# CENTRAL SETS IN THE ANNIHILATING-IDEAL GRAPH OF COMMUTATIVE RINGS

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**Abstract**

Let $R$ be a commutative ring with identity and $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. The *annihilating-ideal graph* of $R$ is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices $I_1$ and $I_2$ are adjacent if and only if $I_1I_2 = (0)$. In this paper, we study some connections between the graph-theoretic properties of $\mathbb{AG}(R)$ and algebraic properties of commutative ring $R$.

**Keywords:** zero-divisor graph, annihilating-ideal graph, semiprimitive ring, domination parameters.

## 1 Introduction

The study of algebraic structures, using the properties of graphs, became an exciting research topic in the past twenty years, leading to many fascinating results and questions. In the literature, there are many papers assigning graphs to rings, groups and semigroups. Let $R$ be a commutative ring with identity. In [2], D. F. Anderson and P. S. Livingston associate a graph called *zero-divisor graph*, $\Gamma(R)$ to $R$ with vertices $Z(R)^*$, the set of non-zero zero-divisors of $R$, and for two distinct $x, y \in Z(R)^*$, the vertices $x$ and $y$ are adjacent if and only if $xy = 0$ in $R$. In ring theory, the structure of a ring $R$ is closely tied to ideal's behavior more than elements, and so it is deserving to define a graph with vertex set as ideals instead of elements. Recently M. Behboodi and Z. Rakeei[4, 5] have introduced and investigated the annihilating-ideal graph of a commutative ring. For a non-domain commutative ring $R$, let $J(R)$ be the Jacobson radical of $R$, $\langle x \rangle$ be the ideal of $R$ generated by $x$ and $\mathbb{A}^*(R)$ be the set of non-zero ideals with non-zero annihilators. We call an ideal $I_1$ of $R$, an *annihilating-ideal* if there exists a non-zero ideal $I_2$ of $R$ such that $I_1I_2 = (0)$. The *annihilating-ideal graph* of $R$ is defined as the graph $\mathbb{AG}(R)$ with the vertex set $\mathbb{A}^*(R)$ and two distinct vertices $I_1$ and $I_2$ are adjacent if and only if $I_1I_2 = (0)$. We investigate the interplay between the graph-theoretic properties of $\mathbb{AG}(R)$ and the ring-theoretic properties of $R$. An ideal $I$ of $R$ is called *nil-ideal* if there exists a positive integer $n$ such that $I^n = 0$ and $I^{n-1} \neq (0)$. An *Artinian* ring is a ring that satisfies the descending chain condition on ideals. An Artinian ring has only finite number of maximal ideals and every ideal in an Artinian ring is a *nil-ideal*. A ring $R$ is called *semiprimitive* if its Jacobson radical $J(R)$ is the zero ideal. For basic definitions on rings, one may refer [3, 8].

Let $G = (V, E)$ be a simple graph. The *distance* between two vertices $x$ and $y$, denoted $d(x, y)$, is the length of the shortest path from $x$ to $y$. The *diameter* of a connected graph $G$ is the maximum distance between two distinct vertices of $G$. For any vertex $x$ of a connected graph $G$, the *eccentricity* of $x$, denoted $e(x)$, is the maximum of the distances from $x$ to the other vertices of $G$. The set of vertices with minimum eccentricity is called the *center* of the graph $G$, and this minimum eccentricity value is the *radius* of $G$. The *clique number* $\omega(G)$ is the number of vertices in a largest complete subgraph of $G$. The *corona* of two graphs $G_1$ and $G_2$ is the graph $G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i^{th}$ vertex of $G_1$ is adjacent to every vertex in the $i^{th}$ copy of $G_2$. For $v \in V(G)$, the neighborhood(degree) of $v$, denoted by $N(v)(deg_G(v))$, is the set (number) of vertices other than $v$ which are adjacent to $v$ and $N[v] = N(v) \cup \{v\}$. We denote by $\delta(G)$ and $\Delta(G)$ the minimum and maximum degrees of the vertices in $G$ respectively. For basic definitions on graphs, one may refer[6].

A set $D \subseteq V$ of vertices in $G = (V, E)$ is called a *dominating set* if for every vertex $v \in V - D$, there exists a vertex $v \in D$ such that $v$ is adjacent to $u$. A dominating set $D$ is said to be minimal if no proper subset of $D$ is a dominating set. The *domination number* of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. A dominating set $D$ in $G$ with cardinality $\gamma$ is called $\gamma$-set of $G$. A dominating set $D$ is a *connected dominating set* if the subgraph $\langle D \rangle$ induced by $D$ is a connected subgraph of $G$. The *connected domination number* of $G$, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set of $G$. A dominating set $D$ is a *total dominating set* if for every vertex $v \in V$, there exists a vertex $u \in D$ such that $v \neq u$ and $v$ is adjacent to $u$. The *total domination number* of $G$, denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of $G$. A dominating set $D$ is a *clique dominating set* if the subgraph $\langle D \rangle$ induced by $D$ is complete in $G$. The *clique domination number* $\gamma_{cl}(G)$ of $G$ equals the minimum cardinality of a clique dominating set of $G$. A dominating set $D$ is a *paired-dominating set* if the subgraph induced $\langle D\rangle$ by $D$ has a perfect matching. The *paired-domination number* $\gamma_{pr}(G)$ of $G$ equals the minimum cardinality of a paired-dominating set of $G$. A set $S$ of vertices is said to be *irredundant set* of $G$ if for every vertex $v \in S$, $pn[v,S] = N[v] - N[S-v] \neq \emptyset$. An irredundant set $S$ is a *maximal irredundant* set if for every vertex $u \in V-S$, the set $S\cup\{u\}$ is not irredundant. The *irredundance number* $ir(G)$ is the minimum cardinality of maximal irredundant sets. There are so many domination parameters in the literature and for more details one can refer[7]. In this paper, apart from domination number, we are also concerned with *connected domination number* $\gamma_c(G)$, *total domination number* $\gamma_t(G)$, *clique domination number* $\gamma_{cl}(G)$, *paired-domination number* $\gamma_{pr}(G)$ and *irredundance number* $ir(G)$ of the annihilating-ideal graph of certain non-domain commutative rings.

The purpose of this article is to study the basic graph-theoretical properties of $\mathbb{AG}(R)$. In section 2, we characterize the degree of a maximal ideal in the annihilating-ideal graph of a commutative semiprimitive Artinian ring with identity. Also, we find the maximum degree of the annihilating-ideal graph of a commutative Artinian ring. In sequel, we prove that if $R_1 = \mathbb{Z}_2^n$ and $R_2 = \oplus_{i=1}^n F_k$ where $F_i$ are fields and $n \geq 2$, then $\mathbb{AG}(R_2) \cong \Gamma(R_1)$. In section 3, a dominating set of $\mathbb{AG}(R)$ is constructed using elements of the center when R is a commutative Artinian ring. Also we prove that the domination number of $\mathbb{AG}(R)$ is equal to the number of factors in the Artinian decomposition of $R$ and we also find several domination parameters of $\mathbb{AG}(R)$. In section 4, we characterize all commutative Artinian rings(non local rings) with identity for which $\mathbb{AG}(R)$ is planar. In this attempt, we obtain the clique number of $\mathbb{AG}(R)$ when $R$ is a commutative semiprimitive Artinian ring. The following results are useful for further reference in this paper.

**Theorem 1.1.** [3, Theorem 8.7] *An Artinian ring is uniquely (up to isomorphism) a finite direct product of Artinian local rings.*

**Proposition 1.2.** [7, Proposition 3.9] *Every minimal dominating set in a graph $G$ is a maximal irredundant set of $G$.*

**Theorem 1.3.** [8, Theorem 12.1.22] *Let $R$ be a commutative ring with identity. If $R$ is semiprimitive Artinian, then $R$ is a direct sum of finite number of fields.*

**Corollary 1.4.** [4, Corollary 2.4] *Let $R$ be an Artinian ring such that $R$ is not a field. Then there is a vertex of $\mathbb{AG}(R)$ which is adjacent to every other vertex if and only if either $R = F_1 \oplus F_2$ where $F_1, F_2$ are fields, or $R$ is a local ring.*

Hereafter by $R$ we mean $R$ is a commutative Artinian ring. By Theorem 1.1, $R = \oplus_{i=1}^n R_i$ where $R_i$'s are Artinian local rings. Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. Note that $Max(R) = \{M_1, \ldots, M_n : M_i = R_1 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in $R$ and $n \geq 2$. Also throughout this paper, we take $S$ as a commutative semiprimitive Artinian ring. By Theorem 1.3, $S$ is a finite direct sum of fields. Let us take $S = \oplus_{i=1}^{n} F_i$ where $F_i$'s are fields. Due to this, the number of proper non-zero ideals in $S$ is $2^n - 2$ and so $|\mathbb{A}^*(S)| = 2^n - 2$. Also note that $Max(S) = \{M'_1, \ldots, M'_n : M'_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in $S$ and every ideal of $S$ is of the form $I_1 \oplus I_2 \oplus \cdots \oplus I_n$ where $I_i$ is an ideal of $F_i$, $n \geq 2$.

# 2 Properties of annihilating-ideal graph

In this section, we study about annihilating-ideal graph and use the same further.

**Theorem 2.1.** *Let $S$ be a commutative semiprimitive Artinian ring and let $M$ be an ideal in $S$. Then $M$ is maximal if and only if $deg_{\mathbb{A}\mathrm{G}(S)}(M) = 1$.*

*Proof.* As mentioned above, $Max(S) = \{M'_1, \ldots, M'_n : M'_i = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in $S$. Note that $M'_i M'_j \neq (0)$ for all $i \neq j$. Suppose $M$ is maximal in $S$. Then $M = M'_i$ for some $i$. Clearly $MI = (0)$, where $I = (0) \oplus \cdots \oplus (0) \oplus F_i \oplus (0) \oplus \cdots \oplus (0)$ and so $deg_{\mathbb{A}\mathrm{G}(S)}(M) \geq 1$.

Suppose $deg_{\mathbb{A}\mathrm{G}(S)}(M) > 1$. Then there exists a non-zero ideal $I'$ of $S$ such that $MI' = (0)$, $I' \neq (0) \oplus \cdots \oplus (0) \oplus F_i \oplus (0) \oplus \cdots \oplus (0)$ and $I' \neq M'_j$ for all $j$. Note that every ideal of $S$ is of the form $I_1 \oplus I_2 \oplus \cdots \oplus I_n$ where $I_i$ is an ideal of $F_i$. Therefore $I' = \oplus_{i=1}^{n} I_i$ where $I_i \in \{(0), F_i\}$. Since $I' \neq (0)$ and $I' \neq (0) \oplus \cdots \oplus (0) \oplus F_i \oplus (0) \oplus \cdots \oplus (0)$, $I_j = F_j$ for some $j \neq i$. From this $(0) \oplus \cdots \oplus (0) \oplus F_j \oplus (0) \oplus \cdots \oplus (0) \subset MI'$ and so $MI' \neq (0)$, a contradiction. Hence $deg_{\mathbb{A}\mathrm{G}(S)}(M) = 1$.

Conversely, let $deg_{\mathbb{A}\mathrm{G}(S)}(M) = 1$. Suppose $M$ is not maximal. Then there exists a maximal ideal $M'$ in $S$ such that $M \subset M'$. By the previous part, $deg_{\mathbb{A}\mathrm{G}(S)}(M') = 1$ and let $M' = M'_j$ for some $j$. Suppose $M = \oplus_{i=1}^{n} I_i$. Then $I_j = (0)$. Since $M' \cdot ((0) \oplus \cdots \oplus (0) \oplus F_j \oplus (0) \oplus \cdots \oplus (0)) = (0)$, $M((0) \oplus \cdots (0) \oplus F_j \oplus (0) \cdots \oplus (0)) = (0)$. Since $M$ is not maximal, there exists $k \neq j$ such that $I_k = (0)$. From this $M \cdot ((0) \oplus \cdots \oplus (0) \oplus F_k \oplus (0) \oplus \cdots \oplus (0)) = (0)$ and so $deg_{\mathbb{A}\mathrm{G}(S)}(M) > 1$, a contradiction. □

In view of Theorem 2.1, we have following corollary.

**Corollary 2.2.** *Let $S$ be a commutative semiprimitive Artinian ring. Then $\mathbb{A}\mathrm{G}(S)$ is neither Eulerian nor Hamiltonian.*

**Lemma 2.3.** *Let $R$ be a commutative Artinian ring with identity and $R = \oplus_{i=1}^{n} R_i$ where $R_i$'s are Artinian local rings and $n \geq 2$. Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. Then $deg_{\mathbb{AG}(R)}(M_i) = deg_{\mathbb{AG}(R_i)}(J_i)$ where $M_i$ is a maximal ideal in $R$ for $1 \leq i \leq n$.*

*Proof.* Note that $M_i = R_1 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n$ for $1 \leq i \leq n$. Clearly $N_{\mathbb{AG}(R)}(M_i) = \{(0) \oplus \cdots \oplus (0) \oplus I_i \oplus (0) \oplus \cdots \oplus (0) : I_i J_i = (0), I_i$ is a non-zero ideal in $R_i\}$ and hence $deg_{\mathbb{AG}(R)}(M_i) = deg_{\mathbb{AG}(R_i)}(J_i)$ for $1 \leq i \leq n$. □

**Theorem 2.4.** *Let $R$ be a commutative Artinian ring with identity and $R = \oplus_{i=1}^{n} R_i$ where $R_i$'s are Artinian local rings and $n \geq 2$. Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. If $D_i = (0) \oplus \cdots \oplus (0) \oplus J_i^{n_i-1} \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$, then $\Delta(\mathbb{AG}(R)) = deg_{\mathbb{AG}(R)}(D_i)$ for some $i$, $1 \leq i \leq n$.*

*Proof.* As mentioned above, $Max(R) = \{M_1, \ldots, M_n : M_i = R_1 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n, 1 \leq i \leq n\}$ is the set of all maximal ideals in $R$. By the assumption $J_i^{n_i} = (0)$ and $J_i^{n_i-1} \neq (0)$ for all $i$. Then $N_{\mathbb{AG}(R)}[D_i] = \{I = \oplus_{i=1}^{n} I_i : I \neq (0), I_i$ is an ideal in $R_i$ and $I_i \neq R_i\}$. Thus $deg_{\mathbb{AG}(R)}(D_i) = |N_{\mathbb{AG}(R)}[D_i]| - 1$. Let $m_i$ be the number of ideals in $R_i$ for $1 \leq i \leq n$. Rearrange the indices such that $m_1 \leq m_2 \leq \ldots \leq m_n$. Note that $J_i^{n_i-1}.R_i \neq (0)$ for every $i$. This implies that $deg_{\mathbb{AG}(R)}(D_i) = m_1 m_2 \ldots m_{i-1}(m_i - 1)m_{i+1} \ldots m_n - 1$ and so $deg_{\mathbb{AG}(R)}(D_i) \leq deg_{\mathbb{AG}(R)}(D_n)$ for all $1 \leq i \leq n-1$.

Let $I$ be any non-zero ideal of $\mathbb{AG}(R)$ and $I \neq D_i$ for $1 \leq i \leq n$. Then $I = \oplus_{i=1}^{n} I_i$ for some ideals $I_i$ in $R_i$ and $I_i \neq (0)$ for some $i$.

**Case 1.** If $I$ is maximal, then $I = M_i$ for some $i$ and so $deg_{\mathbb{AG}(R)}(I) < m_i - 1 < deg_{\mathbb{AG}(R)}(D_n)$.

**Case 2.** If $I = I_1 \oplus I_2 \oplus \cdots \oplus I_n$, then $I_i \subseteq J_i$ for all $i$. Then $deg_{\mathbb{AG}(R)}(I) < (m_1 - 1)(m_2 - 1) \cdots (m_n - 1) < deg_{\mathbb{AG}(R)}(D_n)$.

**Case 3.** If $I_i = R_i$ for some $i$, then $deg_{\mathbb{AG}(R)}(I) < deg_{\mathbb{AG}(R)}(D_n)$.

Thus, for any non-zero ideal $I$ in $R$, $I \neq D_i$ for $1 \leq i \leq n$, $deg_{\mathbb{AG}(R)}(I) < deg_{\mathbb{AG}(R)}(D_n)$. Hence $\Delta(\mathbb{AG}(R)) = deg_{\mathbb{AG}(R)}(D_n) = m_1 m_2 \cdots m_{n-1}(m_n - 1) - 1$. □

**Corollary 2.5.** *Let $R$ be a commutative Artinian ring with identity and $R = \oplus_{i=1}^{n} R_i$ where $R_i$'s are Artinian local rings and $n \geq 2$. Then $\mathbb{AG}(R)$ is not Eulerian.*

*Proof.* Let $m_i$ be the number of ideals in $R_i$ for $1 \leq i \leq n$. Rearrange the indices such that $m_1 \leq m_2 \leq \cdots \leq m_n$. By Theorem 2.4, $deg_{\mathbb{AG}(R)}(D_i) = m_1 m_2 \cdots m_{i-1}(m_i - 1)m_{i+1} \cdots m_n - 1$ for $1 \leq i \leq n$. Note that at least one of the products $m_1 m_2 \cdots m_{i-1}(m_i - 1)m_{i+1} \cdots m_n - 1$ is odd no matter whether $m_i'$s are odd or even and so $deg_{\mathbb{AG}(R)}(D_i)$ is odd for exactly one $i$

and $deg_{\mathbb{AG}(R)}(D_j)$ is even in all other cases. Hence $\mathbb{AG}(R)$ is not Eulerian. □

One can prove the following in analogous to the above.

**Theorem 2.6.** *Let $S$ be a commutative semiprimitive Artinian ring and $S = \oplus_{i=1}^{n} F_i$ where $F_i$'s are fields and $n \geq 2$. If $D_i = (0) \oplus (0) \oplus \cdots \oplus F_i \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$, then $\Delta(\mathbb{AG}(S)) = deg_{\mathbb{AG}(S)}(D_i) = 2^{n-1} - 1$ for all $i$, $1 \leq i \leq n$.*

Now we prove that the annihilating-ideal graph of one particular ring is isomorphic to the zero-divisor graph of another ring.

**Theorem 2.7.** *Let $R_1 = \mathbb{Z}_2^n$, $R_2 = \oplus_{k=1}^{n} F_k$ where $F_i$ are fields and $n \geq 2$. Let $\Gamma(R_1)$ be the zero-divisor graph of $R_1$. Then $\mathbb{AG}(R_2) \cong \Gamma(R_1)$.*

*Proof.* Note that $V(\mathbb{AG}(R_2)) = \{I = \oplus_{i=1}^{n} I_i : I_i \in \{(0), F_i\}, 1 \leq i \leq n\} \smallsetminus \{(0), R_2\}$, $V(\Gamma(R_1)) = \{a = (a_1, a_2, \ldots, a_n) : a_i \in \{0,1\}, 1 \leq i \leq n\} \smallsetminus \{(0,0,\ldots,0), (1,1\ldots,1)\}$ and $|V(\mathbb{AG}(R_2))| = |V(\Gamma(R_1))| = 2^n - 2$.

Define $f : V(\mathbb{AG}(R_2)) \longrightarrow V(\Gamma(R_1))$ by $f(\prod_{i=1}^{n} I_i) = (a_1, a_2, \ldots, a_n)$ where

$$a_i = \begin{cases} 1 & \text{if } I_i = F_i \\ 0 & \text{if } I_i = (0) \end{cases}$$

Clearly $f$ is well-defined and bijective. Let $I = \oplus_{i=1}^{n} I_i$ and $I' = \oplus_{i=1}^{n} I_i'$ be two non-zero ideals in $R_2$. Suppose $I$ and $I'$ are adjacent in $\mathbb{AG}(R_2)$. Then $II' = (0)$ and so $I_i I_i' = (0)$ for all $i$. Hence $I_i = (0)$ or $I_i' = (0)$ for all $i$. Suppose $f(I) = (b_1, b_2, \ldots, b_n)$ and $f(I') = (c_1, c_2, \ldots, c_n)$. Then either $b_i = 0$ or $c_i = 0$ and so $b_i c_i = 0$ for all $i$. i.e, $f(I)f(I') = 0$ and so $f(I)$ and $f(I')$ are adjacent in $\Gamma(R_1)$. Similarly one can prove that $f$ preserves non-adjacency also. Hence $\mathbb{AG}(R_2) \cong \Gamma(R_1)$. □

**Theorem 2.8.** *Let $R_1 \cong \mathbb{Z}_{p_1^{a_1}} \times \mathbb{Z}_{p_2^{a_2}} \times \cdots \times \mathbb{Z}_{p_n^{a_n}}$ and $R_2 \cong \mathbb{Z}_{q_1^{a_1}} \times \mathbb{Z}_{q_2^{a_2}} \times \cdots \times \mathbb{Z}_{q_m^{a_m}}$, where $p_i$ and $q_i$ are distinct prime integers and $a_i \geq 1$ is an integer for $1 \leq i \leq n$. If $n = m$, then $\mathbb{AG}(R_1) \cong \mathbb{AG}(R_2)$.*

*Proof.* If $R_1 \cong R_2$, then the result is obvious. Suppose $R_1 \ncong R_2$. Note that $V(\mathbb{AG}(R_1)) = \{I_1 \times \cdots \times I_n : I_i \in \{\langle 0\rangle, \langle p_i\rangle, \langle p_i^2\rangle, \ldots, \langle p_i^{a_i-1}\rangle, \mathbb{Z}_{p_i^{a_i}}\}, 1 \leq i \leq n\} \smallsetminus \{(0), R_1\}$ and $V(\mathbb{AG}(R_2)) = \{H_1 \times \cdots \times H_n : H_i \in \{\langle 0\rangle, \langle q_i\rangle, \langle q_i^2\rangle, \ldots, \langle q_i^{a_i-1}\rangle, \mathbb{Z}_{q_i^{a_i}}\}, 1 \leq i \leq n\} \smallsetminus \{(0), R_2\}$.

Define $f : V(\mathbb{AG}(R_1)) \longrightarrow V(\mathbb{AG}(R_2))$ by $f(\prod_{i=1}^{n} I_i) = \prod_{i=1}^{n} I_i'$ where

$$I_i' = \begin{cases} \langle q_i^{\alpha} \rangle & \text{if } I_i = \langle p_i^{\alpha} \rangle \\ (0) & \text{if } I_i = (0) \\ \mathbb{Z}_{q_i^{\alpha_i}} & \text{if } I_i = \mathbb{Z}_{p_i^{\alpha_i}} \end{cases}$$

Clearly $f$ is well-defined and bijective. Let $A, B \in V(\mathbb{AG}(R_1))$, $A = \prod_{i=1}^{n} A_i$ and $B = \prod_{i=1}^{n} B_i$. Suppose $A$ and $B$ are adjacent in $\mathbb{AG}(R_1)$. Then $A_iB_i = (0)$ for all $i$. Let $f(A) = \prod_{i=1}^{n} A_i'$ and $f(B) = \prod_{i=1}^{n} B_i'$. Since $A_iB_i = (0)$, $A_i'B_i' = (0)$ for all $i$ and so $f(A)f(B) = (0)$. i.e., $f(A)$ and $f(B)$ are adjacent in $\mathbb{AG}(R_2)$. Similarly one can prove the non-adjacency also. Hence $\mathbb{AG}(R_1) \cong \mathbb{AG}(R_2)$. □

Now we have the following corollary.

**Corollary 2.9.** *Let $R_1 \cong \oplus_{i=1}^{n} F_i$ and $R_2 \cong \oplus_{j=1}^{m} K_j$ where $F_i$ and $K_j$ are fields for $1 \le i \le n$ and $1 \le j \le m$. If $R_1 \ncong R_2$, then $n = m$ if and only if $\mathbb{AG}(R_1) \cong \mathbb{AG}(R_2)$.*

# 3 Center sets of annihilating-ideal graph

In this section, we find certain central sets in the annihilating-ideal graph and use the same to obtain the value certain domination parameters of the annihilating-ideal graph.

**Theorem 3.1.** *Let $R$ be a commutative Artinian local ring with identity. Assume that $M$ is the unique maximal ideal $R$. Then the radius of $\mathbb{AG}(R)$ is 0 or 1 and the center of $\mathbb{AG}(R)$ is $\{I \subseteq ann(M) : I \neq (0) \text{is an ideal in } R\}$.*

*Proof.* If $M$ is the only non-zero proper ideal of $R$, then $\mathbb{AG}(R) \cong K_1$, $e(M) = 0$ and the radius of $\mathbb{AG}(R)$ is 0. Assume that the number of non-zero proper ideals of $R$ is greater than 1. Since $R$ is Artinian, there exists $m \in \mathbb{N}$, $m > 1$ such that $M^m = (0)$ and $M^{m-1} \neq (0)$. For any non-zero ideal $I$ of $R$, $IM^{m-1} \subseteq MM^{m-1} = (0)$ and so $d(M^{m-1}, I) = 1$. Hence $e(M^{m-1}) = 1$ and so the radius of $\mathbb{AG}(R)$ is 1.

Suppose $I$ and $K$ are arbitrary non-zero ideals of $R$ and $I \subseteq ann(M)$. Then $IK \subseteq IM = (0)$ and hence $e(I) = 1$. Suppose $(0) \neq I' \nsubseteq Ann(M)$. Then $I'M \neq (0)$ and so $e(I') > 1$. Hence the center of $\mathbb{AG}(R)$ is

$$\{I \subseteq ann(M) : I \text{ is a non-zero ideal in } R\}$$

□

In view of Theorem 3.1, we have the following corollary.

**Corollary 3.2.** *Let $R$ be a commutative Artinian local ring with identity and $M$ is the unique maximal ideal of $R$. Then the following hold good.*
*(i)* $\gamma(\mathbb{AG}(R)) = 1$
*(ii) $D$ is a $\gamma$-set of $\mathbb{AG}(R)$ if and only if $D \subseteq ann(M)$.*

*Proof.* (i) Trivial from Theorem 3.1.

(ii) Let $D = \{I\}$ be a $\gamma$-set of $\mathbb{AG}(R)$. Suppose $I \nsubseteq ann(M)$. Then $IM \neq (0)$ and so $M$ is not dominated by $I$, a contradiction. Conversely, suppose $D \subseteq ann(M)$. Let $I$ be an arbitrary vertex in $\mathbb{AG}(R)$. Then $IJ \subseteq MJ = (0)$ for every $J \in D$. i.e., Every vertex $I$ is adjacent to every $J \in D$. If $|D| > 1$, then $D - \{K\}$ is also a dominating set of $\mathbb{AG}(R)$ for some $K \in D$ and so $D$ is not minimal. Thus $|D| = 1$ and so $D$ is a $\gamma$-set by (i). □

**Theorem 3.3.** *Let $R$ be a commutative Artinian ring with identity. Then the radius of $\mathbb{AG}(R)$ is 2 and the center of $\mathbb{AG}(R)$ is $\{I : I$ is an ideal of $R$, $I \neq (0), I \subseteq J(R)\}$.*

*Proof.* Since $R$ is a commutative Artinian ring with identity, $R = \oplus_{i=1}^{n} R_i$ where $R_i$'s are Artinian local rings and $n \geq 2$. Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. Note that $Max(R) = \{M_1, \ldots, M_n : M_i = R_1 \oplus \cdots \oplus R_{i-1} \oplus J_i \oplus R_{i+1} \oplus \cdots \oplus R_n,\ 1 \leq i \leq n\}$ is the set of all maximal ideals in $R$. Consider $D_i = (0) \oplus \cdots \oplus (0) \oplus J_i^{n_i-1} \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$. Note that $J(R) = J_1 \oplus \cdots \oplus J_n$ is the Jacobson radical of $R$ and any non-zero ideal in $R$ is adjacent to $D_i$ for some $i$.
Let $I$ be any non-zero ideal of $R$. Then $I = \oplus_{i=1}^{n} I_i$ where $I_i$ is an ideal in $R_i$.
**Case 1.** If $I = M_i$ for some $i$, then $ID_j \neq (0)$ and $IM_j \neq (0)$ for all $j \neq i$. Note that $N(I) = \{(0) \oplus \cdots \oplus (0) \oplus K_i \oplus (0) \oplus \cdots \oplus (0) : J_i K_i = (0)$, $K_i$ is a non-zero ideal in $R_i\}$. Clearly $N(I) \cap N(M_j) = (0)$, $d(I, M_j) \neq 2$ and so $I - D_i - D_j - M_j$ is a path in $\mathbb{AG}(R)$. Therefore $e(I) = 3$ and so $e(M) = 3$ for all $M \in Max(R)$.
**Case 2.** If $I \neq D_i$ and $I_i \subseteq J_i$ for all $i$. Then $ID_i = (0)$ for all $i$. Let $K$ be any non-zero ideal of $R$ with $IK \neq (0)$. Then $KD_j = (0)$ for some $j$, $I - D_j - K$ is a path in $\mathbb{AG}(R)$ and so $e(I) = 2$.
**Case 3.** If $I_i = R_i$ for some $i$, then $ID_i \neq (0)$, $IM_i \neq (0)$ and $ID_j = (0)$ for some $j \neq i$. Thus $I - D_j - D_i - M_i$ is a path in $\mathbb{AG}(R)$, $d(I, M_i) = 3$ and so $e(I) = 3$. Thus $e(I) = 2$ for all $I \subseteq J(R)$.

Further note that in all the cases center of $\mathbb{AG}(R)$ is $\{I :\ I$ is a non-zero ideal of $R$ and $I \subseteq J(R)\}$. □

In view of Theorems 3.1 and 3.3, we have following corollary.

**Corollary 3.4.** *Let $S$ be a commutative semiprimitive Artinian ring and $S = \oplus_{i=1}^{n} F_i$ where $F_i$'s are fields and $n \geq 2$. Then the radius of $\mathbb{AG}(S)$ is 1 or 2 and the center of $\mathbb{AG}(S)$ is $\cup_{i=1}^{n} D_i$, where $D_i = (0) \oplus \cdots \oplus (0) \oplus F_i \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$.*

**Theorem 3.5.** *Let $R$ be a commutative Artinian ring with identity and $R = \oplus_{i=1}^{n} R_i$ where $R_i$ are Artinian local rings and $n \geq 2$. Then $\gamma(\mathbb{AG}(R)) = n$.*

*Proof.* Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. Let $\Omega = \{D_1, D_2, \ldots, D_n\}$, where $D_i = (0) \oplus \cdots \oplus (0) \oplus J_i^{n_i - 1} \oplus (0) \oplus \cdots \oplus (0)$ for $1 \leq i \leq n$. Note that any non-zero ideal in $R$ is adjacent to $D_i$ for some $i$. Therefore $N[\Omega] = V(\mathbb{AG}(R))$, $\Omega$ is a dominating set of $\mathbb{AG}(R)$ and so $\gamma(\mathbb{AG}(R)) \leq n$. Suppose $S$ is a dominating set of $\mathbb{AG}(R)$ with $|S| < n$. Then there exists $M \in Max(R)$ such that $MI \neq (0)$ for all $I \in S$, a contradiction. Hence $\gamma(\mathbb{AG}(R)) = n$. □

In view of Theorem 3.5, we have following corollary.

**Corollary 3.6.** *Let $R$ be a commutative Artinian ring with identity and $|Max(R)| = n \geq 2$. Then*
*(a)* $ir(\mathbb{AG}(R)) = n$
*(b)* $\gamma_c(\mathbb{AG}(R)) = n$
*(c)* $\gamma_t(\mathbb{AG}(R)) = n$
*(d)* $\gamma_{cl}(\mathbb{AG}(R)) = n$
*(e)* $\gamma_{pr}(\mathbb{AG}(R)) = \begin{cases} n & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd.} \end{cases}$

*Proof.* Consider the $\gamma$-set $\Omega$ of $\mathbb{AG}(R)$ identified in the proof of Theorem 3.5. By Proposition 1.2, $\Omega$ is a maximal irredundant set with minimum cardinality and so $ir(\mathbb{AG}(R)) = n$. Clearly $\langle \Omega \rangle$ is a complete subgraph of $\mathbb{AG}(R)$. Hence $\gamma_c(\mathbb{AG}(R)) = \gamma_t(\mathbb{AG}(R)) = \gamma_{cl}(\mathbb{AG}(R)) = n$.

If $n$ is even, then $\langle \Omega \rangle$ has a perfect matching and so $\Omega$ is a paired dominating set of $\mathbb{AG}(R)$. Thus $\gamma_{pr}(\mathbb{AG}(R)) = n$. If $n$ is odd, then $\langle \Omega \cup \{I\} \rangle$ has a perfect matching for some $I \in V(\mathbb{AG}(R)) - \Omega$ and so $\Omega \cup \{I\}$ is a paired dominating set of $\mathbb{AG}(R)$. Thus $\gamma_{pr}(\mathbb{AG}(R)) = n$ if $n$ even and $\gamma_{pr}(\mathbb{AG}(R)) = n + 1$ if $n$ is odd. □

Suppose $R$ is a commutative Artinian ring. Then by Theorem 3.3, radius of $\mathbb{AG}(R)$ is 2. Further, by Theorem 3.5, the domination number of $\mathbb{AG}(R)$ is equal to $n$, where $n$ is the number of distinct maximal ideals of $R$. However, this need not be true if the radius of $\mathbb{AG}(R)$ is 1. For, consider the ring $R = F_1 \oplus F_2$ where $F_1$ and $F_2$ are fields. Then $\mathbb{AG}(R)$ is a star graph and so has radius 1, whereas $R$ has two distinct maximal ideals. The following corollary shows that a more precise relationship between the domination number of $\mathbb{AG}(R)$ and the number of maximal ideals in $R$, when $R$ is finite. Now we generalize the Corollary 1.4 proved by M. Behboodi and Z. Rakeei[4].

**Corollary 3.7.** *Let $R$ be a finite commutative ring with identity that is not a domain and $\gamma(\mathbb{AG}(R)) = n$. Then either $R = F_1 \oplus F_2$ where $F_1, F_2$ are fields or $R$ has $n$ maximal ideals.*

*Proof.* When $\gamma(\mathbb{AG}(R)) = 1$, proof follows from Corollary 1.4. When $\gamma(\mathbb{AG}(R)) = n$, then $R$ cannot be $F_1 \oplus F_2$ where $F_1, F_2$ are fields. Hence $R = \oplus_{i=1}^{m} R_i$ where $R_i$ are Artinian local rings. By Theorem 3.5, $\gamma(\mathbb{AG}(R)) = m$. Hence by assumption $m = n$. i.e, $R = \oplus_{i=1}^{n} R_i$ where each $R_i$ is a Artinian local ring and $n \geq 2$. One can see now that $R$ has $n$ maximal ideals. □

# 4 Planarity of annihilating-ideal graph

In this section, we discuss about the planarity condition of the annihilating-ideal graph of a commutative ring. In that attempt, at the first instance we find the clique number of the annihilating-ideal graph.

**Lemma 4.1.** *Let $S$ be a commutative semiprimitive Artinian ring with identity, $S = \oplus_{i=1}^{n} F_i$ where $F_i$ are fields and $n \geq 2$. Then $\omega(\mathbb{AG}(S)) = n$.*

*Proof.* Let $Max(S) = \{M_1', \ldots, M_n' : M_i' = F_1 \oplus \cdots \oplus F_{i-1} \oplus (0) \oplus F_{i+1} \oplus \cdots \oplus F_n, 1 \leq i \leq n\}$. If $n = 2$, then $\mathbb{AG}(S) = K_2$ and so $\omega(\mathbb{AG}(S)) = 2$. If $n = 3$, then $\mathbb{AG}(S) = K_3 \circ K_1$ and hence $\omega(\mathbb{AG}(S)) = 3$. Assume that $n \geq 4$. Let $\Omega = \{D_1, D_2, \ldots, D_n : D_i = (0) \oplus \cdots \oplus (0) \oplus F_i \oplus (0) \oplus \cdots \oplus (0), 1 \leq i \leq n\}$. Then $\langle \Omega \rangle$ is complete. Thus $\omega(\mathbb{AG}(S)) \geq n$. Let $I = \oplus_{i=1}^{n} I_i \in V(\mathbb{AG}(S)) - \Omega$ be any non-zero ideal. If $I$ is maximal, then by Theorem 2.1, $deg_{\mathbb{AG}(S)}(I) = 1$ and so $\langle \Omega \cup \{I\} \rangle$ is not complete. Thus $I$ is not maximal and so $I = \oplus_{i=1}^{n} I_i$ implies that $I_k = F_k$ for at least two $k$ and $I_j = (0)$ for at least two $j$, $k \neq j$ and $1 \leq k, j, \leq n$. Now $ID_k \neq (0)$ for at least two $k$ and so $I$ is not adjacent with at least two $D_k$ and hence $\langle \Omega \cup \{I\} \rangle$ is not complete. Hence $\omega(\mathbb{AG}(S)) = n$. □

**Theorem 4.2.** *Let $S$ be a commutative semiprimitive Artinian ring, $S = \oplus_{i=1}^{n} F_i$ where $F_i$ are fields and $n \geq 2$ is an integer. Then $\mathbb{AG}(S)$ is planar if and only if $n = 2$ or $3$.*

*Proof.* If $n = 2$, then $\mathbb{AG}(S) \cong K_2$. If $n = 3$, then $\mathbb{AG}(S) = K_3 \circ K_1$. Hence in both the cases $\mathbb{AG}(S)$ is planar. Conversely, suppose $\mathbb{AG}(S)$ is planar and $n > 3$. Let $n = 4$. Let $x_1 = F_1 \oplus (0) \oplus (0) \oplus (0)$, $x_2 = (0) \oplus F_2 \oplus (0) \oplus (0)$, $x_3 = F_1 \oplus F_2 \oplus (0) \oplus (0)$, $y_1 = (0) \oplus (0) \oplus F_3 \oplus (0)$, $y_2 = (0) \oplus (0) \oplus (0) \oplus F_4$, $y_3 = (0) \oplus (0) \oplus F_3 \oplus F_4$. Then $x_i y_j = (0)$ for all $i, j$ and so $K_{3,3}$ is a subgraph of $\mathbb{AG}(S)$. Hence $\mathbb{AG}(S)$ is non-planar. Suppose $n > 4$. Then by Lemma 4.1, $\omega(\mathbb{AG}(S)) = n$, $K_5$ is a subgraph of $\mathbb{AG}(S)$ and so $\mathbb{AG}(S)$ is non-planar. Hence $n = 2$ or $n = 3$. □

**Theorem 4.3.** *Let $R$ be a commutative Artinian ring with identity and $R = \oplus_{i=1}^{n} R_i$ where $R_i$ are Artinian local rings and $n \geq 2$. Let $J_i$ be the unique maximal ideal in $R_i$ with nilpotency $n_i$. Then $\mathbb{AG}(R)$ is planar if and only if $n = 2$ and $J_1, J_2$ are only non-zero proper ideals of $R_1$ and $R_2$ respectively and $J_1^2 = J_2^2 = (0)$.*

*Proof.* Suppose $n = 2$, $R = R_1 \oplus R_2$, $J_1, J_2$ are only non-zero proper ideals of $R_1$ and $R_2$ respectively and $J_1^2 = J_2^2 = (0)$. Then $V(\mathbb{AG}(R) = \{J_1 \oplus J_2, J_1 \oplus (0), (0) \oplus J_2, J_1 \oplus R_2, R_1 \oplus J_2, R_1 \oplus (0), (0) \oplus R_2\}$. The corresponding $\mathbb{AG}(R)$ is given in Figure 4.1 and one can verify that $\mathbb{AG}(R)$ is planar.

Conversely assume that $\mathbb{AG}(R)$ is planar. Suppose $n \geq 3$. Consider $\Omega = \{\oplus_{i=1}^{n} J_i, \oplus_{i=1}^{n} J_i^{n_i-1}, (0) \oplus \cdots \oplus (0) \oplus J_i^{n_i-1} \oplus (0) \oplus \cdots \oplus (0) : 1 \leq i \leq n\}$. Since $n \geq 3$, $|\Omega| \geq 5$ and so $K_5$ is a subgraph of $\langle \Omega \rangle$. From this $\mathbb{AG}(R)$ is non-planar, a contradiction. Hence $n = 2$.

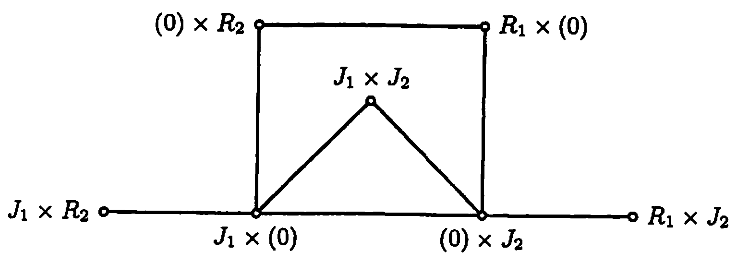


Figure 4.1. $\mathbb{AG}(R)$

Suppose $n_1 > 2$ and $n_2 = 2$. If $\Omega_1 = \{J_1^{n_1-1} \oplus J_2, J_1^{n_1-1} \oplus (0), J_1^{n_1-2} \oplus J_2, J_1^{n_1-2} \oplus (0), (0) \oplus J_2\}$, then $K_5 = \langle \Omega_1 \rangle$ is a subgraph of $\mathbb{AG}(R)$, a contradiction to $\mathbb{AG}(R)$ is planar. Similarly one can get a contradiction in the case of $n_1 = 2$ and $n_2 > 2$. Hence $n_1 = 2$ and $n_2 = 2$.

Suppose $I_1$ is any non-zero proper ideal in $R_1$ and $I_1 \subset J_1$. Since $J_1^2 = (0)$, $I_1^2 = (0)$. Now if $\Omega_2 = \{J_1 \oplus J_2, J_1 \oplus (0), (0) \oplus J_2, I_1 \oplus J_2, I_1 \oplus (0)\}$, then $K_5 = \langle \Omega_2 \rangle$ is a subgraph of $\mathbb{AG}(R)$, which is a contradiction. Hence $J_1$ is the only non-zero proper ideal in $R_1$. Similarly one can prove that $J_2$ is the only non-zero proper ideal in $R_2$. □

**Acknowledgments**

The authors thank the referee for many useful comments and suggestions. The work reported here is supported to the first author by the UGC-SAP(DRS-I) programme for Department of Mathematics, Manonmaniam Sundaranar University and the second author is supported through Faculty Development Programme of University Grants Commission, New Delhi.

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