The spectrum of lattice group divisible 3-designs with block sizes four and six*

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Abstract

In this paper, we give a complete solution of the existence of lattice group divisible 3-designs with block sizes four and six.
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1 Introduction

A *t-wise balanced design* (tBD) is a pair $(X, \mathcal{B})$, where X is a finite set of *points* and $\mathcal{B}$ is a set of subsets of X, called *blocks* with the property that every t-element subset of X is contained in a unique block. A 2BD is usually called a *pairwise balanced design.* If $|X| = v$ and block sizes of $\mathcal{B}$ are all from K, we denote the tBD by $\mathrm{S}(t, K, v)$. When $K = \{k\}$, we simply write k for K. An $\mathrm{S}(t, k, v)$ is called a *Steiner system.* An $\mathrm{S}(3, 4, v)$ is called a *Steiner quadruple system* and denoted by $\mathrm{SQS}(v)$. The existence of an $\mathrm{SQS}(v)$ has been completely determined by Hanani [1].

Theorem 1.1 [1] *An SQS(v) exists if and only if* $v \equiv 2$ *or* $4 (mod\ 6)$.

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When block sizes are 4 and 6, the existence of an S(3, $\{4,6\}, v$) has been determined.

Theorem 1.2 [2] *An* $S(3,\{4,6\},v)$ *exists if and only if* $v \geq 4$ *and* $v \equiv 0 (mod\ 2)$.

Let v be a non-negative integer. Let t be a positive integer and K be a set of some positive integers. A *group divisible t-design* (or t-GDD) of order v and block sizes from K denoted by GDD(t, K, v) is a triple $(X, \mathcal{G}, \mathcal{B})$ such that

(1) X is a set of cardinality v (called *points*),

(2) $\mathcal{G} = \{G_1, G_2, \ldots\}$ is a set of non-empty subsets of X (called *groups*) such that $(X, \mathcal{G})$ is a 1-design,

(3) $\mathcal{B}$ is a family of subsets of X (called *blocks*) each of cardinality from K such that each block intersects any given group in at most one point,

(4) each t-set of points from t distinct groups is contained in exactly one block.

The *type* of the t-GDD is defined as the multiset $\{|G| : G \in \mathcal{G}\}$. If a GDD has n_i groups of size g_i , $1 \leq i \leq r$, then we use the notation $g_1^{n_1} g_2^{n_2} \ldots g_r^{n_r}$ to denote the group type.

Mills [6] showed that for $n \geq 3, n \neq 5$, a GDD($3, 4, ng$) of the type g^n exists if and only if ng is even and $g(n-1)(n-2)$ is divisible by 3, and that for n = 5, a GDD($3, 4, 5g$) of the type g^5 exists if g is divisible by 4 or 6. Recently, Ji [4] improved these results by showing that an GDD($3, 4, 5g$) of the type g^5 exists whenever g is even, $g \neq 2$ and $g \not\equiv 10, 26 \pmod{48}$. When $g = 1$, an GDD($3, 4, n$) of the type 1^n is just a Steiner quadruple system of order n.

Let X be a point set and let $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$, $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_m\}$ be two partitions of the set X. The partitions $\mathcal{G}$ and $\mathcal{G}'$ are called *mutually orthogonal* if and only if $|G_i \cap G'_j| = 1$ for $i \in I_n$ and $j \in I_m$.

Let t, n and m be positive integers and let K be a set of positive integers. Let $(X, \mathcal{G}, \mathcal{B})$ be a uniform group divisible t-design of type m^n which block sizes from K, where $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$. $(X, \mathcal{G}, \mathcal{B})$ is called a *lattice group divisible t-design* (or t-LGDD) with respect to an orthogonal partition $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_m\}$ of X if and only if for any block $B \in \mathcal{B}$ and for any $G'_j, j \in I_m$, either $|B \cap G'_j| < t$ or $B \subseteq G'_j$. We call G_i's, $i \in I_n$, rows and G'_j's, $j \in I_m$, columns. When $t = 3$, a lattice group divisible design with n uniform groups of size g and block size $k \in K$ is denoted by LGDD($n, g, K, 3$) for short.

For a column G'_j of LGDD($n, g, \{4, 6\}, 3$), if a 3-subset from G'_j is contained in a block B, then $B \in G'_j$ from definition. Since $|G'_j| = n$, all the blocks B contained in G'_j consist of the block set of an S($3, \{4, 6\}, n$). This implies that $n \geq 4$ and $n \equiv 0 \pmod 2$ from Theorem 1.2.

In this paper we are concerned with LGDD$(n, g, \{4, 6\}, 3)$ for any positive integer g and prove the following theorem.

Theorem 1.3 *Let g be a positive integer. There exists an LGDD$(n, g, \{4, 6\}, 3)$ if and only if $n \geq 4$ and $n \equiv 0 (mod\ 2)$.*

2 Preliminaries

In this section we shall define some of the auxiliary designs and establish some fundamental results which will be used later.

We begin with some graph theoretical definitions and results. Let G be a graph with g vertices. We can name the vertices of the graph G with the numbers in the range $0, 1, 2, \ldots, g-1$. The *difference* of the edge $e = \{u, v\}$ in G, named so that $u < v$, is defined to be $v - u$ or $g - (v - u)$, whichever is smaller; we denote the difference of $e = \{u, v\}$ by $D_g(u, v)$, or by $D(u, v)$ if the value of g is clear. So if $e = \{u, v\}$ with $u < v$ then

$$D(e) = D(u, v) = min\{v - u, g - (v - u)\}.$$

With this definition, it is clear that $D(u, v) \leq \lfloor g/2 \rfloor$. Notice also that the number of the edges of difference d in the complete graph K_g is g if $d < g/2$, and is $g/2$ if $d = g/2$.

The following graph will be very important to us. For any subset $D \subset \{1, 2, \cdots, \lfloor g/2 \rfloor\}$, define $G(D, g)$ to be the graph on $\mathbb{Z}_g$ with the edge set consisting of all edges having a difference in D; that is, the edge set of $G(D, g)$ is $\{\{u, v\} : D(u, v) \in D\}$. For the existence of the one-factorization of $G(D, g)$, we have the following result.

Lemma 2.1 [8] *Let g be even and D be a set of integers in the range $1, 2, \ldots, g/2$. Then $G(D, g)$ has a one-factorization if and only if $g/gcd(j, g)$ is even for some $j \in D$.*

H.Mohácsy and D.K.Ray-Chaudhuri[7] give the fundamental construction for t-LGDDs.

Theorem 2.2 [7] *Let t, n, a and b be positive integers and let K be a set of positive integers. If for every block A of a t-LGDD of group type a^n there is a t-LGDD of group type $b^{|A|}$ with block sizes from K, then there exists a t-LGDD of group type $(ab)^n$ with block sizes from K.*

We shall give another construction for LGDD designs. We need to introduce two types of new designs. For two mutually orthogonal partitions $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ and $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_m\}$ of the point set X, a triple $(X, \mathcal{G}, \mathcal{A})$ is called a *bias lattice group divisible t-design* if

(1) for any block $A \in \mathcal{A}$ and for any G_i, $i \in I_n$, $|A \cap G_i| = 0$ or 1, for any G'_j, $j \in I_m$, $|A \cap G'_j| = 0$ or 1,

(2) every t-subset from t different rows and t different columns is contained in a unique block of $\mathcal{A}$.

When $t = 3$, a bias lattice group divisible design with n uniform groups of size g and block size from K is denoted by BLGDD$(n, g, K, 3)$ for short.

For two mutually orthogonal partitions $\mathcal{G} = \{G_1, G_2, \ldots, G_n\}$ and $\mathcal{G}' = \{G'_1, G'_2, \ldots, G'_m\}$ of the point set X, a triple $(X, \mathcal{G}, \mathcal{B})$ is called a *curve lattice group divisible t-design* if

(1) for any block $B \in \mathcal{B}$ and for any $G_i, i \in I_n$, $|B \cap G_i| = 0$ or 1, for any G'_j, $j \in I_m$, $|B \cap G'_j| < t$,

(2) except for the t-subset contained in any column, every t-subset from t different rows is contained in a unique block of $\mathcal{B}$.

When $t = 3$, a curve lattice group divisible design with n uniform groups of size g and block size from K is denoted by CLGDD$(n, g, K, 3)$ for short.

It is easy to see that the following result is true.

Lemma 2.3 *If there exist a BLGDD(n, m, K, t), then there exists a BLGDD(m, n, K, t).*

Lemma 2.4 *Let n be even and g be a positive integer. If there exists a BLGDD$(n, g, K, 3)$, then there exists a CLGDD$(n, g, K \cup \{4\}, 3)$. Furthermore, if there exists an $S(3, K', n)$, then there exists an LGDD$(n, g, K \cup K' \cup \{4\}, 3)$.*

Proof. Suppose $(X, \mathcal{G}, \mathcal{B})$ be a BLGDD$(n, g, K, 3)$ on $\mathbb{Z}_g \times \mathbb{Z}_n$ with rows $\mathbb{Z}_g \times \{i\}, i \in \mathbb{Z}_n$, columns $\{j\} \times \mathbb{Z}_n, j \in \mathbb{Z}_g$.

Let E be an edge set of the complete graph on $\mathbb{Z}_g$, $\mathcal{F} = \{F_1, F_2, \ldots, F_{n-1}\}$ be a one-factorization of the complete graph on $\mathbb{Z}_n$. Let $\mathcal{C} = \{\{i_a, i_b, j_c, j_d\} :$ $\{a, b\}$ is the lth edge of F_k,$\{c, d\}$ is the mth edge of F_k, $1 \le l, m \le n/2$, $l \ne m$, $1 \le k \le n-1$, $\{i, j\}$ is the tth edge of E, $1 \le t \le g(g-1)/2$ $\}$. Then $\mathcal{B} \bigcup \mathcal{C}$ is the block set of a CLGDD$(n, g, K \cup \{4\}, 3)$ with rows $\mathbb{Z}_g \times \{i\}, i \in \mathbb{Z}_n$, columns $\{j\} \times \mathbb{Z}_n, j \in \mathbb{Z}_g$.

For any column $Y_i = (\{i\} \times \mathbb{Z}_n), i \in \mathbb{Z}_g$, let $(Y_i, \mathcal{B}'_i)$ be an S$(3, K', n)$. The block set $\mathcal{B} \bigcup \mathcal{C} \bigcup (\bigcup_{0 \le i \le g-1} \mathcal{B}'_i)$ is an LGDD$(n, g, K \cup K' \cup \{4\}, 3)$.
□

An *ordered design* OD(t, k, v) is a $k \times \binom{v}{t} t!$ array with v entries such that

(1) each column has k distinct entries, and

(2) each tuple of t rows contains each column tuple of t distinct entries exactly one time.

L.Teirlinck has worked on OD$(3, 4, v)$. C.Colbourn has revealed by computer search that there is no OD$(3, 4, 7)$. So we know that the following results.

Lemma 2.5 [9], [10] *An* $OD(3,4,v)$ *exists for* $v \notin \{3,7\}$.

Lemma 2.6 [7] *Let* q *be a prime power*
(1) There is an $OD(2,q,q)$.
(2) There is an $OD(3,q+1,q+1)$.

For more information on $\mathrm{OD}(t,k,v)$, the reader can refer to [9], [10] and [5].

If we put all the blocks of $\mathrm{BLGDD}(k,g,k,3)$ together, take every block as a column, then we get an $\mathrm{OD}(3,k,g)$. From Lemma 2.5 and Lemma 2.6, we can get the following results.

Lemma 2.7 *(1) Let* $g \geq 4$ *and* $g \neq 7$. *There exists a* $BLGDD(4,g,4,3)$.
(2) Let q *be a prime power. There is a* $BLGDD(q+1,q+1,q+1,3)$.

Lemma 2.8 *Suppose that there exists an* $S(3,K',v)$. *If there exist a* $BLGDD(g,k,K,3)$ *for any* $k \in K'$, *then there exists a* $BLGDD(g,v,K,3)$. *Furthermore, if* g *is even and there exists an* $S(3,K,g)$, *then there exists an* $LGDD(g,v,K \cup \{4\},3)$.

Proof. We shall construct the desired design on $Y = (X \times \mathbb{Z}_g), X = \mathbb{Z}_v$. Let $(X,\mathcal{B})$ be the given $\mathrm{S}(3,K',v)$. For each block $B \in \mathcal{B}$, construct a $\mathrm{BLGDD}(g,k,K,3)$ on $B \times \mathbb{Z}_g$ with rows $B \times \{i\}, i \in \mathbb{Z}_g$, columns $\{x\} \times \mathbb{Z}_g, x \in B$. Denoted its block set by $\mathcal{C}_B$. Then $\bigcup_{B\in\mathcal{B}} \mathcal{C}_B$ is the block set of a $\mathrm{BLGDD}(v,g,K,3)$ with rows $X \times \{i\}, i \in \mathbb{Z}_g$, columns $\{j\} \times \mathbb{Z}_g, j \in X$.

For any column $Y_i = (\{i\} \times \mathbb{Z}_g), i \in X$, let $(Y_i,\mathcal{B}_i)$ be an $\mathrm{S}(3,K,g)$. Let $\mathcal{F} = \{F_1,F_2,\ldots,F_{g-1}\}$ be a one-factorization of the complete graph on $\mathbb{Z}_g$. Let E be an edge set of the complete graph on X. Let $\mathcal{A} = \{\{i_a,i_b,j_c,j_d\}$: $\{a,b\}$ is the lth edge of F_t,$\{c,d\}$ is the mth edge of F_t, $1 < l,m \leq g/2$, $l \neq m$, $1 \leq t \leq g-1$, $\{i,j\}$ is the kth edge of $E, 1 \leq k \leq v(v-1)/2\}$. Then $(\bigcup_{B\in\mathcal{B}} \mathcal{C}_B) \bigcup (\bigcup_{i\in X} \mathcal{B}_i) \bigcup \mathcal{A}$ is the block set of an $\mathrm{LGDD}(g,v,K \cup \{4\},3)$ with rows $X \times \{i\}, i \in \mathbb{Z}_g$, columns $\{j\} \times \mathbb{Z}_g, j \in X$. □

Lemma 2.9 *There exists a* $BLGDD(6,5,4,3)$.

Proof. The $\mathrm{BLGDD}(6,5,4,3)$ is constructed on $\mathbb{Z}_{30}$ with rows $G_i = \{i+6j : j \in \mathbb{Z}_5\}, i \in \mathbb{Z}_6$ and columns $G'_j = \{i+6j : i \in \mathbb{Z}_6\}, j \in \mathbb{Z}_5$. Developing the following base blocks by (+6 mod 30) yield the block set of the required design.

$\{0,7,14,21\}$, $\{0,7,15,20\}$, $\{0,7,16,26\}$, $\{0,7,17,27\}$,
$\{0,7,22,29\}$, $\{0,7,23,28\}$, $\{0,8,13,22\}$, $\{0,8,15,25\}$,
$\{0,8,16,19\}$, $\{0,8,17,28\}$, $\{0,8,21,29\}$, $\{0,8,23,27\}$,
$\{0,9,13,23\}$, $\{0,9,14,25\}$, $\{0,9,16,29\}$, $\{0,9,17,19\}$,
$\{0,9,20,28\}$, $\{0,9,22,26\}$, $\{0,10,13,26\}$, $\{0,10,14,23\}$,

$\{0,10,15,29\}$,	$\{0,10,17,20\}$,	$\{0,10,19,27\}$,	$\{0,10,21,25\}$,
$\{0,11,13,27\}$,	$\{0,11,14,28\}$,	$\{0,11,15,22\}$,	$\{0,11,16,21\}$,
$\{0,11,19,26\}$,	$\{0,11,20,25\}$,	$\{0,13,20,29\}$,	$\{0,13,21,28\}$,
$\{0,14,19,29\}$,	$\{0,14,22,27\}$,	$\{0,15,19,28\}$,	$\{0,15,23,26\}$,
$\{0,16,20,27\}$,	$\{0,16,23,25\}$,	$\{0,17,21,26\}$,	$\{0,17,22,25\}$,
$\{1,8,16,29\}$,	$\{1,8,21,28\}$,	$\{1,8,22,27\}$,	$\{1,9,17,28\}$,
$\{1,9,20,29\}$,	$\{1,9,23,26\}$,	$\{1,10,14,29\}$,	$\{1,10,15,23\}$,
$\{1,10,17,21\}$,	$\{1,11,14,22\}$,	$\{1,11,15,28\}$,	$\{1,11,16,20\}$,
$\{1,14,23,27\}$,	$\{1,15,22,26\}$,	$\{1,16,21,26\}$,	$\{1,17,20,27\}$,
$\{2,9,16,23\}$,	$\{2,9,17,22\}$,	$\{2,10,17,27\}$,	$\{2,11,16,27\}$.

□

Begin with a design in Example 3.22 [3] and delete its two blocks of size n, we get a CLGDD$(n,2,4,3)$.

Lemma 2.10 [3] *For any even integer $n \geq 4$, there exists a CLGDD$(n,2,4,3)$.*

Lemma 2.11 *There exists a CLGDD$(6,3,\{4,6\},3)$.*

Proof. The CLGDD$(6,3,\{4,6\},3)$ is constructed on $\mathbb{Z}_{18}$ with rows $G_i = \{i+6j : j \in \mathbb{Z}_3\}$, $i \in \mathbb{Z}_6$ and columns $G'_j = \{i+6j : i \in \mathbb{Z}_6\}$, $j \in \mathbb{Z}_3$. Developing the following base blocks by $(+6 \bmod 18)$ yield the block set of the required design.

$\{0,1,8,9,16,17\}$,	$\{0,2,7,10,15,17\}$,	$\{0,3,7,11,14,16\}$,
$\{0,4,8,11,13,15\}$,	$\{0,5,9,10,13,14\}$,	$\{0,1,10,11\}$,
$\{0,1,14,15\}$,	$\{0,2,9,11\}$,	$\{0,2,13,16\}$,
$\{0,3,8,10\}$,	$\{0,3,13,17\}$,	$\{0,4,7,9\}$,
$\{0,4,14,17\}$,	$\{0,5,7,8\}$,	$\{0,5,15,16\}$,
$\{1,2,9,10\}$,	$\{1,3,8,11\}$,	$\{1,4,15,17\}$,
$\{1,5,14,16\}$,	$\{2,3,16,17\}$.	

□

Lemma 2.12 *There exists an LGDD$(4,7,4,3)$.*

Proof. The LGDD$(4,7,4,3)$ is constructed on $\mathbb{Z}_{28}$ with rows $G_i = \{i+4j : j \in \mathbb{Z}_7\}$, $i \in \mathbb{Z}_4$ and columns $G'_j = \{i+4j : i \in \mathbb{Z}_4\}$, $j \in \mathbb{Z}_7$. Developing the following base blocks by $(+4 \bmod 28)$ yield the block set of the required design.

$\{0,1,2,3\}$,	$\{0,1,6,7\}$,	$\{0,1,10,11\}$,	$\{0,1,14,15\}$,
$\{0,1,18,19\}$,	$\{0,1,22,23\}$,	$\{0,1,26,27\}$,	$\{0,2,5,7\}$,
$\{0,2,9,11\}$,	$\{0,2,13,15\}$,	$\{0,2,17,19\}$,	$\{0,2,21,23\}$,
$\{0,2,25,27\}$,	$\{0,3,5,6\}$,	$\{0,3,9,10\}$,	$\{0,3,13,14\}$,

$\{0,3,17,18\}$,	$\{0,3,21,26\}$,	$\{0,3,22,25\}$,	$\{0,5,10,19\}$,
$\{0,5,11,22\}$,	$\{0,5,14,23\}$,	$\{0,5,15,26\}$,	$\{0,5,18,27\}$,
$\{0,6,9,27\}$,	$\{0,6,11,13\}$,	$\{0,6,15,17\}$,	$\{0,6,19,21\}$,
$\{0,6,23,25\}$,	$\{0,7,9,14\}$,	$\{0,7,10,21\}$,	$\{0,7,13,18\}$,
$\{0,7,17,22\}$,	$\{0,7,25,26\}$,	$\{0,9,15,22\}$,	$\{0,9,18,23\}$,
$\{0,9,19,26\}$,	$\{0,10,13,27\}$,	$\{0,10,15,25\}$,	$\{0,10,17,23\}$,
$\{0,11,14,21\}$,	$\{0,11,17,26\}$,	$\{0,11,18,25\}$,	$\{0,13,19,22\}$,
$\{0,13,23,26\}$,	$\{0,14,17,27\}$,	$\{0,14,19,25\}$,	$\{0,15,18,21\}$,
$\{0,21,22,27\}$.			

□

From Lemma 2.4, Lemma 2.7 and Lemma 2.12 we have the following result.

Lemma 2.13 *For any positive integer g, there exists an LGDD$(4,g,4,3)$.*

Since S$(3,K,v)$ is an LGDD$(v,1,K,3)$, we have the following result from Theorem 1.2.

Lemma 2.14 *For $n \geq 4$ and $n \equiv 0 (mod\, 2)$, there exists an LGDD$(n,1,\{4,6\},3)$.*

Lemma 2.15 *If there exists an S$(3,K,g)$ which contains a subdesign S$(2,K',g)$, where the block sizes of subdesign from $K' \subset K$ and others come from $K \setminus K'$. If there exists a BLGDD$(n,k,K'',3)$ for every $k \in K \setminus K'$ and a CLGDD$(n,k',K'',3)$ for every $k' \in K'$, then there exists a CLGDD$(n,g,K'',3)$. Furthermore, if there exists an S$(3,K'',n)$, then there exists an LGDD$(n,g,K'',3)$.*

Proof. Let $(X,\mathcal{B})$ be an S$(3,K,g)$ which contained a subdesign S$(2,K',g)$ $(X,\mathcal{A})$. $X=\mathbb{Z}_g$, $|A| \in K'$ if $A \in \mathcal{A}$, $|B| \in K\setminus K'$ if $B \in \mathcal{B}\setminus\mathcal{A}$. For every block $B \in \mathcal{A}$, $|B|=k'$, construct a CLGDD$(n,k',K'',3)$ on $B \times \mathbb{Z}_n$ by rows $\{B \times \{i\}\}$, $i \in \mathbb{Z}_n$ and columns $\{\{x\} \times \mathbb{Z}_n\}$, $x \in B$. We denote its block set by $\mathcal{C}_B$.

For every block $B \in \mathcal{B}\setminus\mathcal{A}$, $|B|=k$, construct a BLGDD$(n,k,K'',3)$ on $B \times \mathbb{Z}_n$ with rows $\{B \times \{i\}\}$, $i \in \mathbb{Z}_n$ and columns $\{\{x\} \times \mathbb{Z}_n\}$, $x \in B$ and denote its block set by $\mathcal{D}_B$.

Then $(\mathbb{Z}_g \times \mathbb{Z}_n, (\bigcup_{B\in\mathcal{A}} \mathcal{C}_B)\bigcup(\bigcup_{B\in\mathcal{B}\setminus\mathcal{A}} \mathcal{D}_B))$ is a CLGDD$(n,g,K'',3)$ with groups $G_i=\{\mathbb{Z}_g \times \{i\}\}$, $i \in \mathbb{Z}_n$ and columns $G'_j=\{j\}\times\mathbb{Z}_n, j \in \mathbb{Z}_g$.

For $Y=\mathbb{Z}_n$, let $(Y,\mathcal{E})$ is a $S(3,K'',n)$. Then $(\mathbb{Z}_g \times \mathbb{Z}_n, (\bigcup_{B\in\mathcal{A}} \mathcal{C}_B)\bigcup(\bigcup_{B\in\mathcal{B}\setminus\mathcal{A}} \mathcal{D}_B)\bigcup\{\{k\}\times B : k \in \mathbb{Z}_g, B \in \mathcal{E}\})$ is an LGDD$(n,g,K'',3)$ with rows $G_i=\{\mathbb{Z}_g \times \{i\}\}$, $i \in \mathbb{Z}_n$ and columns $G'_j=\{\{j\}\times\mathbb{Z}_n\}, j \in \mathbb{Z}_g$. □

3 Main Result

In this section, we will discuss the existence of $LGDD(n, g, \{4, 6\}, 3)$. We first give the special case when the block size is four.

Lemma 3.1 *Let g be a positive integer. There exists an $LGDD(n, g, 4, 3)$ for $n \equiv 2, 4 (mod\ 6)$.*

Proof. For a point set X, $|X| = n$, let $(X, \mathcal{G}, \mathcal{B})$ be an LGDD$(n, 1, 4, 3)$. Apply Theorem 2.2 with an LGDD$(4, g, 4, 3)$ from Lemma 2.13 to obtain the desired design. □

Lemma 3.2 *There exist an $LGDD(6, g, \{4, 6\}, 3)$ for $g \equiv 1 (mod\ 2)$.*

Proof. Let $(X \cup \{\infty\}, \mathcal{B})$ be an S$(3, \{4, 6\}, g + 1)$ in Theorem 1.2, where $|X| = g \equiv 1(mod\ 2)$. Let $\mathcal{A} = \{B : B \in \mathcal{B}, \infty \notin B\}$, $\mathcal{A}' = \{B \setminus \{\infty\} : B \in \mathcal{B}, \infty \in B\}$, then $(X, \mathcal{A} \cup \mathcal{A}')$ is an $S(3, \{3, 4, 5, 6\}, g)$ which contained a subdesign $S(2, \{3, 5\}, g)$ $(X, \mathcal{A}')$ and for every block $B \in \mathcal{A}$, $|B| = 4$ or 6.

From Lemma 2.3 and Lemma 2.7 to get a BLGDD$(6, m, \{4, 6\}, 3)$ for $m = 4, 6$. From Lemma 2.11, Lemma 2.4 and Lemma 2.9 to get a CLGDD(6, $n, \{4, 6\}, 3)$ for $n = 3, 5$. Apply Lemma 2.15 with an $S(3, \{3, 4, 5, 6\}, g)$ and the designs above, we obtain the desired design. □

Lemma 3.3 *There exists an $LGDD(6, g, \{4, 6\}, 3)$ for $g \equiv 0(mod\ 2)$.*

Proof. For the case $g = 2$, begin with a CLGDD$(6, 2, 4, 3)$, add the two columns as two new blocks, we get an LGDD$(6, 2, \{4, 6\}, 3)$.

For $g \geq 4$ and $g \equiv 0$(mod 2), apply Lemma 2.8 with $S(3, \{4, 6\}, g)$ in Theorem 1.2 we obtain the desired design. We need a BLGDD$(k, g, \{4, 6\}, 3)$ for $k = 4, 6$ and an S$(3, \{4, 6\}, 6)$ as input design, the formers come from Lemma 2.7 and Lemma 2.3 the latter comes from Theorem 1.2 . □

Lemma 3.4 *Let g be a positive integer. There exists an $LGDD(n, g, \{4, 6\}, 3)$ for $n \equiv 0(mod\ 6)$.*

Proof. For a point set X, $|X| = n$ and $n \equiv 0$(mod 6), let $(X, \mathcal{G}, \mathcal{B})$ be an LGDD$(n, 1, \{4, 6\}, 3)$ which exists by Lemma 2.14. Apply Theorem 2.2 with an LGDD$(4, g, 4, 3)$ from Lemma 2.7 and LGDD$(6, g, \{4, 6\}, 3)$ from Lemma 3.2 and Lemma 3.3, we obtain the desired design. □

Combining Lemma 3.1 and Lemma 3.4, we have established the following result.

Theorem 3.5 *Let g be a positive integer. There exists an $LGDD(n, g, \{4, 6\}, 3)$ if and only if $n \geq 4$ and $n \equiv 0(mod\ 2)$.*

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