A Combinatorial Interpretation of the Lucas-Nomial Coefficients in Terms of Tiling of Rectangular Boxes

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Abstract

Generalized binomial coefficients are considered. The aim of this paper is to provide a new general combinatorial interpretation of the Lucas-nomial and (p,q)-nomial coefficients in terms of tiling of d-dimensional rectangular boxes. The recurrence relation of these numbers is proved in a combinatorial way. To this end, our results are extended to the case of corresponding multi-nomial coefficients.

1 Introduction

We assume $\mathbb{N} = \{1,2,\ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $F = (F_0, F_1, F_2, \ldots)$ be a sequence of positive integers with $F_0 = 0$. Fix $n, k \in \mathbb{N}_0$ such that $n \geq k$. Then by the *F-nomial coefficient* we mean

$$\binom{n}{k}_F = \frac{F_n F_{n-1} \cdots F_{n-k+1}}{F_1 F_2 \cdots F_k}, \tag{1}$$

where $\binom{n}{0}_F = 1$.

For example, if we set $F_n = n$ we obtain ordinary binomial coefficients. With the setting $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0$, $F_1 = 1$ we obtain the Fibonomial coefficients [4, 5]. These generalized binomial coefficients have been intensively studied in the literature, starting from Carmichael [1], Jarden and Motzkin [6]. The general form of the F-nomial coefficients is considered by Kwaśniewski [9, 10] in terminology of special "cobweb" posets.

In this paper we show that for the Lucas sequence [12] we have a new combinatorial interpretation of the corresponding Lucas-nomial coefficients in terms of tiling of d-dimensional rectangular boxes. Recall, the Lucas sequence of the first kind $\{U_n(p,q)\}_{n\geq 0}$ is defined by the following recurrence relation

$$U_n(p,q) = pU_{n-1}(p,q) - qU_{n-2}(p,q), \qquad \text{for } n \geq 2, \tag{2}$$

with initial values $U_0(p,q) = 0$, $U_1(p,q) = 1$ and arbitrary parameters p, q. Therefore, the F-nomial coefficients reduce to the Lucas-nomial coefficients with the setting $F_n = U_n(p,q)$ for $n \geq 0$.

Let λ and ρ be two functions $\mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$. Suppose that there is a sequence $F = (F_0, F_1, \ldots)$ such that for any fixed $n \in \mathbb{N}_0$ and any $m, k \in \mathbb{N}_0$ such that $m + k = n$ we have

$$F_n = \lambda(m,k)F_m \ + \ \rho(m,k)F_k. \tag{3}$$

Moreover, we show that F is uniquely designated by λ and ρ (see Corollary 1). Denote by $\mathcal{F}$ family of all sequences F for which we can define such functions λ and ρ with the above property.

Consider $N = (0,1,2,\ldots)$, it is easy to see that $N \in \mathcal{F}$. In this case the functions λ and ρ are constant and equal to one. Family $\mathcal{F}$ contains also Lucas sequences (see Section 4).

Simple algebraic modifications of (3) gives us the following recurrence relation for the F-nomial coefficients

$$\binom{n}{m}_F = \lambda(m,k)\binom{n-1}{m-1}_F + \rho(m,k)\binom{n-1}{m}_F \tag{4}$$

with $\binom{n}{0}_F = 1$.

2 Tiling of m-dimensional boxes

We follow the notation of [11]. Take $F \in \mathcal{F}$. Let $n, k \in \mathbb{N}$ such that $n \geq k$. Then a rectangular m-dimensional box of sizes

$$V_{k,n} : F_k \times F_{k+1} \times \cdots \times F_n$$

is called the *m-dimensional F-box* and denoted by $V_{k,n}$, where $m = n-k+1$. By the *m-dimensional F-brick*, denoted by V_m, we mean an m-dimensional F-box of sizes

$$V_m : F_1 \times F_2 \times \cdots \times F_m.$$

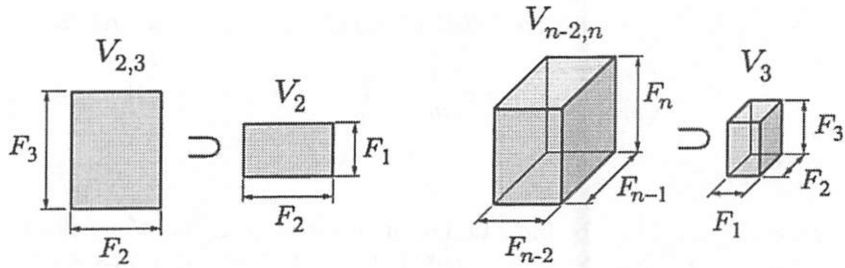

Figure 1: Exemplary F-boxes and its F-bricks.

Following de Bruijn [2], by the *tiling* of the F-box $V_{k,n}$ we mean the set of translated and rotated F-bricks V_m which interiors are pairwise disjoint and the union is the entire F-box $V_{k,n}$ (compare with Fig. 2).

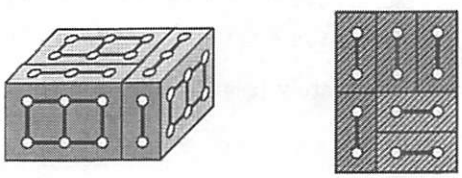

Figure 2: Exemplary tilings of F-boxes.

The next observation is due to Kwaśniewski [9, 10] (see also references therein). He proposes a new general combinatorial interpretation for a wide family of generalized binomial coefficients. Here we reformulate it in terms of tilings of F-boxes.

Observation 1. *If an m-dimensional F-box $V_{k,n}$ is tiled with F-bricks V_m then the number of these bricks is equal to $\binom{n}{m}_F$, where $m = n - k + 1$.*

Proof. Observe that the "volume" of the F-box $V_{k,n}$ is equal to $F_n F_{n-1} \cdots F_k$ and the "volume" of any F-brick V_m is $F_1 F_2 \cdots F_m$. Finally, the number of bricks of the tiling is equal to

$$\frac{\textit{volume of } V_{k,n}}{\textit{volume of } V_m} = \frac{F_n F_{n-1} \cdots F_k}{F_1 F_2 \cdots F_m} = \binom{n}{m}_F.$$

□

Theorem 1. *Let $F \in \mathcal{F}$ and $m, n \in \mathbb{N}$ such that $n \geq m$, set $k = n - m$. Then any m-dimensional F-box $V_{k+1,n}$ can be tiled with F-bricks V_m and* *the number of these bricks satisfies the following recurrence relation*

$$\binom{n}{m}_F = \lambda(m,k)\binom{n-1}{m-1}_F + \rho(m,k)\binom{n-1}{m}_F \tag{5}$$

with $\binom{n}{0}_F = 1$.

Proof. The proof is by induction on n. For $n = 1$ the box $V_{1,1}$ has a trivial tiling. Suppose $n > 1$. Assume that any F-box $V_{i,n-1}$ has a tiling by F-bricks V_{n-i} for $1 \le i \le n-1$.

Consider the last size of the box $V_{k+1,n}$ which is equal to F_n. By the definition of the family $\mathcal{F}$, we have that F_n is the sum of two numbers $\lambda(m,k)F_m$ and $\rho(m,k)F_k$ for certain functions λ and ρ, where $n = m + k$. Therefore, we may "cut" the box $V_{k+1,n}$ into two disjoint sub-boxes A and B of sizes

$$A : F_{k+1} \times F_{k+2} \times \cdots \times F_{n-1} \times (\lambda(m,k) \cdot F_m),$$
$$B : F_{k+1} \times F_{k+2} \times \cdots \times F_{n-1} \times (\rho(m,k) \cdot F_k),$$

and we handle them separately (see Fig. 3).

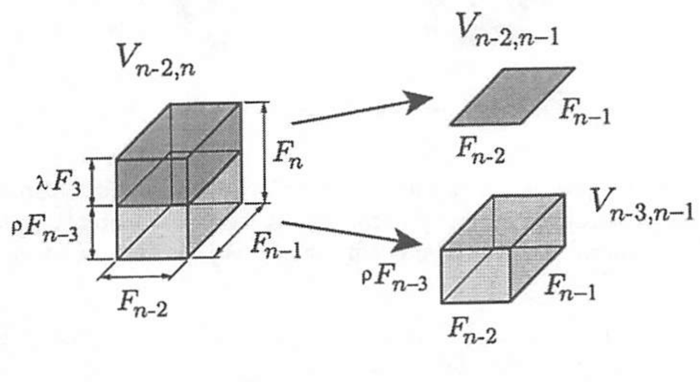

Figure 3: An illustration of the proof for the 3-dimensional case.

Step 1: Tiling the box A.
Observe that the first $(m-1)$ sizes of A define the box $V_{k+1,n-1}$ and by the induction hypothesis, it can be tiled with bricks V_{m-1}. The last size of A might be covered by the last size of the brick V_m exactly $\lambda(m,k)$ times. Therefore, the whole A might be tiled.

Step 2: Tiling the box B.
Note that the last size of B is $\rho(m,k)$ times greater than F_k. Therefore, let

us divide again the box B into $\rho(m,k)$ boxes along this coordinate. Since we are using rotated bricks V_m, we permute sizes of B to get $\rho(m,k)$ boxes of sizes $F_k \times F_{k+1} \times \cdots \times F_{n-1}$. And by the induction hypothesis, it can be tiled with bricks V_m.

We have divided the box $V_{k+1,n}$ into two disjoint sub-boxes $V_{k+1,n-1}$ and $V_{k,n-1}$ and tiled them separately in two steps. Therefore, the whole box $V_{k+1,n}$ might be tiled. If we sum up the number of bricks in corresponding tilings of sub-boxes A and B we obtain the recurrence relation (5) which completes the proof. □

Now we give another formula for F-nomial coefficients which follows from the recurrence relation (5). Fix $n,k \in \mathbb{N}_0$ such that $n \geq k$ and let $\pi \in \mathcal{P}_k(n)$ be a k-subset of the n-set. By $\overline{\pi}$ we mean the set $\{1,2,\ldots,n\}\setminus\pi$. Denote by $w^{n,k}(\pi)$ the product

$$w^{n,k}(\pi) = \prod_{i=1}^{k} \lambda(i, \pi_i - i) \prod_{i=1}^{n-k} \rho(\overline{\pi}_i - i, i).$$

Theorem 2. *Let $F \in \mathcal{F}$ and $n,k \in \mathbb{N}_0$. Then we have*

$$\binom{n}{k}_F = \sum_{\pi \in \mathcal{P}_k(n)} w^{n,k}(\pi). \tag{6}$$

Proof. The proof is by induction on n. The case $n = 0$ is trivial. Assume that the formula (6) holds for $n-1$ and $k = 1,2,\ldots,n-1$. Then consider the right-hand side of (6). Let us separate the family $\mathcal{P}_k(n)$ into two disjoint classes: A_k with these subsets that contain the last element n and B_k without n, respectively.

First, consider $A_k = \{\{\pi_1,\ldots,\pi_k\} \in \mathcal{P}_k(n) : \pi_k = n\}$. Let $\overline{\pi} = [n] \setminus \pi$, then we have

$$\sum_{\pi \in A_k} w^{n,k}(\pi) = \sum_{\pi \in A_k} \lambda(k, n-k) \prod_{i=1}^{k-1} \lambda(i, \pi_i - i) \prod_{i=1}^{n-k} \rho(\overline{\pi}_i - i, i).$$

Note, the summation over elements of A_k may be considered as the sum over all $(k-1)$ subsets of the set $[n-1]$. Therefore,

$$\sum_{\pi \in A_k} w^{n,k}(\pi) = \lambda(k, n-k) \binom{n-1}{k-1}_F. \tag{7}$$

In the same way we deal with the class $B_k = \mathcal{P}_k(n) \setminus A_k$. Now, we have

$$\sum_{\pi \in B_k} w^{n,k}(\pi) = \rho(k, n-k) \binom{n-1}{k}_F. \tag{8}$$

Finally, if we add (7) to (8) and use the recurrence relation (4) we obtain (6). □

3 The multi-nomial coefficients

In this section we show how our results can be extended to the multi-tiling of hyper boxes and corresponding multi-nomial coefficients.

Let $F = \{F_n\}_{n \geq 0}$ be a sequence of positive integers with $F_0 = 0$ and let $\langle b_1, b_2, \ldots, b_k \rangle$ be a composition of a fixed number $n \in \mathbb{N}$ into k non-zero parts. Then by the *multi F-nomial coefficient* we mean

$$\binom{n}{b_1, b_2, \ldots, b_k}_F = \frac{F_n!}{F_{b_1}! F_{b_2}! \cdots F_{b_k}!}, \tag{9}$$

where $F_s! = F_s F_{s-1} \cdots F_1$ and $F_0! = 1$.

We can easily see that if the values of the F-nomial coefficients are natural numbers for any $n, k \in \mathbb{N}$ such that $n \geq k$ then also the values of the multi F-nomial coefficients are natural numbers. Indeed,

$$\binom{n}{a, b, c}_F = \binom{n}{a}_F \binom{n-a}{b}_F \binom{n-a-b}{c}_F.$$

In general, the opposite conclusion is not true.

Here and subsequently β stands for a composition $\langle b_1, b_2, \ldots, b_k \rangle$ of a fixed number $n \in \mathbb{N}$ into k non-zero parts.

Proposition 1. *Let $F \in \mathcal{F}$. Then*

$$F_n = \sum_{i=1}^{k} \alpha_i(\beta)\, F_{b_i}, \tag{10}$$

where

$$\alpha_i(\beta) = \lambda(b_i, b_{i+1} + \cdots + b_k) \prod_{j=1}^{i-1} \rho(b_j, b_{j+1} + \cdots + b_k), \tag{11a}$$

$$\alpha_i(\beta) = \rho(b_{i+1} + \cdots + b_k, b_i) \prod_{j=1}^{i-1} \lambda(b_{j+1} + \cdots + b_k, b_j), \tag{11b}$$

Proof. It is a straightforward algebraic exercise due to the property (3) of sequences from family $\mathcal{F}$. We only outline the proof. The first form (11a) of the coefficients $\alpha_i(\beta)$ follows from the rule $\big(b_1 + (n - b_1)\big) \Rightarrow (b_1) + \big(b_2 + (n - b_1 - b_2)\big)$, and the second one (11b) from $\big((n - b_k) + b_k\big) \Rightarrow \big((n - b_k - b_{k-1}) + b_{k-1}\big) + (b_k)$. The rest of the proof is left to the reader and can be done by induction on k. □

Taking the composition $\beta = \langle 1, 1, \ldots, 1\rangle$ of a number $n \in \mathbb{N}$ we obtain the following result.

Corollary 1. *For any $F \in \mathcal{F}$ and $n \in \mathbb{N}$ we have*

$$F_n = \sum_{k=1}^{n} \lambda(1, n-k) \prod_{i=1}^{k-1} \rho(1, n-i), \tag{12a}$$

$$F_n = \sum_{k=1}^{n} \rho(n-k, 1) \prod_{i=1}^{k-1} \lambda(n-i, 1). \tag{12b}$$

Corollary 2. *For $F \in \mathcal{F}$ we have*

$$\binom{n}{b_1, b_2, \ldots, b_k}_F = \sum_{i=1}^{k} \alpha_i(\beta) \binom{n-1}{b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_k}_F. \tag{13}$$

where $\alpha_i(\beta)$ are specified in Proposition 1.

Recall $\beta = \langle b_1, b_2, \ldots, b_k\rangle$. By the n-dimensional multi F-brick $V_n(\beta)$ we mean the F-brick of sizes

$$\overbrace{\underbrace{F_1 \times \cdots \times F_{b_1}}_{b_1} \times \underbrace{F_1 \times \cdots \times F_{b_2}}_{b_2} \times \cdots \times \underbrace{F_1 \times \cdots \times F_{b_k}}_{b_k}}^{n}.$$

And finally, by the *multi-tiling* we mean a tiling of the F-box $V_{1,n}$ with multi F-bricks $V_n(\beta)$.

Observation 2. *Let $F \in \mathcal{F}$. If an F-box $V_{1,n}$ is tiled with multi F-bricks $V_n(\beta)$ then the number of these bricks is equal to*

$$\binom{n}{b_1, b_2, \ldots, b_k}_F. \tag{14}$$

where $\beta = \langle b_1, b_2, \ldots, b_k\rangle$.

Proof. The proof is analogous to the proof of Observation 1. □

Theorem 3. *Let $F \in \mathcal{F}$. Then any F-box $V_{1,n}$ can be tiled into multi F-bricks $V_n(\beta)$ and the number of these bricks satisfies* (13).

Proof. The proof is by induction on n. (Compare with the proof of Theorem 1.) The case of $n = 1$ is trivial. Suppose then $n > 1$ and assume that the F-box $V_{1,n-1}$ might be tiled into any multi-bricks $V_{n-1}(\beta')$, where β' is a composition of the number $(n-1)$ into k non-zero parts.

Take the F-box $V_{1,n}$. We need to tile the box into multi-bricks $V_n(\beta)$. Consider the last n-th size of the box $V_{1,n}$ which is equal to F_n. From Proposition 1 we know that the number F_n might be expressed as the sum $F_n = \alpha_1(\beta)F_{b_1} + \cdots + \alpha_k(\beta)F_{b_k}$.

Therefore, we divide the F-box $V_{1,n}$ into k sub-boxes $B_1, \ldots, B_k$ of sizes

$$\begin{aligned} B_1 &: F_1 \times F_2 \times \cdots \times F_{n-1} \times (\alpha_1(\beta)F_{b_1}), \\ B_2 &: F_1 \times F_2 \times \cdots \times F_{n-1} \times (\alpha_2(\beta)F_{b_2}), \\ &\vdots \\ B_k &: F_1 \times F_2 \times \cdots \times F_{n-1} \times (\alpha_k(\beta)F_{b_k}). \end{aligned}$$

Next, we tile these k sub-boxes independently in the following k steps. Let $i = 1, 2, \ldots, k$.

Step i: Tiling the box B_i.
Observe that the box designated by the first $(n-1)$ sizes of B_i forms F-box $V_{1,n-1}$ and it can be tiled into $(n-1)$-dimensional multi-bricks by the induction hypothesis. What is left is to cover the last size $(\alpha_i(\beta)F_{b_i})$ of F-box by the $(b_1 + \cdots + b_i)$-th size of the multi F-brick exactly $\alpha_i(\beta)$ times. In the next induction step we use $(n-1)$-dimensional multi F-bricks $V_{n-1}(\beta^{(i)})$ of sizes

$$\overbrace{\underbrace{F_1 \times \cdots \times F_{b_1}}_{b_1} \times \cdots \times \underbrace{F_1 \times \cdots \times F_{b_i - 1}}_{b_i - 1} \times \cdots \times \underbrace{F_1 \times \cdots \times F_{b_k}}_{b_k}}^{n},$$

where $\beta^{(i)} = \langle b_1, \ldots, b_{i-1}, b_i - 1, b_{i+1}, \ldots, b_k \rangle$. The rest of the proof goes similar as the proof of Theorem 1. □

4 Remarks and examples

This note is a partial answer to the Kwaśniewski tiling problem [10, Problem II, p.12] originally expressed in terms of cobweb posets and its tilings.

The question is to find all sequences $\mathcal{T}$ for which we have such "tiling interpretation" of the F-nomial coefficients. Now, we know that the family $\mathcal{T}$ encompass, among others, Fibonacci, Lucas sequences and (p,q)-analogues. However, the problem of characterization of the whole family $\mathcal{T}$ is still open and related to the general problem of filling rectangular hyper boxes.

Next, we present a few examples of the sequences $F \in \mathcal{F}$ that gives us a combinatorial interpretation of corresponding F-nomial coefficients.

Example 1 (Lucas sequence). Let p, q be arbitrary numbers. Then we define Lucas sequence as $U_0 = 0$, $U_1 = 1$ and

$$U_n = pU_{n-1} - qU_{n-2}.$$

It is the well-known that the Lucas sequences satisfy the following recurrence relation

$$U_{m+k} = U_{k+1}U_m - qU_{m-1}U_k.$$

Therefore, we have

$$\binom{n}{k}_U = U_{k+1}\binom{n-1}{k}_U - qU_{m-1}\binom{n-1}{k-1}_U.$$

If $p \in \mathbb{N}$ and $-q \in \mathbb{N}$ then we have a combinatorial interpretation for the (p,q)-Lucas nomial coefficients expressed in terminology of tilings.

Example 2 (Fibonacci numbers). One of the most famous example of Lucas sequences is the sequence of Fibonacci numbers where $p = 1$ and $q = -1$, i.e., $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Therefore, we have a new combinatorial interpretation for the Fibonomial coefficients which explains also their recurrence relation

$$\binom{n}{k}_{Fib} = F_{k+1}\binom{n-1}{k}_{Fib} + F_{m-1}\binom{n-1}{k-1}_{Fib}.$$

For a deeper discussion of an interpretation of the Fibonomial coefficients, we refer the reader to Kwaśniewski [8], Sagan and Savage [13], Knuth and Wilf [7].

Example 3. Let α, β be natural numbers. Then we define so-called (α, β)-analogues as $A_0 = 0$, $A_1 = 1$ and

$$A_n = (\alpha + \beta)A_{n-1} - (\alpha \cdot \beta)A_{n-2}.$$

These sequences generalize, among-others, q-Gaussian integers where $\alpha = 1$ and $\beta = q$ is a power of a prime number. If $\alpha = \beta$ then we have $A_n =$

$n\alpha^{n-1}$, otherwise $A_n = (\alpha^n - \beta^n)/(\alpha - \beta)$. We can show that these numbers satisfy

$$A_{m+k} = \alpha^k A_m + \beta^m A_k.$$

Finally, we have

$$\binom{n}{k}_A = \alpha^k \binom{n-1}{k}_A + \beta^m \binom{n-1}{k-1}_A.$$

This geometrical phenomenon of $\mathcal{F}$-nomial coefficients is a starting point to new questions. For example, we can ask about geometric proofs for many of binomial-like identities. Another way might follows us to the problem of tilings' counting of certain $\mathcal{F}$-box and to special kinds of the Stirling numbers.

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