 comes from the existence of two different perfect matchings in C_4 (i.e., $1, 2$ and $3, 4$; $1, 4$ and $2, 3$) so is a multi-edge in the graph.

Taking large powers and reading off the first entry or reading off the trace we get the number of perfect matchings in $C_4 \square P_n$ and $C_4 \square C_n$.

n	$pm(C_4 \square P_n)$
1	2
2	9
3	32
4	121
5	450
6	1681
7	6272
8	23409
9	87362
10	326041

n	$pm(C_4 \square C_n)$
3	50
4	272
5	722
6	3108
7	10082
8	39952
9	140450
10	537636

We note that $pm(C_4 \square P_n)$ corresponds to A006253 in the OEIS [4], while $pm(C_4 \square C_n)$ has not yet appeared. In general many of the sequences in the OEIS involving tilings of boards with one dimension fixed can be found using this method, and many exotic new sequences could be formed. For example, the sequence counting the number of perfect matchings in $\mathbf{P} \square P_n$, starting with $n = 1$, where $\mathbf{P}$ is the Petersen graph, is

$$6, 472, 14508, 616945, 23310528, 919890493, 35726458296, \ldots.$$

and the sequence counting the number of perfect matchings in $Q_3 \square C_n$, starting with $n = 3$, where Q_3 is the hypercube of dimension three, is

$$13680, 589185, 8569929, 275875712, 5108424393, 145226575873, \ldots.$$

We also note that having the matrix gives us a lot of useful information about these sequences. For example, suppose that $f(x)$ is the minimal polynomial for $\mathcal{A}$ (or $\mathcal{B}$). Then since $f(\mathcal{A}) = \mathbf{0}$ it follows that $f(\mathcal{A})_{S,S} = 0$, so that $f(x)$ is a recurrence for the desired sequences. For example the minimal polynomial for the above matrix for $G = C_4$ for perfect matchings is

$$f(x) = x^8 - 4x^7 - 6x^6 + 28x^5 - 28x^3 + 6x^2 + 4x - 1.$$

So that if a_n is one of the above sequences then it will satisfy

$$0 = a_{n+8} - 4a_{n+7} - 6a_{n+6} + 28a_{n+5} - 28a_{n+3} + 6a_{n+2} + 4a_{n+1} - a_n.$$

Going a little further, we can also use the matrix to find explicit solutions, namely we have $\mathcal{A} = \sum_i \lambda_i \mathcal{P}_i$ where $\mathcal{P}_i$ is the projection onto the ith eigenspace. It follows that $\mathcal{A}^n = \sum_i \lambda_i^n \mathcal{P}_i$, from which we can recover our exact values by summing appropriate combinations of powers of the λ_i and entries of $\mathcal{P}_i$.

Tiling twisted tori

The difference between $G\,\Box\,P_n$ and $G\,\Box\,C_n$ is that we loop back around and glue the two ends in the natural way. However we can decide to glue slightly askew and produce new graphs. Let the vertices of $G\,\Box\,P_n$ be represented by (i,j) where $i \in \{1,2,\ldots,k\}$ and $j \in \{1,2,\ldots,n\}$, and let π be a permutation of $\{1,2,\ldots,k\}$. Then the graph $(G\,\Box\,P_n)_\pi$ will be the graph $G\,\Box\,P_n$ with the addition of edges of the form $(i,n)\sim(\pi(i),1)$. So as noted above we have $G\,\Box\,C_n = (G\,\Box\,P_n)_e$. Another well known graph is the Möbius ladder which is $(P_2\,\Box\,P_n)_{(12)}$.

We can count matchings in $(G\,\Box\,P_n)_\pi$ by allowing ourselves to allow edges to stick out at the two ends of $G\,\Box\,P_n$ but ensure that they are consistent when we connect the two ends as dictated by $(G\,\Box\,P_n)_\pi$. It follows that

$$pm\big((G\,\Box\,P_n)_\pi\big) = \sum_S (\mathcal{B}^n)_{S,\pi(S)}.$$

We obtain a similar variation when we count general matchings. (Note that when $\pi = e$ then this becomes the trace and so is consistent with what was done before for $G\,\Box\,C_n$.)

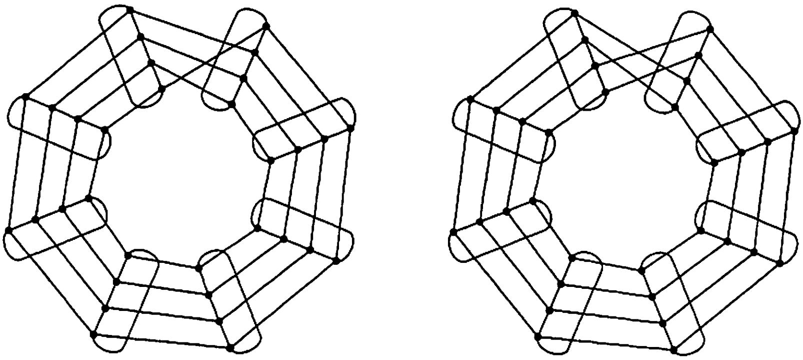
By way of comparison, in Figure 2 we show the graphs $(C_4\,\Box\,P_8)_{(1234)}$ and $(C_4\,\Box\,P_8)_{(13)(24)}$. We can count the perfect matchings in these various graphs using the above formula. Below we compare their values and we see there are differences, sometimes substantial, in the number of perfect matchings.

n	$pm\big((G\,\Box\,P_n)_e\big)$	$pm\big((G\,\Box\,P_n)_{(1234)}\big)$	$pm\big((G\,\Box\,P_n)_{(13)(24)}\big)$
3	50	80	54
4	272	194	260
5	722	888	726
6	3108	2702	3096
7	10082	11040	10086
8	39952	37634	39940
9	140450	146024	140454
10	537636	524174	537624

Counting statistics

We can modify the matrices to count some basic statistics about the resulting tilings. This is done by using polynomials for the entries of the matrix instead of numerical values where we interpret the polynomial appropriately.

As an example, suppose we want to count the number of "vertical" tiles used when tiling a $m\times n$ board with dominoes. Then when we construct the



$(C_4 \square P_8)_{(1234)}$ $(C_4 \square P_8)_{(13)(24)}$

Figure 2: An example of the results of twisting the graph.

auxiliary graph $\mathcal{G}$, as in the proof of Theorem 1, we assign a weight of x^k to the edges where k is the number of tiles used in covering the remaining vertices of G. Therefore the entries of our matrix become polynomials in x. If we were to evaluate this polynomial at $x = 1$, it would reduce to the polynomial we have already discussed.

If we carry this out for $P_6 \square P_6$ and look at the corresponding entry of the matrix we get the polynomial

$$x^{18}+30x^{16}+281x^{14}+1064x^{12}+1988x^{10}+1988x^{8}+1064x^{6}+281x^{4}+30x^{2}+1.$$

The exponent corresponds to the number of vertical tiles and the coefficient corresponds to the number of tiles using that number of vertical tiles. So for example, we can conclude that there are 1064 tilings of a 6×6 board that uses 12 vertical tiles.

Similarly we can count the number of tilings using a fixed number of monomoes in a tiling of a board using dominoes and monomoes. In this case we weight each edge according to the number of monomoes which appear in the corresponding matching. So for example of the 2989126727 ways there are to tile the 6×6 board using dominoes and monomoes we can further refine the count by the number of monomoes used to get the following:

# monimoes	# tilings	# monimoes	# tilings
0	6728	20	146702793
2	363536	22	48145820
4	5580152	24	11785382
6	39277112	26	2135356
8	154396898	28	281514
10	377446076	30	26172
12	613605045	32	1622
14	693650988	34	60
16	562203148	36	1
18	333518324		

3 Counting tilings with squares

We can expand our repertoire of tiles to include squares (or C_4's if we think of them as a graph). This requires some modification to the construction of the matrices.

Theorem 2. *Let G be a graph containing no four-cycle on the vertex set $[k] = \{1, 2, \ldots, k\}$. Let C be a matrix indexed by the set of all possible disjoint union of edges (i.e., no vertex can be used more than once). Further, the entries are defined as follows: for any subsets S and T of $[k]$,*

$$C_{S,T} = \begin{cases} 0 & \textit{if } S \textit{ and } T \textit{ share a common vertex,} \\ 1 & \textit{otherwise,} \end{cases}$$

then the number of ways to tile $G \,\square\, P_n$ using squares and monomoes is $(C^n)_{\emptyset,\emptyset}$ while the number of ways to tile $G \,\square\, C_n$ using squares and monomoes is trace(C^n).

Let $\mathcal{D}$ and $\mathcal{E}$ be matrices indexed by the set of all possible disjoint union of edges and vertices (i.e., no vertex can be used more than once). Further, the entries are defined as follows: for any subsets S and T of $[k]$,

$$\mathcal{D}_{S,T} = \begin{cases} 0 & \textit{if } S \textit{ and } T \textit{ share a common vertex,} \\ m(G[V \setminus (S \cup T)]) & \textit{otherwise,} \end{cases}$$

$$\mathcal{E}_{S,T} = \begin{cases} 0 & \textit{if } S \textit{ and } T \textit{ share a common vertex,} \\ pm(G[V \setminus (S \cup T)]) & \textit{otherwise,} \end{cases}$$

then the number of ways to tile $G \,\square\, P_n$ using monomoes, dominoes and squares is $(\mathcal{D}^n)_{\emptyset,\emptyset}$ while the number of ways to tile $G \,\square\, C_n$ using monomoes, dominos and squares is trace$(\mathcal{D}^n)$. *Similarly, the number of ways to tile $G \,\square\, P_n$ using dominoes and squares is $(\mathcal{E}^n)_{\emptyset,\emptyset}$ while the number of ways to tile $G \,\square\, C_n$ using dominos and squares is* trace$(\mathcal{E}^n)$.

The techniques to prove Theorem 2 are the same as those used in Theorem 1 and so we will omit the proof. We note in passing that by assumption, the squares must bridge between two copies of G. Hence, we keep track of the edges in G which correspond to the squares in a bridge between copies of G, as well as the vertices in G which correspond to dominoes in a bridge between copies of G.

The above theorem allows us to count many different tilings. By way of comparison we have the following counts (M, D and S correspond to monomoes, dominoes and squares respectively):

Tilings of $P_8 \square P_n$	Using M, S	Using D, S	Using M, D, S
$n = 1$	1	1	34
$n = 2$	34	171	12190
$n = 3$	171	1037	2326760
$n = 4$	2115	48605	527889422
$n = 5$	16334	550969	114411435032
$n = 6$	159651	16231655	25111681648122
$n = 7$	1382259	242436361	5492577770367562
$n = 8$	12727570	5811552169	1202536689448371122

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