Maximal Flat Regular Antichains

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February 23, 2012

Abstract

Let $2^{[m]}$ be ordered by set inclusion, and let $\mathcal{B} \subseteq 2^{[m]}$ be an antichain. An antichain $\mathcal{B}$ is called k-regular ($k \in \mathbb{N}$) if for each $i \in [m]$ there are exactly k blocks $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. An antichain is called flat if there exists a positive integer l such that $l \leq |B| \leq l+1$ for all $B \in \mathcal{B}$, and we call an antichain maximal if the collection of sets $\mathcal{B} \cup \{B\}$ is not an antichain for all $B \notin \mathcal{B}$. We call a maximal k-regular antichain $\mathcal{B} \subseteq \binom{[m]}{2} \cup \binom{[m]}{3}$ a (k,m)-MFRAC. In this paper we analyze (k,m)-MFRACs in the cases $m \leq 7$, $k = m$, $k = m-1$ and $k = m-2$. We provide some constructions, give necessary conditions for existence and mention some open problems.

Keywords: Extremal Set Theory, Regular Antichain, (Triangular) Graphs

1 Definitions and notation

For positive integers a, m with $a \leq m$, let $[a,m] := \{a, a+1, \ldots, m\}$ and $[m] := [1,m]$.

Let $\mathcal{B}$ be a subset of $2^{[m]}$, the power set of $[m]$. The size of $\mathcal{B}$ is $n := |\mathcal{B}|$. We call $\mathcal{B}$ an **antichain (AC)** if there are no two sets in $\mathcal{B}$ which are comparable under set inclusion. An antichain $\mathcal{B}$ is called **k-regular** ($k \in \mathbb{N}$), if for each $i \in [m]$ there are exactly k blocks $B_1, B_2, \ldots, B_k \in \mathcal{B}$ containing i. In this case we say that $\mathcal{B}$ is a **(k,m,n)-antichain**.

JCMCC **88** (2014), pp. 237-261

An antichain is called **flat** if there exists a positive integer l such that $l \leq |B| \leq l+1$ for all $B \in \mathcal{B}$. So $\mathcal{B} \subseteq \binom{[m]}{l} \cup \binom{[m]}{l+1}$. We call an antichain $\mathcal{B}$ **maximal** if $\mathcal{B} \cup \{B\}$ is not an antichain for all $B \notin \mathcal{B}$. If an antichain $\mathcal{B} \subseteq \binom{[m]}{2} \cup \binom{[m]}{3}$ is maximal and regular we say that $\mathcal{B}$ is an **MFRAC**. A (k,m)-**MFRAC** is a k-regular MFRAC on $[m]$.

A **Completely Separating System (CSS)** $\mathcal{C}$ on $[n]$ is a collection of blocks of $[n]$ such that for any distinct points $x, y \in [n]$, there exist blocks $A, B \in \mathcal{C}$ such that $x \in A - B$ and $y \in B - A$. A CSS on $[n]$ without restrictions on the size of the blocks in the collection is said to be an (n)CSS. An (n,k)**Completely Separating System** ((n,k)**CSS**) is an (n)CSS in which each block is of size k.

The **volume** of a collection $\mathcal{C}$ of sets is $v(\mathcal{C}) := \sum_{C \in \mathcal{C}} |C|$. For a (k,m,n)-antichain $\mathcal{B}$, $v(\mathcal{B}) = km$. Often we omit brackets and commas in our notation for sets. For example, we write 1345 instead of $\{1,3,4,5\}$.

Let $\mathcal{C}$ be a collection of subsets of $[m]$, and let $i \in [m]$ be an arbitrary fixed element. Then we define $\mathcal{C}_i := \{C \in \mathcal{C} : i \in C\}$ to be the collection of sets which contain this element i.

Let $\mathcal{C}$ be a collection of subsets of $[m]$. We say that a set $U \subseteq [m]$ is **not covered** by $\mathcal{C}$, if $U \nsubseteq C$ for all $C \in \mathcal{C}$.

Let $\mathcal{B}$ and $\mathcal{B}'$ be collections of sets of A, and of A', respectively. We say that $\mathcal{B}$ and $\mathcal{B}'$ are **isomorphic** to each other, if there exists a set preserving bijection f between A and A', i.e. B is in $\mathcal{B}$ if and only if $f(B)$ is in $\mathcal{B}'$, where $f(B) := \{f(b) : b \in B\}$.

Let $G = (V, E)$ be a graph, i.e. a pair of vertices V and edges E. The number of triangles containing x is $t(x) := |\{yz \in E : xy, xz \in E\}|$. The value c of a vertex x is defined by $c(x) := t(x) - d(x)$. A graph $T = (V, E)$ is called a **c-triangular graph** (or short triangular graph) if for all $xy \in E$ there is a $z \in V$ such that xyz is a triangle and if $c = c(x)$ for all $x \in V$.

2 Motivation

A cornerstone in Sperner Theory is the Sperner Theorem [12]:

Theorem 1 (Sperner's Theorem). *Let $\mathcal{B}$ be an antichain on $[m]$. Then*

(a) $|\mathcal{B}| \leq \binom{m}{\lfloor \frac{m}{2} \rfloor}$.

(b) Equality holds if and only if

$$\mathcal{B} = \begin{cases} \{B \subseteq [m] : |B| = \frac{m}{2}\} & \text{for } m \text{ even,} \\ \{B \subseteq [m] : |B| = \frac{m-1}{2}\} \text{ or } \{B \subseteq [m] : |B| = \frac{m+1}{2}\} & \text{for } m \text{ odd.} \end{cases}$$

In particular, these middle level antichains are maximal flat regular antichains.
The interest in k-regular antichains on $[m]$ comes from the dual problem. Therefore, we need the following definition:
Let $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ be a collection of subsets of $[n]$. We define the dual $\mathcal{C}^*$ of $\mathcal{C}$ to be the collection $\mathcal{C}^* := \{C_1^*, C_2^*, \ldots, C_n^*\}$ of subsets of $[m]$ given by $C_i^* := \{j \in [m] : i \in C_j\}$ $(i = 1, \ldots, n)$.
A consequence of this duality is the following lemma, which was proven by Spencer [11] in 1970.

Lemma 2. *If $\mathcal{C}$ is a CSS then its dual $\mathcal{C}^*$ is an antichain and vice versa.*

For given parameters k, m, n, it follows that:

Lemma 3. *If $\mathcal{C}$ is an (n,k)CSS of size m, then its dual $\mathcal{C}^*$ is a (k,m,n)-AC and vice versa.*

Completely Separating Systems were motivated by work of Rényi [7] in 1961 and Katona [4] in 1966 and first introduced by Dickson [2] in 1969. One important question in the area of Completely Separating Systems is to determine the value $R(n,k) := \min\{|\mathcal{C}| : \mathcal{C} \text{ is an } (n,k)\text{CSS}\}$. Using Lemma 3 we obtain that for given values k and n, a (k,m,n)-AC exists if and only if $m \geq R(n,k)$. On the other hand if we know the value $N := N(k,m) := \max\{n : \exists\ (k,m,n)\text{-AC}\ \}$, then we know that $R(N,k) \leq m$ and $R(i,k) > m$ for all $i > N$. Some work has been done in this area of combinatorics (see for example [1], [9], [10]). So it is useful to know more about maximal regular antichains.

We call two antichains $\mathcal{B}$ and $\mathcal{B}'$ over the same ground set equivalent if and only if $v(\mathcal{B}) = v(\mathcal{B}')$ and $|\mathcal{B}| = |\mathcal{B}'|$. The Flat Antichain Theorem by Lieby [6] and Kisvölcsey [5] increased the interest in flat antichains.

Theorem 4 (Flat Antichain Theorem). *Let $\mathcal{B}$ be an arbitrary antichain, then there exists an equivalent flat antichain $\mathcal{B}'$ over the same ground set.*

It is conjectured that every regular antichain has also an equivalent flat regular one (see also [9]). So it is interesting to learn more about maximal flat k-regular antichains on $[m]$ and especially about (k,m)-MFRACs.

3 The associated graph

In the following we are only interested in (k,m)-MFRACs, i.e. in maximal flat k-regular antichains which are subsets of $\binom{[m]}{2} \cup \binom{[m]}{3}$. Using this constraint with a given (k,m)-MFRAC we associate the graph $G(\mathcal{B})$ (see also [3]). This is quite useful, because some constructions are easier to understand when they are formulated in this way.
Let $\mathcal{B}$ be a (k,m)-MFRAC. Then $G(\mathcal{B}) := (V,E)$ with

$$V := [m],$$
$$xy \in E :\Leftrightarrow xy \subsetneq B \text{ for some } B \in \mathcal{B}$$
$$(\text{equivalently: } xy \in E \Leftrightarrow xy \notin \mathcal{B})$$

is the associated graph. Using the fact that xy is not in E if and only if xy is in $\mathcal{B}$, it follows that $G(\mathcal{B})$ is a triangular graph. The fact that $G(\mathcal{B})$ is $(k-m+1)$-triangular follows from the definition $c = c(x) = t(x) - d(x)$ and $k = t(x) + |\{y \in [m] - \{x\} : xy \notin E\}| = t(x) + m - 1 - d(x)$. It is also clear that the associated antichain of $G(\mathcal{B})$ is $\mathcal{B}$, if we define the associated antichain of a triangular graph inversely in the natural way. We obtain the following lemma:

Lemma 5. *A collection $\mathcal{B}$ of sets is a (k,m)-MFRAC if and only if $G(\mathcal{B})$ is a $(k-m+1)$-triangular graph on $[m]$.*

4 Non-isomorphic MFRAC on a small ground set

The following lemmas are (trivial) consequences of the maximality of an MFRAC. We list them here, because they are useful in subsequent proofs.

Lemma 6. *Let $\mathcal{B}$ be a (k,m)-MFRAC, and let a, b, $c \in [m]$ be three pairwise distinct elements. If there exist $A, B, C \in \mathcal{B}$ with $ab \subsetneq A$, $bc \subsetneq B$ and $ac \subsetneq C$, then $abc \in \mathcal{B}$.*

Lemma 7. *Let $\mathcal{B}$ be a (k,m)-MFRAC, and let a, b, c, x, $y \in [m]$ five pairwise distinct elements with $\{abx, acy\} \subseteq \mathcal{B}$ and $abc \notin \mathcal{B}$. Then $bc \in \mathcal{B}$.*

Lemma 8. *If $\mathcal{B}$ is an MFRAC on $[m]$ and $\mathcal{B} \neq \binom{[m]}{3}$, then there exists a 2-set in $\mathcal{B}$.*

Corollary 9. *If $\mathcal{B} \neq \binom{[m]}{3}$ is a (k,m)-MFRAC with $m \geq 5$, then $k \leq 1 + \binom{m-2}{2}$.*

The following three lemmas are useful for the proofs in the following subsections.

Lemma 10. *Let $m \geq 4$, and let $\mathcal{C} \subseteq \binom{[m]}{3}$ be a collection of sets which has the following three properties:*

(a) $|\mathcal{C}| = m - 2$,

(b) $|\bigcap_{C \in \mathcal{C}} C| \geq 1$,

(c) $\bigcup_{C \in \mathcal{C}} C = [m]$.

There exists an $(m-2, m)$-MFRAC $\mathcal{B} \supseteq \mathcal{C}$ if and only if $\mathcal{C}$ has the following property (d):

(d) $ijx,\ ijy \in \mathcal{C} \Rightarrow iyx \notin \mathcal{C}$.

Proof. Assume $\mathcal{C}$ has property (d). Let $\mathcal{D} := \mathcal{D}(\mathcal{C}) := \{X \in \binom{[m]}{2} : \{X\} \cup \mathcal{C}$ is an antichain$\}$ the set of all 2-sets, which are not covered by $\mathcal{C}$. Now, we show that $\mathcal{B} := \mathcal{C} \cup \mathcal{D}$ is an $(m-2, m)$-MFRAC.
Using property (d) and the definition of $\mathcal{D}$, the collection $\mathcal{B}$ of sets is a maximal flat antichain. For every $c \in \bigcap_{C \in \mathcal{C}} C$, $|\mathcal{B}_c| = m - 2$ (properties (a) and (c)). Using (b) we can assume that $\{1\} \subseteq \bigcap_{C \in \mathcal{C}} C$. Let $j \notin \bigcap_{C \in \mathcal{C}} C$ and $l \notin \{1, j\}$. Then either jl is in $\mathcal{D}$ or $1jl$ is in $\mathcal{C}$. So $|\mathcal{B}_j| \geq m - 2$. Because of (d), $|\mathcal{B}_j| \leq m - 1$. Using (c) we know that $|\mathcal{B}_j| = m - 2$, and so $\mathcal{B}$ is an $(m-2, m)$-MFRAC.

Now, let $\mathcal{C}$ be a collection of sets which fulfills the conditions (a) – (c), but which does not fulfill (d). So there are $i \in \bigcap_{C \in \mathcal{C}} C$ and $j, x, y \in [m]$ with ijx, ijy, $iyx \in \mathcal{C}$. Because of maximality of $\mathcal{B}$, jyx has to be in $\mathcal{B}$. Because $ijx \cap ijy \cap iyx = i$, the elements j, x, y can not be in $\bigcap_{C \in \mathcal{C}} C$. Using (c), for all $l \in [m] - \{i, j, x, y\}$, we obtain that either jl is in $\mathcal{B}$ (maximality) or ijl is in $\mathcal{C} \subseteq \mathcal{B}$ (otherwise $\mathcal{B}$ is not maximal or $|\mathcal{B}_i| \geq |\mathcal{C}_i| + 1 = m - 1$). But that implies $|\mathcal{B}_j| \geq 3 + (m - 4) = m - 1$. And this is a contradiction to the regularity of $\mathcal{B}$. ∎

Remark.

1. When we start with a collection $\mathcal{C}$ of sets with properties (a) – (d), then the MFRAC $\mathcal{B} := \mathcal{C} \cup \mathcal{D}$ is unique. Otherwise there exists an element $i \in \bigcap_{C \in \mathcal{C}} C$ and a 3-set $x_1x_2x_3$ which does not contain i. Because of (c) and the fact that i is just in 3-sets, we know that there exists a set $B_j \in \mathcal{C}$ with $ix_j \subsetneq B_j$ for every $j \in [3]$. Using Lemma 6 we get that ix_1x_2, ix_1x_3 and ix_2x_3 are in $\mathcal{C}$, but this is a contradiction to property (d).
2. There exist collections of sets such that $|\bigcap_{C \in \mathcal{C}} C| > 1$ in (b). For example $\mathcal{C} = \{123, 124, 125, 126\}$, and so $\mathcal{B} = \{123,\ 124,\ 125,\ 126,\ 34,\ 35,\ 36,\ 45,\ 46,\ 56\}$ is a $(4, 6)$-MFRAC.

Lemma 11. *Let $\mathcal{B}$ be an $(m-1, m)$-MFRAC. If an element i is in one 3-set in $\mathcal{B}$, then i is in at least three 3-sets in $\mathcal{B}$.*

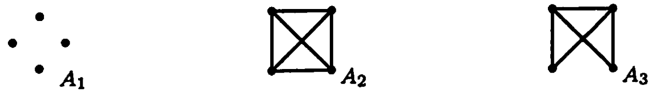

Figure 1: All triangular graphs on [4]

Proof. Assume $\mathcal{B}$ is an $(m-1,m)$-MFRAC and $i \in [m]$ is in exactly one (i) or two (ii) 3-sets. Then in (i), $|\mathcal{B}_i| = 1+(m-3) = m-2$ and in (ii), $|\mathcal{B}_i| \leq 2+(m-4) = m-2$ and this is a contradiction, because it implies $m-1 = k = |\mathcal{B}_i| \leq m-2$. ■

Lemma 12. *Let $m \geq 4$, and let $\mathcal{C} \subseteq \binom{[m]}{3}$ be a collection of sets with the following properties:*

(a) $|\mathcal{C}| = m-1$,

(b) $\bigcap_{C \in \mathcal{C}} C = \{1\}$,

(c) $\bigcup_{C \in \mathcal{C}} C = [m]$.

If there exists an $(m-1,m)$-MFRAC $\mathcal{B} \supseteq \mathcal{C}$, then $|\mathcal{C}_x| = 2$ for every $x \in [2,m]$.

Proof. Using (c) every element of $[m]$ has to be in at least one 3-set. Assume that there is an $(m-1,m)$-MFRAC $\mathcal{B} \supseteq \mathcal{C}$ and an element $x \in [2,m]$ with $|\mathcal{C}_x| = 1$. If x is not in any further 3-set, then this would be a contradiction to Lemma 11. Let $1xy \in \mathcal{C}$ and $xab \in \mathcal{B} - \mathcal{C}$. We can assume that $y \neq a$. Using Lemma 6 and property (c), also 1xa is in $\mathcal{C}$. But this is a contradiction to $|\mathcal{C}_x| = 1$.
If there exists an element $z \in [2,m]$ with $|\mathcal{C}_z| \geq 3$, then because of the pigeonhole principle we also know that there is at least one element $z' \in [2,m]$ with $|\mathcal{C}_{z'}| = 1$. So we are done. ■

Now, we characterize all non-isomorphic MFRACs on $[m]$ and non-isomorphic triangular graphs $T = ([m], E)$ with $m \leq 7$. If $m \leq 4$, then the only triangular graphs are the empty graph E_m, the complete graph K_m for $m \geq 3$ and the complete graph on four vertices with one missing edge (see Figure 1).

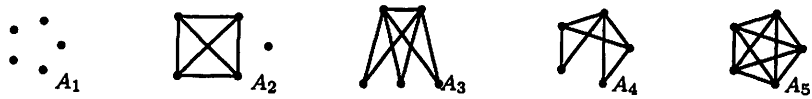

Figure 2: All triangular graphs on [5]

4.1 The case $m = 5$

Theorem 13. *There are exactly five non-isomorphic MFRACs on* [5] *(see Figure 2).*

Proof. We prove this theorem using a case-by-case analysis, i.e. we analyze the possible values $tr := \max_{i\in[m]} |\{B \in \mathcal{B}_i : |B| = 3\}|$. W.l.o.g. we can assume that $|\{B \in \mathcal{B}_1 : |B| = 3\}| = tr$.

$tr = 0$ There is only one possibility: $A_1 = \binom{[5]}{2}$.

$tr = 1$ We can assume that 123, 14, 15 are in $\mathcal{B}$. Since $tr = 1$ we know that 234, 235, 245 or 345 are not in $\mathcal{B}$, so we have to add 24, 25, 34, 35 and 45. But this antichain is not regular since $|\mathcal{B}_1| = 3$ and $|\mathcal{B}_5| = 4$.

$tr = 2$ We get two possibilities for $\mathcal{B}_1$: (a) 123, 124, 15 or (b) 123, 145. In (a), we have to add 25, 34, 35 and 45. So $|\mathcal{B}_5| > |\mathcal{B}_1|$. In (b), because of maximality of $\mathcal{B}$ we have to add 24 and 25. Otherwise we would get a contradiction to $tr = 2$. But then $|\mathcal{B}_2| > |\mathcal{B}_1|$. So there is not an MFRAC in this case.

$tr = 3$ We have to analyze three cases (a) $\mathcal{B}_1^a = 123, 124, 134, 15$, (b) $\mathcal{B}_1^b = 123, 124, 125$, (c) $\mathcal{B}_1^c = 123, 124, 145$.
In (a), we have to add 234 and since $tr = 3$ we can not add any further 3-set. So we have to add all 2-sets which are not covered, and obtain the MFRAC A_2. In the cases (b) and (c), it is only possible to add all 2-sets, which are not covered (Lemma 10). We get two MFRACs A_3 and A_4.

$tr = 4$ Using Lemma 12 we just have to analyze the following case 123, 124, 135, 145. Because of $tr = 4$ and maximality of $\mathcal{B}$, we have to add all 2-sets which are not covered. But then $|\mathcal{B}_1| > |\mathcal{B}_2|$ and this is a contradiction.

$tr = 5$ Using Corollary 9, there can not be any MFRAC.

$tr = 6$ Using Sperner's Theorem there is only one possibility A_5. ∎

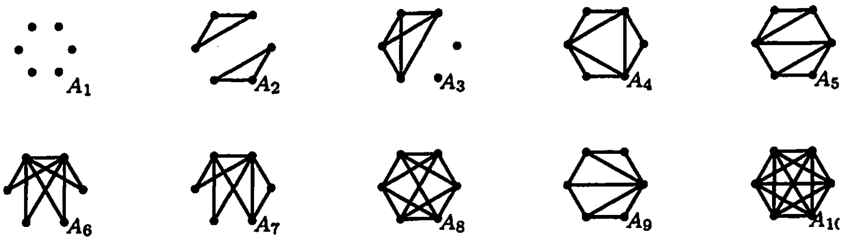

Figure 3: All triangular graphs on [6]

4.2 The case $m = 6$

Theorem 14. *There are ten non-isomorphic MFRACs on [6] (see Figure 3).*

Proof. We use again a case-by-case analysis to prove this theorem, i.e. we analyze again the possible values $tr := \max_{i \in [m]} |\{B \in \mathcal{B}_i : |B| = 3\}|$. We can again assume that $|\{B \in \mathcal{B}_1 : |B| = 3\}| = tr$.

$tr = 0$ There is only one possibility $A_1 = \binom{[6]}{2}$.

$tr = 1$ If there is an MFRAC $\mathcal{B}$ we obtain w.l.o.g. the following structure: 123, 14, 15, 16, 24, 25, 26, 34, 35, 36. It is impossible that 45 is in $\mathcal{B}$ because otherwise 46 is also in $\mathcal{B}$ and we would obtain that $|\mathcal{B}_4| > |\mathcal{B}_1|$. If we add 456 we get the MFRAC A_2.

$tr = 2$ We have to analyze two subcases (a) $\mathcal{B}^a = 123, 124, 15, 16$ and (b) $\mathcal{B}^b = 123, 145, 16$. In both cases we will get a contradiction to the assumption that there is an MFRAC. In (a), we have to add 25, 26 and 34. Since $|\mathcal{B}_1^a| = 4$, we also have to add 35, 36, 45 and 46. If we now add 56 we violate the regularity of $\mathcal{B}^a$ but otherwise $\mathcal{B}^a$ is not maximal. In (b), we have to add 24, 25. Because of maximality of $\mathcal{B}^b$ we also have to put in 26 or 236. In both cases we get that $|\mathcal{B}_2^b| > |\mathcal{B}_1^b|$ which is a contradiction.

$tr = 3$ We have to analyze the following four cases: (a) $\mathcal{B}^a = 123, 124, 125, 16$; (b) $\mathcal{B}^b = 123, 124, 134, 15, 16$; (c) $\mathcal{B}^c = 123, 124, 156$; (d) $\mathcal{B}^d = 123, 124, 135, 16$. In (a), since $tr = 3$ we can not add any further 3-set. But, if we put in all 2-sets which are not covered, then we do not obtain an MFRAC. In (b), we have to put in 234. Since $tr = 3$, we can just add all 2-sets which are not covered, and we have to do that. We obtain the MFRAC A_3. In (c), we have to put in 25 and 26 and get a contradiction to the regularity of $\mathcal{B}^c$. In (d), we have to add 25, 34 and 45. Now, we have to analyze three subcases: (d1) 236, (d2) 246 or (d3) 26 are in $\mathcal{B}^d$. In (d1), we have to add 46 and 56 and get the MFRAC A_4. In (d2), we have to put in 36 and 56 and get the MFRAC A_5. In (d3), we have to add 356 as well as 46 and get an MFRAC which is isomorphic to the MFRAC A_5.

$tr = 4$ We have to analyze the following cases: (a) $\mathcal{B}^a$ = 123, 124, 125, 126; (b) $\mathcal{B}^b$ = 123, 124, 125, 134, 16; (c) $\mathcal{B}^c$ = 123, 124, 125, 136; (d) $\mathcal{B}^d$ = 123, 124, 134, 156; (e) $\mathcal{B}^e$ = 123, 124, 135, 145, 16; (f) $\mathcal{B}^f$ = 123, 124, 135, 146. Using Lemma 10 we know that (a), (c) and (f) deliver three non-isomorphic MFRACs A_6, A_7 and A_9 and we also know that in (d) there is none. So we just have to analyze (b) and (e). In (b), we have to add 234, 35 and 45. Up to now the element 5 is in three sets. We can only add either 56 or 256. So $|\mathcal{B}^b_5| < |\mathcal{B}^b_1|$ and this is a contradiction. In (e), we have to add 25 and 34. The possible sets, which we can add and which also contain the element 2, are 236, 246 or 26. Because of regularity of $\mathcal{B}^e$ we have to put in 236 and 246. Now, all possible sets which we can add and which contain 6, are 356, 456 or 56. So again because of regularity of $\mathcal{B}^e$ we have to add 356 and 456 and obtain the MFRAC A_8.

$tr = 5$ We have to analyze the following cases: (a) 123, 124, 125, 126, 134; (b) 123, 124, 125, 134, 135, 16; (c) 123, 124, 125, 134, 136; (d) 123, 124, 125, 136, 146; (e) 123, 124, 125, 134, 156; (f) 123, 124, 135, 146, 156. In every case we will obtain that there can not be any MFRAC. Using Lemma 12 we know that in the cases (a), (c), (d) and (e) there can not be any MFRAC. In (b), we have to put in 234 and 235. So we also have to add 26 and 36. But then $|\mathcal{B}_6| < |\mathcal{B}_1|$. In (f), there also can not be any further 3-set. Otherwise $tr > 5$. But on the other hand if we add all 2-sets which are not covered, then $|\mathcal{B}_i| = 4 < |\mathcal{B}_1|$ for all $i \in [2, 6]$.

$tr \geq 6$ We get that $v(\mathcal{B}) = 6k$. So the number of 2-sets is congruent 0 mod 3. If there are not any further 2-sets we get $A_{10} = \binom{[6]}{3}$. If there are three 2-sets, then there are two possibilities: Either all elements are in exactly one 2-set, but this yields the MFRAC, which we get in $tr = 4$ subcase (e), or there exists at least one element i which is in more than one 2-set. But then $|\mathcal{B}_i| \leq 5 < 6 \leq tr$ which is a contradiction. ■

4.3 The case $m = 7$

Theorem 15. *There are twenty-one non-isomorphic MFRACs on* [7].

Proof. Again, we prove this theorem with the help of a case-by-case analysis. We can assume that every element is in at least one 3-set (we call this property PR). Otherwise we get a $(6, 7)$-MFRAC which we also would obtain by adding one single element and all 2-sets which are not covered to a $(5, 6)$-MFRAC. In that way we obtain three non-isomorphic MFRACs A_1, A_2 and A_3.
If there is a $(k, 7)$-MFRAC with the property that one element is in at least three 2-sets, then $k \leq 3 + \binom{3}{2} = 6$.
If every element is in at most two 2-sets, we can easily check that for every MFRAC either every element is just in 3-sets (A_{21}) or there is at least one element which is in exactly two 2-sets.
Let us assume that the 2-sets 12 and 13 are in a $(k, 7)$-MFRAC $\mathcal{B}$. Now, we will show that there is at least one further 2-set in $[4, 7]$. Otherwise, $|\mathcal{B}_1| = 8$ and consequently $v(\mathcal{B}) = 56$. So the number of 2-sets has to be congruent 1 mod 3. Because of that we would have at least two further 2-sets. But we can also have at most two, otherwise, because of the pigeonhole principle, there would be an element in $\{2, 3\}$ which is in at least three 2-sets. So we get the two subcases (a) 24, 34 or (b) 24, 35 are in $\mathcal{B}$. In both cases $|\mathcal{B}_7| = 11$ and this is a contradiction to $|\mathcal{B}_1| = 8$.
So in this case, where the element 1 is in exactly two 2-sets, $k = |\mathcal{B}_1| \leq 2 + \binom{4}{2} - 1 = 7$.
Using Theorem 27 we also know that there does not exist a $(7, 7)$-MFRAC. So up to now, we showed that if $\mathcal{B}$ is a $(k, 7)$-MFRAC, then $\mathcal{B} = \binom{[7]}{3}$ or $k \leq 6$.

- $tr \leq 1$ Because of property PR there is no MFRAC in this case.
- $tr = 2$ We have the two cases (a) 123, 124 and (b) 123, 145. In (a), the elements 3 or 4 can not be in any further 3-set. Otherwise $|\mathcal{B}_3| < |\mathcal{B}_1|$ and $|\mathcal{B}_4| < |\mathcal{B}_1|$. Because of property PR we have to add 567 and all 2-sets which are not covered, and get the $(5, 7)$-MFRAC A_4.
  In (b), we have to add 16, 17, 24, 25, 34, 35. We can not add 26 and 27. Otherwise $|\mathcal{B}_2| > |\mathcal{B}_1|$. So we have to add 267. Similarly, we obtain that we have to add 367, 467, 567. But this is a contradiction, because $|\mathcal{B}_7| > |\mathcal{B}_1|$.
- $tr = 3$ We have five cases. We can always assume that $123 \in \mathcal{B}$. In (a) 124, 125, 16, 17; in (b) 124, 134, 15, 16, 17; in (c) 124, 135, 16, 17; in (d) 124, 156, 17 and in (e) 145, 167 are also in $\mathcal{B}$.
  In (a), we have to add 26, 27, 34, 35, 45 and get a problem with 67.

This set can not be an element (property PR) or a subset of an element of $\mathcal{B}$, otherwise $|\mathcal{B}_i| < |\mathcal{B}_1|$ for an $i \in [3,5]$.
In (b), we have to add 234. If we would add all 2-sets which are not covered, then we destroy property PR. Since $tr = 3$ we also can not add any further 3-set which contain 2, 3 or 4. So we have to add 567 and all 2-sets which are not covered. But then we do not obtain a regular collection of sets.
In (c), we have to put in 25, 34 and 45. Now, we have a look at the elements 2 and 3: Because of $tr = 3$ and isomorphism, we have to look at the following five subcases: (c1) 26, 27, 36, 37, (c2) 236, 27, 37, (c3) 246, 27, 36, 37, (c4) 246, 27, 356, 37 and (c5) 246, 27, 357, 36. In (c1), (c2) and (c3), because of property PR and up to isomorphism we have to add 467. But then $|\mathcal{B}_1| > |\mathcal{B}_4|$. In (c4), because of property PR and regularity of $\mathcal{B}$ we have to add 467 and 567. But this is a contradiction to $tr = 3$. In (c5), either we can add 47, 56 as well as 67 to obtain the $(5,7)$-MFRAC A_5, or 467 and 567 to obtain the $(5,7)$-MFRAC A_6.
In (d), because of maximality of $\mathcal{B}^d$ we have to add 34, 45, 46 and a set which contains at least 4 and 7. So we get a contradiction to the regularity of $\mathcal{B}$, because $|\mathcal{B}_4| > |\mathcal{B}_1|$.
In (e), we have to add all 2-sets which are not covered. But then $|\mathcal{B}_i| > |\mathcal{B}_1|$ for all $i \in [2,7]$.

$tr = 4$ Up to isomorphism we have the following cases: 123, 124 together with (a) 125, 126 or (b) 125, 134 or (c) 125, 136 or (d) 125, 167 or (e) 134, 156 or (f) 135, 167 or (g) 135, 146 or (h) 156, 157 or (i) 135, 145.
In (a), since $tr = 4$ we have to add all 2-sets which are not covered. But then we destroy the regularity of $\mathcal{B}$ and also obtain a contradiction to property PR.
In (b), we add 234, 35 and 45. Since $tr = 4$ we can not add 256 and 257. So the element 5 can be in at most five sets, but 1 is in six.
In (c), we have to add 17, 26, 34, 35, 45, 46, 56. Actually 2, 3, 4, 5 and 6 are contained in four sets. So because of regularity of $\mathcal{B}$ we have to add one further 3-set and three 2-sets, all containing the element 7. Therefore, we have three non-isomorphic possibilities: (c1) 237, 47, 57, 67 (A_7); (c2) 247, 37, 57, 67 (A_8); (c3) 367, 27, 47, 57 (A_9), and we obtain three $(5,7)$-MFRACs.
In (d), we have to add all 2-sets which are not covered, and get a contradiction to the regularity of $\mathcal{B}$.
In (e), we have to add 25, 26 and a set, which contains the elements 2 and 7 as well as the set 234. So $|\mathcal{B}_2| \geq 6 > 5 = |\mathcal{B}_1|$.

In (f), we have to add 25, 26 and 27. We obtain the contradiction $|\mathcal{B}_2| \geq 5 > 4 = |\mathcal{B}_1|$. In the exact same way, we get a contradiction in (h).

In (g), we will obtain the two non-isomorphic MFRACs A_{10} and A_{11}. We have to add 17, 25, 26, 34, 36, 45, 56. Because of property PR we have to add at least one further 3-set which contains the element 7. Up to isomorphism and because of $|\mathcal{B}_i| = 5$ as well as $tr = 4$, we have two possibilities, either we add 237 or 357, and obtain the two non-isomorphic (5, 7)-MFRACs 237, 47, 57, 67 (A_{10}) or 357, 27, 47, 67 (A_{11}).

In (i), we have to add 16, 17, 25 and 34. If 67 is in $\mathcal{B}$ then because of $|\mathcal{B}_1| = 6$ and property PR we have to add 236, 237, 246, 247, 356, 357, 456 and 457. Otherwise $|\mathcal{B}_i| < 6$ for all $i \in [6, 7]$. But on the other hand if we add these sets, then $|\mathcal{B}_i| > 6$ for all $i \in [2, 5]$. So we can assume that 67 is not in $\mathcal{B}$. Because of $v(\mathcal{B}) = 42$ we can assume that we have at least two further 2-sets, and we also can assume that one of them is 26. We look at the element 2. Because of regularity of $\mathcal{B}$ we also have to add 237 and 247. Either we can add 36 or 356. If we assume that also 36 is in $\mathcal{B}$, then we have to add 357 (because $|\mathcal{B}_3|$ has to be six) and 456, 467 as well as 567 (because $|\mathcal{B}_6|$ has to be six). But then the element 7 is in at least five 3-sets and we obtain a contradiction to $tr = 4$. So we can assume that 356 and 357 are in $\mathcal{B}$. But then there can be only one 2-set in $\mathcal{B}$ which can contain the element 5. So $|\{B \in \mathcal{B}_5 : |B| = 3\}| \geq 6 - 1 = 5 > tr$ which is a contradiction.

$tr = 5$ Using Lemma 10 we know that 123, 124 together with (a) 125, 126, 127 (A_{12}); (b) 125, 126, 137 (A_{13}); (c) 125, 136, 137 (A_{14}); (d) 125, 136, 147 (A_{15}); (e) 135, 146, 157 (A_{16}), (f) 135, 145, 167 (A_{17}), (g) 125, 136, 167 (A_{18}) deliver seven non-isomorphic (5, 7)-MFRACs and (h) 125, 134, 167, (i) 134, 156, 157 deliver none. So we just have to analyze (j) 125, 126, 134; (k) 125, 136, 146; (l) 125, 134, 136; (m) 135, 146, 156, (n) 125, 134, 135 and (o) 125, 134, 156.

In (j), (k), (l) and (o), we use Lemma 11 to obtain a contradiction: In (j), the element 6 can be in at most two 3-sets; in (k), 5 can be in at most two 3-sets; in (l), 6 can be in at most two 3-sets and in (o), 6 can be in at most two 3-sets.

In (m), we can construct a (6, 7)-MFRAC A_{19}. We have to add 17, 25, 26, 34, 36, 45. The element 7 has to be in at least three 3-sets and also every $i \in [2, 6]$ has to be in at least one further 3-set. So we also have to add 237, 247, 357, 467, 567.

In (n), using Theorem 27 we know that there is no MFRAC.

k \ m	2	3	4	5	6	7
1	1	1	-	-	-	-
2	-	1	1	-	-	-
3	-	-	2	2	-	-
4	-	-	-	2	6	-
5	-	-	-	-	3	15
6	-	-	-	1	-	5
10	-	-	-	-	1	-
15	-	-	-	-	-	1
$\sum$	1	2	3	5	10	21

c \ m	2	3	4	5	6	7
-1	-	1	1	2	6	15
0	1	1	2	2	3	5
1	-	-	-	-	-	-
2	-	-	-	1	-	-
5	-	-	-	-	1	-
9	-	-	-	-	-	1
	1	2	3	5	10	21

Table 1: Overview of (k, m)-MFRACs, and c-triangular graphs on $[m]$ respectively with $m \leq 7$

$tr = 6$ Using the argument from the beginning, we know that $k \leq 6$. So we just have to analyze the collections of sets which has the properties of Lemma 12: (a) 123, 124, 135, 146, 157, 167; (b) 123, 124, 134, 156, 157, 167. In (a), we have to add 25, 26 and 27 and can not add any further set which contains the element 2, so $|\mathcal{B}_2| = 5 < 6 = |\mathcal{B}_1|$. In (b), we have to put in 234, 25, 26, 27, 35, 36, 37, 45, 46, 47, 567 and obtain a $(6, 7)$-MFRAC A_{20}. ■

4.4 Overview

In Table 1 we give a short overview of the number of non-isomorphic (k, m)-MFRACs, and c-triangular graphs on $[m]$ respectively. We want to remark that Karsten Schölzel wrote a computer program and checked the correctness of these values.

5 Necessary conditions for the existence of (k, m)-MFRACs

Before we analyze special values of c, and k, m respectively, we note that for every negative integer c there exists a c-triangular graph. This is the content of the next theorem which is inspired by a construction that can be found in [3]. A natural question is how small the value k can be in terms of m.

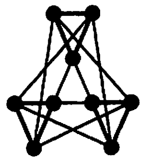

Figure 4: (-2)-triangular graph on $V = [9]$

Theorem 16. *Let $c < 0$. Then there exists a c-triangular graph $T = (V, E)$.*

Proof. We define $c' := -c$, $V := (\mathbb{Z}_3)^{c'}$ and

$$\begin{aligned} &\{(i_1, i_2, \ldots, i_{c'}), (j_1, j_2, \ldots, j_{c'})\} \in E \\ \Leftrightarrow \quad &\exists k \in [c'] \text{ with } i_k \neq j_k \text{ and } \forall l \in [c'] - \{k\} : i_l = j_l. \end{aligned}$$

For all $x \in V$ we get $d(x) = 2c'$, $t(x) = c'$, and $c(x) = t(x) - d(x) = c' - 2c' = c$. ■

An example for $c = -2$ is shown in Figure 4.

Lemma 17. *Let $\mathcal{B}$ be a (k, m)-MFRAC. Then*

$$k \geq \left\lfloor \frac{3}{4}(m-1) \right\rfloor.$$

Proof. Let $\mathcal{B}$ be an arbitrary (k, m)-MFRAC. Choose $x \in [m]$. We define $\mathcal{B}_x^2 := \{y \in [m] : xy \in \mathcal{B}\}$ and $\mathcal{B}_x^3 := \{y \in [m] - \{x\} : \exists B \in \mathcal{B} \text{ with } xy \subsetneq B\}$. Because of maximality of $\mathcal{B}$:

$$|\mathcal{B}_x^2| + |\mathcal{B}_x^3| = m - 1.$$

We analyze two distinct cases:

First, let us assume that

$$|\mathcal{B}_x^3| > \frac{m-1}{2}.$$

Choose an $i \in \mathcal{B}_x^3$ with the property that

$$(xiy, xiz \in \mathcal{B} \implies y = z).$$

If there does not exist such an element i, then

$$|\mathcal{B}_x| \geq 2\frac{|\mathcal{B}_x^3|}{2} + |\mathcal{B}_x^2| = m - 1$$

and this fulfills our proposition. So for these fixed elements x and i, there exists exactly one element j in $[m]$ such that $xij \in \mathcal{B}$. Because of maximality of $\mathcal{B}$ we get

$$\forall y \in \mathcal{B}_x^3 - \{i, j\} : \quad iy \in \mathcal{B}.$$

This implies $|\mathcal{B}_x^3| - 2 \leq |\mathcal{B}_i^2|$, and we obtain

$$\begin{aligned}
|\mathcal{B}_i| &\geq |\mathcal{B}_i^2| + \frac{|\mathcal{B}_i^3|}{2} \\
&= \frac{|\mathcal{B}_i^2|}{2} + \frac{|\mathcal{B}_i^2| + |\mathcal{B}_i^3|}{2} \\
&\geq \frac{1}{2}(|\mathcal{B}_x^3| - 2) + (\frac{|\mathcal{B}_i^2| + |\mathcal{B}_i^3|}{2}) \\
&\geq \frac{1}{2}|\mathcal{B}_x^3| - 1 + \frac{m-1}{2} \\
&> \frac{m-1}{4} - 1 + \frac{m-1}{2} \\
&= \frac{3}{4}(m-1) - 1.
\end{aligned}$$

Secondly, we analyze the case when

$$|\mathcal{B}_x^3| \leq \frac{m-1}{2}$$

and so $|\mathcal{B}_x^2| \geq \frac{m-1}{2}$. We obtain:

$$\begin{aligned}
|\mathcal{B}_x| &\geq \frac{|\mathcal{B}_x^3|}{2} + |\mathcal{B}_x^2| \\
&= \frac{|\mathcal{B}_x^3| + |\mathcal{B}_x^2|}{2} + \frac{|\mathcal{B}_x^2|}{2} \\
&\geq \frac{m-1}{2} + \frac{m-1}{4} \\
&= \frac{3}{4}(m-1).
\end{aligned}$$

■

Remark. This inequality is sharp, because there exists a (6, 9)-MFRAC. But, it does not seem to be the best bound if m is large. We can improve the bound to $k \geq \frac{5}{6}m - \frac{3}{2}$ if for all distinct elements x, y with $xy \subsetneq B \in \mathcal{B}$

for some $B = xyz$, $\mathcal{B}_x^3 \cap \mathcal{B}_y^3 = \{z\}$. This bound is also attained for the (6, 9)-MFRAC.

Lemma 18. *Let $\mathcal{B}$ be a (k, m)-MFRAC with the property that every 2-set of $[m]$ is a subset of at most one element B of $\mathcal{B}$, then*

$$m - 1 \geq k \geq \frac{5}{6}m - \frac{3}{2}.$$

Proof. It is obvious that $k \leq m - 1$. So we just have to prove the other inequality.
Let $123 \in \mathcal{B}$. Then every element in $\{4, 5, \ldots, m\}$ can be with at most one of the elements 1, 2, 3 in at most one further 3-set. Otherwise, if there is an element i which is with two of them in one further 3-set, then we obtain a contradiction to the property that every 2-set is a subset of at most one element B of $\mathcal{B}$.
Using the pigeonhole principle we know that $|\mathcal{B}_j^3| \leq 2 + \frac{m-3}{3} = \frac{m}{3} + 1$ for at least one $j \in [3]$. For this j we get that $|\mathcal{B}_j^2| = (m-1) - |\mathcal{B}_j^3| \geq \frac{2m}{3} - 2$ and so:

$$\begin{aligned} |\mathcal{B}_j| &= \frac{|\mathcal{B}_j^3|}{2} + |\mathcal{B}_j^2| \\ &= \frac{|\mathcal{B}_j^3| + |\mathcal{B}_j^2|}{2} + \frac{|\mathcal{B}_j^2|}{2} \\ &\geq \frac{m-1}{2} + \frac{m}{3} - 1 \\ &= \frac{5}{6}m - \frac{3}{2}. \end{aligned}$$

∎

Corollary 19. *Let $\mathcal{B}$ a (k, m)-MFRAC with the property that there exists a 3-set $xyz \in \mathcal{B}$ with $\mathcal{B}_x^3 \cap \mathcal{B}_y^3 = \{z\}$, $\mathcal{B}_x^3 \cap \mathcal{B}_z^3 = \{y\}$ and $\mathcal{B}_y^3 \cap \mathcal{B}_z^3 = \{x\}$. Then*

$$k \geq \frac{5}{6}m - \frac{3}{2}.$$

6 General constructions

The following construction is important in the context of (k, m, n)-ACs (see for example [1]).

Theorem 20. *Let $\mathcal{B}$ be an $(m_1 + t, m_1, n_1)$-AC and let $\mathcal{B}'$ be an $(m_2 + t, m_2, n_2)$-AC with $t \geq -\min\{m_1, m_2\}$ and with the property that each element of $\mathcal{B} \cup \mathcal{B}'$ has at least cardinality two. Then there exists an $(m_1 + m_2 + t, m_1 + m_2, n_1 + n_2 + m_1 m_2)$-AC $\mathcal{C}$.*

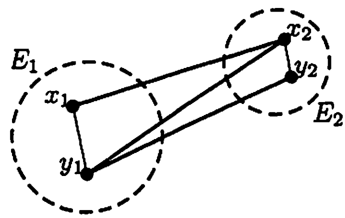

Figure 5: Z-construction

When we look at the proof of Theorem 20 (see [1], for example), we immediately obtain the following result for MFRACs:

Corollary 21. *Let $\mathcal{B}^i$ be a (k_i, m_i)-MFRAC $(i = 1, 2)$. If $k_1 - m_1 = k_2 - m_2$, then there exists an $(m_1 + k_2, m_1 + m_2)$-MFRAC.*

The dual form of this corollary looks simple:

Corollary 22. *Let c be an integer, and let $T_i = (V_i, E_i)$ be a c-triangular graph on $[m_i]$ $(i = 1, 2)$. Then there also exists a c-triangular graph T' on $[m_1 + m_2]$.*

The following is slightly stronger:

Lemma 23. *Let c be an integer, and let $T_i = (V_i, E_i)$ be a connected c-triangular graph on $m_i > 1$ vertices $(i = 1, 2)$. Then there also exists a connected c-triangular graph T' on $[m_1 + m_2]$.*

Proof (Z-construction). As $E_i \neq \emptyset$ $(i = 1, 2)$, we can pick an edge $e_i = x_i y_i \in E_i$ $(i = 1, 2)$ and define $T' := (V_1 \cup V_2, E')$ with

$$E' := E_1 \cup E_2 \cup \{x_1 x_2, x_2 y_1, y_1 y_2\}.$$

In Figure 5 this construction is illustrated. ∎

7 (m, m)-MFRACs

7.1 Existence of (m, m)-MFRACs

In the following we describe two constructions which we need to prove the sufficient conditions for the existence of (m, m)-MFRACs.

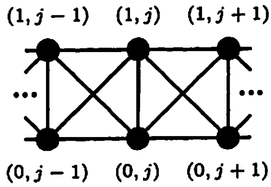

Figure 6: Detail of a 1-triangular graph on $2r$ vertices

Theorem 24. *Let $m \geq 8$ be an even positive integer. Then there exists a 1-triangular graph on $[m]$ (see Figure 6).*

Proof. Let $m = 2r$, and let $G = (V, E)$ be a simple graph with $V := \mathbb{Z}_2 \times \mathbb{Z}_r$ and

$$\begin{aligned} &\{(i_1, j_1), (i_2, j_2)\} \in E \\ \Leftrightarrow\, &(i_1 \neq i_2 \text{ and } j_1 = j_2) \text{ or } \{j_1, j_2\} \in C, \end{aligned}$$

where $C := \{\{1,2\},\{2,3\},\{3,4\},\ldots,\{r-1,r\},\{r,1\}\}$. For all $x \in V$, $d(x) = 5$ and $t(x) = 6$. So G is a 1-triangular graph. ■

Corollary 25. *Let $m \geq 8$ be an even positive integer. Then there exists an (m, m)-MFRAC.*

Lemma 26. *Let $T = ([m], E)$ be a (connected) 1-triangular graph. Then there exists a (connected) 1-triangular graph T' on $V(T') = [m+3]$.*

Proof. Let $x, y \in [m]$ with $xy \notin E$. We define $T' := ([m+3], E')$ with

$$\begin{aligned} E' := E &\cup \{ij : i \in \{x, y\}, j \in \{m+1, m+2, m+3\}\} \\ &\cup \{(m+1)(m+2), (m+1)(m+3), (m+2)(m+3)\}. \end{aligned}$$

We obtain that $d_{T'}(m+i) = 4$, $t_{T'}(m+i) = 5$ $(i = 1,2,3)$, $d_{T'}(a) = d_T(a)+3$, $t_{T'}(a) = t_T(a)+3$ $(a \in \{x, y\})$ and $d_{T'}(b) = d_T(b)$, $t_{T'}(b) = t_T(b)$ for all $b \in [m] - \{x, y\}$. So T' is a 1-triangular graph on $m+3$ vertices. See also Figure 7. ■

7.2 Non-existence of (m, m)-MFRACs

Theorem 27. *There does not exist any $(7, 7)$-MFRAC.*

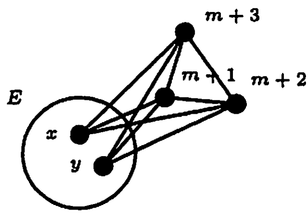

Figure 7: 1-triangular graph on $V = [m+3]$

Proof. Suppose there is a (7,7)-MFRAC $\mathcal{B}$. Then $v(\mathcal{B}) = 49$. So there are exactly two 2-sets (case 1), or five 2-sets (case 2), or more than eight 2-sets (case 3).
Before we analyze these cases we remark (*) that no element is in more than two 2-sets. Otherwise this element is in at most $3 + \binom{3}{2} = 6 < 7$ sets.

case 1: Using the pigeonhole principle we know that there exists an element j, which is only in 3-sets. We get that $7 = |\mathcal{B}_j| = \binom{6}{2} - 2 = 13$ and this is a contradiction.

case 2: Using (*) we know that there is none element which is in three or more 2-sets. So there has to be at least one element, which is in exactly two 2-sets. W.l.o.g. we get that 12, 13, 145, 146, 147, 156, 157 are in $\mathcal{B}$. Because of maximality and regularity of $\mathcal{B}$, $67 \in \mathcal{B}$ as well as 456, 457.
We know that every element $i \in [7]$, especially the elements 4 and 5, have to be in at least one 2-set. Otherwise $|\mathcal{B}_i| = \binom{6}{2} - 5 = 10 > 7$. Because of isomorphism we can assume that 24 and 35 are also in $\mathcal{B}$. The possible 3-sets we could add and which contain the element 2, are 236, 237, 256 and 257. So $7 = |\mathcal{B}_2| \leq 6$, which is a contradiction.

case 3: Again, using the pigeonhole principle, we know that there is an element j, which is in at least three 2-sets, and this is a contradiction to (*). ■

Theorem 28. *There does not exist any (9,9)-MFRAC.*

Proof. We assume that there exists a (9,9)-MFRAC $\mathcal{B}$. Then $v(\mathcal{B}) = 81$. So the number n_2 of 2-sets is divisible by three. (1)
As $R(n, 9) \geq 10$ for all $n \geq 33$ (see [8]), the size of $\mathcal{B}$ is smaller than or equal to 32, and so $n_2 \leq 15$. (2) Otherwise we can not reach a volume of 81.

If there is an element of $\mathcal{B}$, which is only in 3-sets, then there are exactly $\binom{8}{2} - 9 = 28$ 2-sets. This would be a contradiction to (1). If there is an element $j \in [9]$, which is in exactly one 2-set ji, then there are exactly $\binom{7}{2} - (9-1) = 13$ 2-sets on $[m] - \{i,j\}$ and consequently n_2 is at least $1 + 13 = 14$. According to (1) and (2), n_2 has to be 15. So i has to be in exactly one further 2-set il and so there are $\binom{6}{2} - (9-2) = 8$ 2-sets on $[m] - \{i,j,l\}$. Then l has to be in $13 - 8 + 1 = 6$ 2-sets. But this is a contradiction because then $9 = k = |\mathcal{B}_l| \leq 6 + \binom{2}{1} = 8$. So every element is in at least two 2-sets (3).
Using these three facts we have to look at the following cases: (a) $n_2 = 9$, (b) $n_2 = 12$, (c) $n_2 = 15$.

(a) Using the pigeonhole principle and (3), we know that every element is in exactly two 2-sets. Up to isomorphism there are four subcases (a) 12, 13, 24, 35, 46, 57, 68, 79, 89, (b) 12, 13, 24, 35, 46, 56, 78, 79, 89, (c) 12, 13, 24, 35, 45, 67, 68, 79, 89 and (d) 12, 13, 23, 45, 46, 56, 78, 79, 89. But in every subcase it is easy to verify that there does not exist a $(9,9)$-MFRAC.

(b) We know that there must be at least one element, which is in exactly two 2-sets. W.l.o.g. this is the element 1 and the two 2-sets are 12 and 13. So there are exactly $\binom{6}{2} + 2 - 9 = 8$ 2-sets on $[4,9]$ and $12 - 2 - 8 = 2$ 2-sets on $[2,9]$, which have at least one element from $\{2,3\}$. So we get four subcases: (i) 12, 13, 23, 24; (ii) 12, 13, 24, 35; (iii) 12, 13, 24, 34; (iv) 12, 13, 24, 25.

We start with (i) and look at the element 2. We know that there are exactly $\binom{5}{2} + 3 - 9 = 4$ 2-sets on $[5,9]$ and therefore also four further 2-sets, which contain the element 4. We obtain that $|\mathcal{B}_4| \leq 5 + \binom{3}{2} = 8$, which is a contradiction.

Now, we have a look at (ii) and especially at the elements 2 and 3. We know that there are $\binom{6}{2} + 2 - 9 = 8$ 2-sets on $\{5,6,7,8,9\}$ and also eight 2-sets on $\{4,6,7,8,9\}$. So we know, there are eight 2-sets on $[6,9]$ and this is obviously a contradiction.

Also in case (iii), we get a contradiction. With the same argument as before we know that there are eight 2-sets on $[5,9]$. So there is at least one element, which is in at least four 2-sets. But then this element can only be contained in $4 + (\binom{4}{2} - 4) = 6$ sets.

In (iv), there can not be any MFRAC, because the element 3 is just in one 2-set and this is a contradiction to (3).

(c) At first we assume that there is an element, which is in exactly two 2-sets. Again w.l.o.g this is the element 1, and the sets 12, 13 are in $\mathcal{B}$. We obtain that there are eight 2-sets on $[4,9]$ and five further ones on $[2,9]$, which contain at least one of the elements 2 or 3. No element can be in five or more 2-sets. We use the pigeonhole principle, and up to isomorphism we obtain the following two cases: (i) 12, 13, 23, 24, 25, (ii) 12, 13, 24, 25, 26.

We start with (i) and have a look at the element 2. We obtain that there is $4 + \binom{4}{2} - 9 = 1$ 2-set on $[6,9]$. W.l.o.g. this is 67. Nine 2-sets are missing. Exactly two of them contain 3 and so at least $9 - 2 = 7$ have to contain 4 or 5. Using the pigeonhole principle we know that 4 or 5 is in at least $4 + 1 = 5$ 2-sets. This is a contradiction.
In case (ii), the argument is nearly the same. We look at the element 3. Two subcases are possible, we have to add two 2-sets containing 3 and exactly one on $\{3,7,8,9\}$. Up to isomorphism we either can add $\{34,37\}$ or $\{34,35,78\}$. In the first subcase we have to add $\binom{5}{2} - 2 - (9 - 3) = 2$ further 2-sets on $\{2,5,6,8,9\}$. Hence, we have to add exactly two 2-sets on $\{5,6,8,9\}$. Therefore we have to add $15 - 5 - 2 - 2 = 6$ further 2-sets, which contain 4 or 7. So 4 or 7 is in at least five 2-sets.
In the other subcase we have to add $\binom{5}{2} - 2 - (9-3) = 2$ further 2-sets on $\{2,6,7,8,9\}$. So we have to add two further 2-sets on $\{6,7,8,9\}$ and five 2-sets, which contain 4 or 5. But again this is not possible. Otherwise the element 4 or the element 5 are in at least five two 2-sets. So we get again a contradiction.

From now on, we can assume that every element is in at least three 2-sets. We obtain that there are exactly three elements, which are in four 2-sets and six elements which are in three 2-sets. W.l.o.g we obtain that 12, 13, 14, 15 and 67 are in $\mathcal{B}$.

We have to add ten further 2-sets, none of which is a subset of $\{6,7,8,9\}$. Since $|\mathcal{B}_i| \geq 3$ for all $i \in [6,9]$, we obtain that every further 2-set contains exactly one element from $[6,9]$, and so also exactly one element from $[2,5]$. Two elements of $\{2,3,4,5\}$ are in three further 2-sets. W.l.o.g. we can assume that 2 is one of them. Up to isomorphism we get the following two cases (*) 26, 27, 28 or (**) 27, 28, 29 are in $\mathcal{B}$. In (*), the element 2 can be at most in the following 3-sets 234, 235, 239, 245, 249, 259. We have to add five of these six 3-sets. But this is a contradiction, because then the element 9 can be in at most one 2-set, otherwise $|\mathcal{B}_2| < 9$. In (**), the element 2 can be in the following 3-sets 234, 235, 236, 245, 246, 256. But this is also a contradiction, because then the element 6 can be in at most two 2-sets.

So we are done. ∎

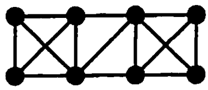

Figure 8: A connected 0-triangular graph on $V = [8]$

Theorem 29. *An (m, m)-MFRAC and a (connected) 1-triangular graph T on m vertices exist if and only if $m = 8$ or $m \geq 10$.*

Proof. That this criterion is necessary follows immediately from Section 4, Theorem 27 and Theorem 28. Using Theorem 25 and Lemma 26 we also know that it is sufficient. ∎

8 $(m-1, m)$-MFRACs

In this section we analyze if for given m a connected 0-triangular graph on $[m]$ exists. If we do not require that our graph is connected, then for every m we can choose a graph with empty edge set.
The result of this section will be that there is a connected 0-triangular graph on $[m]$ if and only if $m = 4$ or $m \geq 6$. In Section 4 we presented connected 0-triangular graphs on $V \in \{[4], [6], [7]\}$. Using these three graphs and recursive constructions, we will show by induction that for every $m \geq 8$ a connected 0-triangular graph exists. It follows from Section 4 that there does not exist one for $m = 5$ and $m < 4$.

Using the complete graph on four vertices and Lemma 23 we obtain a connected 0-triangular graph on $V = [8]$ (see Figure 8).

Theorem 30. *Let $T = ([m], E)$ be a connected 0-triangular graph. Then there exists a connected 0-triangular graph T' on $[m+3]$.*

Proof. Let $x \in [m]$ be an arbitrary fixed vertex. We define $T' := ([m+3], E')$ with

$$E' := E \cup \{ij : i, j \in \{x, m+1, m+2, m+3\}, i \neq j\}.$$

We obtain $d_{T'}(m+i) = t_{T'}(m+i) = 4$ $(i = 1, 2, 3)$, $d_{T'}(x) = d_T(x) + 3$, $t_{T'}(x) = t_T(x) + 3$ and $d_{T'}(y) = d_T(y)$, $t_{T'}(y) = t_T(y)$ for all $y \in [m] - \{x\}$. This construction is illustrated in Figure 9. ∎

Corollary 31. *A connected 0-triangular graph exists if and only if $m = 4$ or $m \geq 6$.*

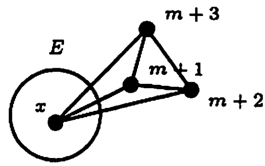

Figure 9: A 0-triangular graph on $V = [m+3]$

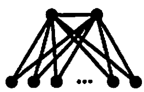

Figure 10: A connected (-1)-triangular graph

Proof. The necessity follows from Section 4.
If $m = 4, 6, 7, 8$ there exists a 0-triangular graph. Using Lemma 30 and the 0-triangular graphs on [6], [7] and [8], we know that there exist also a connected 0-triangular graphs on $[m]$ for every $m \geq 9$. ∎

9 $(m-2, m)$-MFRACs

In this section we briefly show that for every positive integer $m \geq 3$ there is an $(m-2, m)$-MFRAC.

Lemma 32. *Let $m \geq 3$ be a positive integer. Then there exists a connected (-1)-triangular graph $T = ([m], E)$.*

Proof. Let $m \geq 3$ be an arbitrary fixed positive integer. We define $E := \{12\} \cup \{ij : i \in \{1,2\}, j \in \{3,4,\ldots,m\}\}$ (see Figure 10). ∎

Remark. We remark that Lemma 32 is a special dual case of Lemma 10 with $C := \{123, 124, 125, \ldots, 12m\}$. In this way many different constructions can be found. For example, we choose for fixed m the vertex set $V := [m]$ and the edge set $E := \{1j : j \in [2, m]\} \cup \{23, 34, 45, \ldots (m-1)m\}$ or in the dual form $C := \{123, 134, 145, \ldots, 1(m-1)m\}$.

10 Open problems

There are several related questions and open problems, which are unsolved in general:

1. For which parameters k, m, n does there exist a (k, m)-MFRAC of size n?
   Especially: What is the maximum size of a (k, m)-MFRAC?
2. For which parameters k, m does there exist a maximal k-regular antichain on $[m]$?
3. For which parameters k, m does there exist a maximal flat k-regular antichain on $[m]$?
4. For which parameters k, m does there exist a maximal k-regular antichain on $[m]$ on r levels?
5. How many non-isomorphic (k, m)-MFRACs exist?
   Or more generally: How many non-isomorphic maximal k-regular antichains on $[m]$ exist?

Acknowledgement

The author would like to thank Thomas Kalinowski and Karsten Schölzel for valuable help whilst preparing this paper.

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