A variation of the partial parity (g, f)-factor theorem due to Kano and Matsuda*

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Abstract. Let $A_n = (a_1, a_2, \ldots, a_n)$ and $B_n = (b_1, b_2, \ldots, b_n)$ be two sequences of nonnegative integers satisfying $a_1 \geq a_2 \geq \cdots \geq a_n$, $a_i \leq b_i$ for $i = 1, 2, \ldots, n$ and $a_i = a_{i+1}$ implies that $b_i \geq b_{i+1}$ for $i = 1, 2, \ldots, n-1$. Let I be a subset of $\{1, 2, \ldots, n\}$ and $a_i \equiv b_i \pmod 2$ for each $i \in I$. $(A_n; B_n)$ is said to be partial parity graphic with respect to I if there exists a simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that $a_i \leq d_G(v_i) \leq b_i$ for $i = 1, 2, \ldots, n$ and $d_G(v_i) \equiv b_i \pmod 2$ for each $i \in I$. In this paper, we give a characterization for $(A_n; B_n)$ to be partial parity graphic. This is a variation of the partial parity (g, f)-factor theorem due to Kano and Matsuda in degree sequences.

Keywords. degree sequence, partial parity graphic sequence, factor theorem.

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1. Introduction

A non-increasing sequence $\pi = (d_1, d_2, \ldots, d_n)$ of nonnegative integers is said to be *graphic* if it is the degree sequence of a simple graph G on n vertices, and such a graph G is called a *realization* of π. The following well-known theorem due to Erdős and Gallai [2] gives a characterization for π to be graphic. This is a variation of the classical Tutte's f-factor theorem [11] in degree sequences.

Theorem 1.1 (Erdős and Gallai [2]) Let $\pi = (d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers. Then π is graphic if and

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only if $\sum_{i=1}^{n} d_i$ is even and

$$\sum_{i=1}^{t} d_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, d_i\} \text{ for each } t \text{ with } 1 \leq t \leq n.$$

There are several survey articles on the subject of degree sequences of graphs (see Li and Yin [6], Lai and Hu [5] and Rao [10]).

Let $A_n = (a_1, a_2, \ldots, a_n)$ and $B_n = (b_1, b_2, \ldots, b_n)$ be two sequences of nonnegative integers with $a_i \leq b_i$ for $i = 1, 2, \ldots, n$. $(A_n; B_n)$ is said to be *graphic* if there exists a simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that $a_i \leq d_G(v_i) \leq b_i$ for $i = 1, 2, \ldots, n$. If A_n and B_n satisfy $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_i = a_{i+1}$ implies that $b_i \geq b_{i+1}$ for $i = 1, 2, \ldots, n-1$, then A_n and B_n is said to be in *good* order. In [1], Cai et al. presented a characterization for $(A_n; B_n)$ to be graphic, where A_n and B_n are in good order. This is a variation of the classical Lovász's (g, f)-factor theorem [7] in degree sequences, solves a research problem posed by Niessen [9] and generalizes Theorem 1.1 (which corresponds to $a_i = b_i = d_i$ for each i). They defined for $t = 0, 1, \ldots, n$

$$I_t = \{i | i \geq t+1 \text{ and } b_i \geq t+1\}$$

and

$$\varepsilon(t) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for all } i \in I_t \text{ and } \sum_{i \in I_t} b_i + t|I_t| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.2 (Cai et al. [1]) If A_n and B_n are in good order, then $(A_n; B_n)$ is graphic if and only if

$$\sum_{i=1}^{t} a_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, b_i\} - \varepsilon(t) \text{ for each } t \text{ with } 0 \leq t \leq n.$$

Let G be a simple graph and let $g, f : V \to Z^+$ be two functions such that $g(v) \leq f(v)$ for all $v \in V$, where $V = V(G)$ is the vertex set of G and Z^+ denotes the set of nonnegative integers. Let U be a subset of V, and let $g(v) \equiv f(v) \pmod 2$ for all $v \in U$. A spanning subgraph F of G is called a *partial parity (g, f)-factor with respect to U* if $g(v) \leq d_F(v) \leq f(v)$ for all $v \in V$ and $d_F(v) \equiv f(v) \pmod 2$ for all $v \in U$. For any two disjoint subsets P and Q of V, we write $e_G(P, Q)$ for the number of edges of G joining P to Q.

Theorem 1.3 (Kano and Matsuda [4]) Let G be a simple graph and U a subset of $V(G)$. Let $g, f : V \to Z^+$ be two functions satisfying $g(v) \leq f(v)$

for all $v \in V$, and $g(v) \equiv f(v) \pmod 2$ for all $v \in U$. Then G has a partial parity (g, f)-factor with respect to U if and only if

$$\sum_{v \in S} f(v) - \sum_{v \in T} g(v) + \sum_{v \in T} d_{G \setminus S}(v) - \omega(S,T) \geq 0$$

for all disjoint sets $S, T \subseteq V$, where $\omega(S,T)$ denotes the number of components C of $G - (S \cup T)$ such that $g(v) = f(v)$ for all $v \in V(C) \setminus U$ and $\sum_{v \in V(C)} f(v) + e_G(V(C), T)$ is odd.

Note that if $U = \emptyset$, then Theorem 1.3 is the classical Lovász's (g, f)-factor theorem, and if $U = V(G)$, then Theorem 1.3 is the classical Lovász's parity (g, f)-factor theorem [8]. We now consider a variation of Theorem 1.3 in degree sequences. Let I be a subset of $\{1, 2, \ldots, n\}$ and $a_i \equiv b_i \pmod 2$ for each $i \in I$. $(A_n; B_n)$ is said to be *partial parity graphic with respect to* I if there exists a simple graph G with vertices $v_1, v_2, \ldots, v_n$ such that $a_i \leq d_G(v_i) \leq b_i$ for each i and $d_G(v_i) \equiv b_i \pmod 2$ for each $i \in I$. The purpose of this paper is to give a characterization for $(A_n; B_n)$ to be partial parity graphic with respect to I, where A_n and B_n are in good order. For $0 \leq t \leq n$, we define

$$L_t = \{i | i \geq t+1 \text{ and } b_i \geq t+1\}$$

and

$$\zeta(t) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for all } i \in L_t \setminus I \text{ and } \sum_{i \in L_t} b_i + t|L_t| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.4 If A_n and B_n are in good order, I is a subset of $\{1, 2, \ldots, n\}$ and $a_i \equiv b_i \pmod 2$ for each $i \in I$, then $(A_n; B_n)$ is partial parity graphic with respect to I if and only if

$$\sum_{i=1}^{t} a_i \leq t(t-1) + \sum_{i=t+1}^{n} min\{t, b_i\} - \zeta(t) \text{ for each } t \text{ with } 0 \leq t \leq n. \quad (1)$$

It is easy to see that if $I = \emptyset$, then Theorem 1.4 is a variation of the classical Lovász's (g, f)-factor theorem in degree sequences, that is Theorem 1.2, and if $I = \{1, 2, \ldots, n\}$, then Theorem 1.4 is a variation of the classical Lovász's parity (g, f)-factor theorem in degree sequences, that is Theorem 1.4 of [3].

2. Proof of Theorem 1.4

The proof technique of this paper closely follows that of [3]. We first give a lemma, which is a generalization of Lemma 2.1 of [3]. We define for

$J \subseteq \{1, 2, \dots, n\}$

$$L_J = \{i | i \notin J \text{ and } b_i \geq |J| + 1\}$$

and

$$\zeta(J) = \begin{cases} 1 & \text{if } a_i = b_i \text{ for all } i \in L_J \setminus I \text{ and } \sum_{i \in L_J} b_i + |J||L_J| \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.1 Let A_n and B_n be in good order. If $(A_n; B_n)$ is partial parity graphic with respect to I, then

$$\sum_{i \in J} a_i \leq |J|(|J| - 1) + \sum_{i \notin J} min\{|J|, b_i\} - \zeta(J). \tag{2}$$

Proof. Without loss of generality, we may assume that $b_i \geq 1$ for $1 \leq i \leq n$. There exists a simple graph G with vertices $v_1, v_2, \dots, v_n$ such that $a_i \leq d_G(v_i) \leq b_i$ and $d_G(v_i) \equiv b_i \pmod 2$ for each $i \in I$. If $J = \emptyset$, then $L_\emptyset = \{1, 2, \dots, n\}$. If $\zeta(\emptyset) = 1$, then $a_i = d_G(v_i) = b_i$ for each $i \in \{1, 2, \dots, n\} \setminus I$ and $\sum_{i=1}^{n} b_i$ is odd. However,

$$\begin{aligned} \sum_{i=1}^{n} d_G(v_i) &= \sum_{i \in I} d_G(v_i) + \sum_{i \in \{1,2,\dots,n\} \setminus I} d_G(v_i) \\ &\equiv (\sum_{i \in I} b_i + \sum_{i \in \{1,2,\dots,n\} \setminus I} b_i) \ (mod\ 2) \\ &= \sum_{i=1}^{n} b_i \ (mod\ 2) \equiv 1 \ (mod\ 2), \end{aligned}$$

a contradiction. Thus $\zeta(\emptyset) = 0$ and therefore, (2) holds in this case. If $J \neq \emptyset$, it follows from theorem 1.1 that

$$\begin{aligned} \sum_{i \in J} a_i &\leq \sum_{i \in J} d_G(v_i) \\ &\leq |J|(|J| - 1) + \textstyle\sum_{i \notin J} min\{|J|, d(v_i)\} \\ &\leq |J|(|J| - 1) + \textstyle\sum_{i \notin J} min\{|J|, b_i\}. \end{aligned} \tag{3}$$

Clearly, (2) holds when $\zeta(J) = 0$. Now suppose $\zeta(J) = 1$. Then $a_i = d_G(v_i) = b_i$ for all $i \in L_J \setminus I$ and $\sum_{i \in L_J} b_i + |J||L_J|$ is odd. If (2) does not hold for the J, then it follows from (3) that

$$\sum_{i \notin J} min\{|J|, d_G(v_i)\} = \sum_{i \notin J} min\{|J|, b_i\}, \tag{4}$$

and

$$\sum_{i \in J} d_G(v_i) = |J|(|J| - 1) + \sum_{i \notin J} min\{|J|, d_G(v_i)\}. \tag{5}$$

For convenience, let $\{i|i \notin J\} = L_J \cup A_3$ and $L_J \cap I = A_1 \cup A_2$, where $A_1 = \{i|i \in L_J \cap I \text{ and } d_G(v_i) \geq |J|+1\}$ and $A_2 = \{i|i \in L_J \cap I \text{ and } d_G(v_i) \leq |J|\}$. Since $min\{|J|, d_G(v_i)\} \leq min\{|J|, b_i\}$ for each $i \notin J$, by (4), we have that $min\{|J|, d_G(v_i)\} = min\{|J|, b_i\}$ for each $i \notin J$. Thus $d_G(v_i) = |J|$ for each $i \in A_2$. Let F be the induced subgraph of $\{v_i|i \in L_J\}$ in G. The sum of the degrees of vertices in F equals $\sum_{i \in L_J} d_G(v_i) - |J||L_J| = (\sum_{i \in L_J} d_G(v_i) + |J||L_J|) - 2|J||L_J|$ by (5). But this sum must be odd because $\sum_{i \in L_J} d_G(v_i) + |J||L_J| \equiv \sum_{i \in L_J} b_i + |J||L_J| \ (mod\ 2) \equiv 1 \ (mod\ 2)$, a contradiction. □

Proof of Theorem 1.4 Taking $J = \{1, 2, \ldots, t\}$, by Lemma 2.1, the necessity is obvious. For the sufficiency, let $A_n = (a_1, a_2, \ldots, a_n)$ and $B_n = (b_1, b_2, \ldots, b_n)$ be in good order and $a_i \equiv b_i$ (mod 2) for each $i \in I$. Without loss of generality, we may assume that $b_i \geq 1$ for $1 \leq i \leq n$. Call a graph G with vertices $v_1, v_2, \ldots, v_n$ a *subrealization* if $d_G(v_i) \leq b_i$ for all i, and a *realization* if $b_i \geq d_G(v_i) \geq a_i$ for all i. In a subrealization, the critical index r is the largest index such that $d(v_i) \geq a_i$ for $1 \leq i < r$. Initially, we start with n vertices and no edges, so that $r = 1$ unless $a_i = 0$ for all i, in which case the process is complete. While $r \leq n$, we obtain a new subrealization by iteratively removing the deficiency $a_r - d(v_r)$ at vertex v_r without changing the degrees of previous vertices.

Let $Z = \{v_{r+1}, v_{r+2}, \ldots, v_n\}$. We maintain the condition that Z is an independent set, which must hold initially. We write $v_i \leftrightarrow v_j$ for "v_i is adjacent to v_j" and $v_i \not\leftrightarrow v_j$ for "v_i is not adjacent to v_j".

Case 1. Suppose $v_i \not\leftrightarrow v_r$ for some i with $d(v_i) < b_i$, then add edge $v_i v_r$.

Case 2. Suppose $v_i \not\leftrightarrow v_r$ for some $i < r$. Since $d(v_i) \geq a_i \geq a_r > d(v_r)$, there must exist $u \in N(v_i) \setminus N(v_r)$. If $b_r - d(v_r) \geq 2$, replace uv_i with $\{uv_r, v_i v_r\}$. If $b_r - d(v_r) = 1$, then $a_r = b_r$. If $v_r \leftrightarrow v_k$ for some $v_k \in Z$, then remove $v_k v_r$ and apply the argument of the first part of the case. Otherwise, $v_r \not\leftrightarrow v_k$ for all $v_k \in Z$, Case 1 applies unless $d(v_k) = b_k$ for all $v_k \in Z$. In this case, we write $X = \{v_i | 1 \leq i \leq r-1 \text{ and } v_i \leftrightarrow v_r\}$, and $Y = \{v_1, v_2, \ldots, v_{r-1}\} \setminus X$. If $d(v_i) > a_i$ for some $i \in \{1, 2, \ldots, r-1\}$, then replace uv_i with uv_r. If $d(v_i) < b_i$ for some $v_i \in Y$, then add edge $v_i v_r$. We may assume that $d(v_i) = a_i$ for all $i \in \{1, 2, \ldots, r-1\}$ and $d(v_i) = b_i$ for all $v_i \in Y$. If $v_i \leftrightarrow v_k$ for some $v_i \in Y$ and $v_k \in Z$, then replace $v_i v_k$ with $v_i v_r$. So we may further assume that vertices in Y are only adjacent to vertices in $\{v_1, v_2, \ldots, v_{r-1}\}$.

Let $X_1 = \{v_i | v_i \in X, (\exists v_k \in Z)(v_i \leftrightarrow v_k)\}$, $X_2 = \{v_i | v_i \in X \setminus X_1, a_i = b_i\}$ and $X_3 = X \setminus (X_1 \cup X_2)$. Now we claim that there must be some vertex in $X_1 \cup X_3$ that is not adjacent to some vertex in $\{v_1, v_2, \ldots, v_r\}$. By contradiction, suppose that $v_i \leftrightarrow v_j$ for all $v_i \in X_1 \cup X_3$ and $1 \leq j \leq r$.

Let $T = \{i | v_i \in X_1 \cup X_3\}$ and $|T| = t$. Notice that $d(v_k) = b_k \leq t$ for all $v_k \in Z$, since the only vertices adjacent to v_k are in X_1. In addition, $b_i \geq a_i \geq a_r \geq t+1$ for all $i \in \{1, 2, \ldots, r-1\}$ because of $a_r - 1 = d(v_r) \geq t$. Thus, for all $v_i \in X_2 \cup Y$, $d(v_i) = b_i \geq t+1$. Therefore

$$\sum_{i \in T} a_i = t(t-1) + \sum_{i \notin T} min\{t, b_i\}. \tag{6}$$

We shall show that $T = \{1, 2, \ldots, t\}$. We can see that $d(v_i) \leq t$ for all $v_i \in Z$, $d(v_r) \leq t + |X_2|$, $d(v_i) \leq t + |X_2| + |Y| - 1$ for all $v_i \in Y$, $d(v_i) \leq t + |X_2| + |Y|$ for all $v_i \in X_2$, $d(v_i) = t + |X_2| + |Y|$ for all $v_i \in X_3$, and $d(v_i) \geq t + |X_2| + |Y| + 1$ for all $v_i \in X_1$. By $d(v_i) = a_i$ for $1 \leq i < r$ and $a_1 \geq a_2 \geq \cdots \geq a_n$, we have that $\{i | v_i \in X_1\} = \{1, 2, \ldots, |X_1|\}$. We now show that $p < q$ for $p \in \{i | v_i \in X_3\}$ and $q \in \{i | v_i \in X_2\}$. To the contrary, we assume that there exist $v_i \in X_2$ and $v_j \in X_3$ such that $i < j$. Then $a_j = d(v_j) = t + |X_2| + |Y| \geq d(v_i) = a_i$, and hence $a_j = a_i$. On the other hand, it follows from $v_i \in X_2$ and $v_j \in X_3$ that $a_i = b_i$ and $a_j < b_j$. Thus $b_i < b_j$, contradicting the *good* order. Therefore, $p < q$ for all $p \in \{i | v_i \in X_3\}$ and $q \in \{i | v_i \in X_2\}$, which implies that $\{i | v_i \in X_3\} = \{|X_1| + 1, \ldots, t\}$ and $T = \{1, 2, \ldots, t\}$.

In order to complete the proof of the claim, we define

$$f(J) = t(t-1) + \sum_{i \notin J} min\{t, b_i\} - \sum_{i \in J} a_i - \zeta(J).$$

By (1), we can see that $f(T) \geq 0$. Now we consider the graph H induced by $Y \cup X_2 \cup \{v_r\}$. The sum of the degrees of vertices in H is equals to $\sum_{v_i \in V(H)} d(v_i) - t|H| = \sum_{v_i \in V(H)} a_i - t|H| - 1$. Since this sum must be even, $\sum_{v_i \in V(H)} a_i - t|H|$ is odd. Furthermore, L_T equals the set of indices in H, $a_i = b_i$ for all $v_i \in V(H)$ and $\sum_{v_i \in V(H)} b_i + t|H| = (\sum_{v_i \in V(H)} a_i - t|H|) + 2t|H|$ is odd, so $\zeta(T) = 1$, contradicting (6) and $f(T) \geq 0$. Therefore, there exists some vertex in $X_1 \cup X_3$ that not adjacent to some vertex in $\{v_1, v_2, \ldots, v_r\}$. In other words, the claim is proved. We now can decrease the deficiency at v_r in terms of the following cases.

subcase (2.1). Suppose $v_i \not\leftrightarrow v_j$ for some $v_i \in X_1$ and $v_j \in X$, then $v_i \leftrightarrow v_k$ for some $v_k \in Z$. Replace $\{v_i v_k, v_j v_r\}$ with $v_i v_j$. This increases the deficiency at v_r by 1, and the first part of Case 2 applies.

subcase (2.2). Suppose $v_i \not\leftrightarrow v_j$ for some $v_i \in X_1$ and $v_j \in Y$, then $v_i \leftrightarrow v_k$ for some $v_k \in Z$. Note that $|X| = d(v_r) = a_r - 1 < a_j$. So $v_j \leftrightarrow v_\ell$ for some $v_\ell \in Y$. Replace $\{v_i v_k, v_j v_\ell\}$ with $\{v_i v_j, v_\ell v_r\}$.

subcase (2.3). Suppose $v_i \not\leftrightarrow v_j$ for some $v_i \in X_3$ and $v_j \in X_2 \cup X_3$, then replace $v_j v_r$ with $v_i v_j$. This increase the deficiency at v_r by 1, and the first part of Case 2 applies.

subcase (2.4). Suppose $v_i \not\leftrightarrow v_j$ for some $v_i \in X_3$ and $v_j \in Y$, then $v_j \leftrightarrow v_\ell$ for some $v_\ell \in Y$. Replace v_jv_ℓ with $\{v_iv_j, v_\ell v_r\}$.

In the last two cases, we increase $d(v_i)$ by 1, this can be done as $d(v_i) = a_i < b_i$, by the definition of X_3.

Case 3. Suppose $v_1, v_2, \ldots, v_{r-1} \in N(v_r)$ and $d(v_k) \neq min\{r, b_k\}$ for some k with $k > r$. Since $d(v_k) \leq b_k$ and Z is an independent set, we have that $d(v_k) \leq r$, and so $d(v_k) < min\{r, b_k\}$, Case 1 applies unless $v_r \leftrightarrow v_k$. Since $d(v_k) < r$, $v_k \not\leftrightarrow v_i$ for some $i \in \{1, 2, \ldots, r-1\}$. There is $u \in N(v_i) \setminus N(v_r)$. Replace uv_i with $\{uv_r, v_iv_k\}$.

Case 4. Suppose $v_1, v_2, \ldots, v_{r-1} \in N(v_r)$ and $v_i \not\leftrightarrow v_j$ for some $i < j < r$. Then there exists vertices u and w (possibly $u = w$) such that $u \in N(v_i) \setminus N(v_r)$ and $w \in N(v_j) \setminus N(v_r)$. Since $v_1, v_2, \ldots, v_{r-1} \in N(v_r)$, we have that $u, w \in Z$. Replace $\{uv_i, wv_j\}$ with $\{uv_r, v_iv_j\}$.

Case 5. Suppose $d(v_i) > a_i$ for some $i \in \{1, 2, \ldots, r-1\}$, and $v_1, v_2, \ldots, v_{r-1} \in N(v_r)$. Then there exists $u \in N(v_i) \setminus N(v_r)$. Replace uv_i with uv_r.

If none of these cases apply, then $v_1, v_2, \ldots, v_r$ are pairwise adjacent and $d(v_k) = min\{r, b_k\}$ for all $k > r$. Since Z is independent, we have that $\sum_{i=1}^{r} d(v_i) = r(r-1) + \sum_{i=r+1}^{n} min\{r, b_i\}$. By (1) and $d(v_i) = a_i$ for $i < r$, we have that

$$\sum_{i=1}^{r} a_i \leq r(r-1) + \sum_{i=r+1}^{n} min\{r, b_i\} = \sum_{i=1}^{r} d(v_i) = \sum_{i=1}^{r-1} a_i + d(v_r).$$

Thus, $d(v_r) = a_r$. Increase r by 1 and continue.

Finally, we obtain a subrealization G' with $d_{G'}(v_i) = a_i$ for $1 \leq i \leq n$. By $a_i \equiv b_i \pmod 2$ for all $i \in I$, we have that $d_{G'}(v_i) \equiv b_i \pmod 2$ for all $i \in I$. This means that G' is the required realization of $(A_n; B_n)$. In other words, $(A_n; B_n)$ is partial parity graphic with respect to I. □

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