Nested $(2, r)$-regular graphs and their network properties

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Abstract

A graph G is a (t, r)-regular graph if every collection of t independent vertices is collectively adjacent to exactly r vertices. Let p, s, and m be positive integers, where $m \geq 2$ and let G be a $(2, r)$-regular graph. If n is sufficiently large, then G is isomorphic to $G = K_s + mK_p$, where $2(p-1) + s = r$. A nested $(2, r)$-regular graph is constructed by replacing selected cliques in a $(2, r)$-regular graph with a $(2, r')$-regular graph and joining the vertices of the peripheral cliques. We examine the network properties such as the average path length, clustering coefficient, and the spectrum of these nested graphs.

1 Introduction

1.1 Small World Graphs

The concept of a small-world network describes the network property that, despite the network being large in size, in most cases there is a relatively short path between any two vertices. The most popular appearance in the literature is the "six degrees of separation" concept which was studied by the psychologist Stanley Milgram [9]. The idea behind the six degrees of separation is that every person in the world can be linked by acquaintance to another through at most five individuals. The formal notion of small-world networks was introduced in the 1998 paper, *Collective dynamics of 'small-world' networks*, by Strogatz and Watts [13]. These small-world networks tend to have dense subgraphs which they call clusters. The clusters

JCMCC **88** (2014), pp. 27-38

are generally sparsely joined, yet the network still has a low average path length. They defined two network measures to quantify these properties, the clustering coefficient and the average path length. The clustering coefficient is defined as follows. Given a vertex v has k_v neighbors; then there are at most $k_v(k_v - 1)/2$ edges that can exist between them. Let C_v denote the fraction of these allowable edges that actually exist. Define the clustering coefficient C as the average of C_v over all v. Note that the clustering coefficient of a complete graph is equal to one and the clustering coefficient of a path is zero. In particular, the clustering coefficient lies in the range from zero to one [10].The average path length, L, is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices [13].

Researchers, including Strogatz and Watts and later Albert and Barabási [1], have shown that naturally occurring networks such as the power grid of the western United States [13] or social networks are examples of the small-world phenomena. Small-world networks have been found to be generic for many large, sparse networks found in nature. A few examples are the neural network of the worm *Caenorhabditis elegans* and the short-term memory circuits between neurons [11].

1.2 Pseudofractals and Hierarchical Graphs

Barabási, Erzébet Ravasz, and Tamás Vicsek introduced a deterministic algorithm to construct small-world networks [3]. The construction of such networks follows a hierarchical rule, where each iteration uses components that are created in previous steps. The construction can be described as follows. Let $G = K_1$. In the next iteration, add two vertices and connect them to the initial vertex so that we now have constructed a P_3. In the next step, add two more copies of a P_3 and connect the mid-point of the initial P_3 with the outer vertices of the two new P_3's. This construction can be continued indefinitely. The Figure 3 is an example of a hierarchical network using this algorithm [12].

The pseudofractal is another example of a deterministic graph construction, which has been proposed by S. N. Dorogovstev, et al. to model the growth of scale-free networks [4]. The graph is constructed in a similar manner to that of the hierarchical graphs. The graph grows at each step by connecting together three copies of the graph in the previous step. Figure 2 gives an example of one such graph [12].

1.3 (t, r)- regular Graphs

The *join* $G = G_1 + G_2$, sometimes denoted $G = G_1 \bigvee G_2$, has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1)$ and $v \in$

$n = 0$

$n = 1$

$n = 2$

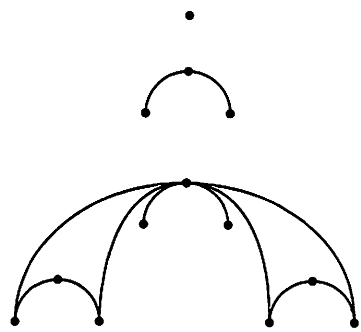

Figure 1: Hierarchical Network

$t = 0$

$t = 1$

$t = 2$

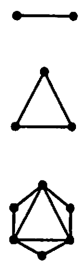

Figure 2: Pseudofractal Graph

$V(G_2)\}$.

A graph G is (t, r)-regular if every collection of t independent vertices is collectively adjacent to exactly r vertices. An r-regular graph is therefore a $(1, r)$-regular graph. In a 1996 paper by R. Faudree and D. Knisley the characterization of large $(2, r)$-regular graphs was given [5].

Theorem 1.1 *Let r, s, and p be nonnegative integers and let G be a $(2, r)$-regular graph of order n. If n is sufficiently large, then G is isomorphic to $K_s + mK_p$ where $2(p-1) + s = r$. There are exactly $\lfloor \frac{r+1}{2} \rfloor$ such graphs.*

Figure 3 is an example of a $(2, 6)$-regular graph.

Jamison and Johnson found that this characterization does not hold for $t \geq 3$, but the structure of such graphs is very similar to the case when $t = 2$ if the order of the graph is sufficiently large [6]. In particular, given the $(2, r)$-regular graph is of the form $K_s + mK_p$, they denote the central clique with s vertices to be the Kernel of the graph and the m peripheral cliques to be the Shell. They show that when $t \geq 3$, the kernel has independence number no greater than $t - 1$ and the shell is still a collection of disjoint

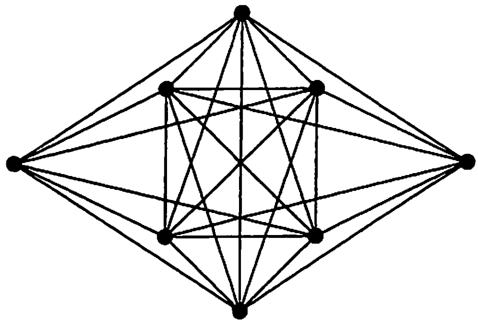

Figure 3: a (2,6)-regular graph ($K_6 + 2K_1$)

cliques that are mostly joined to the kernel [6].

The $(3, r)$-regular graphs were studied by Laffin [8].

2 Nested (2,r)-regular graphs

Recall that the clustering coefficient of a graph ranges from zero to one. In general, a sparse graph is small-world if the clustering coefficient is high and the average path length is low. We find that the $(2, r)$-regular graphs have a high clustering coefficient and short average path length, but they are not sparse.

Motivated by the work on pseudofractal [3] and hierarchical graphs [4], we developed a method to reduce the number of edges in $(2, r)$-regular graphs while maintaining the high clustering coefficient and short average path length properties. In our construction we applied the technique of 'nesting' or replacing specific cliques of the graph with another $(2, r')$-regular graph of the same order as the clique and only joining the vertices of the peripheral cliques. In this section we define the nested 's' and nested 'p' graphs.

2.1 Nested 's' Graphs

For a nested 's' graph we replace the center clique, K_s, with a $(2, r')$-regular graph of the form $G_1 = K_{s_1} + m_1K_{p_1}$. In the formula $n = s + mp$ where $s = s_1 + m_1p_1$ so that we obtain $n = s_1 + m_1p_1 + mp$. The nested 's' graph is now of the form $G_s = (K_{s_1} \bigcup mK_p) + m_1K_{p_1}$. Figure 4 is the general form of the nested 's' graphs.

Consider for example the $(2, 10)$-regular graph G in Figure 5 and the nested graph G_6 in Figure 6 where the K_6 is replaced with a $K_2 + 4K_1$.

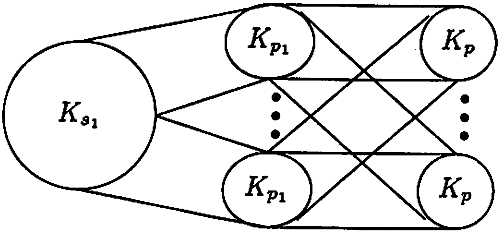

Figure 4: General form of a nested 's' $(2, r)$-regular graph

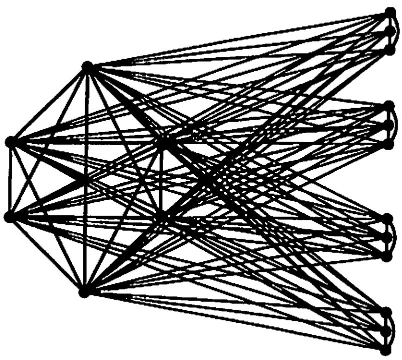

Figure 5: $(2, 10)$-regular graph

2.2 Nested 'p'

For a nested 'p' graph we replace the peripheral cliques, K_p's, with $(2, r')$-regular graphs of the form $G_1 = K_{s_1} + m_1 K_{p_1}$. In the formula $n = s + mp$ where $p = s_1 + m_1 p_1$ we obtain $n = s + m(s_1 + m_1 p_1)$. The nested 'p' graph is now of the form $G_p = (K_{s_1} \bigcup m_1 K_{p_1}) + m(K_{s_1})$. Figure 7 is the general form of the nested 'p' graphs.

3 Network Properties

The clustering coefficient and average path length of $(2, r)$-regular graphs have been previously studied. Knisley et al. proved the following result for $(2, r)$-regular graphs [7].

Theorem 3.1 *Let s, m, and p be nonnegative integers and let $r = 2(p - 1) + s$. Let G be a $(2, r)$-regular graph of the form $K_s + mK_p$. If s and p are fixed constants and $m \to \infty$, then $L \to 2$ and $C \to 1$.*

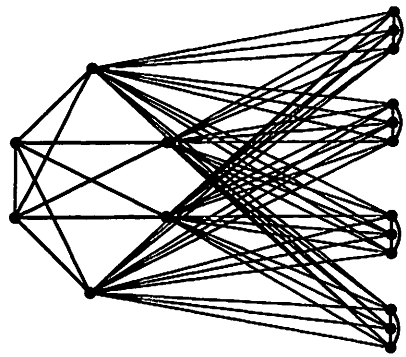

Figure 6: G_6 nested s graph

In the proof of this theorem, the generalized formulas for the average path length and clustering coefficients of the $(2, r)$-regular graph were determined. They found the average path length to be

$$L = \frac{\binom{s}{2} + smp + m\binom{p}{2} + 2[\binom{mp}{2} - m\binom{p}{2}]}{\binom{s+mp}{2}}$$

and the clustering coefficient as

$$C = \frac{s\left(\frac{\binom{s-1}{2}+m\binom{p}{2}+mp(s-1)}{\binom{s+mp-1}{2}}\right) + mp}{s+mp}.$$

We give the corresponding values for the nested $(2, r)$-regular graphs in the following sections.

3.1 Average Path Length of nested $(2, r)$-regular Graphs

Theorem 3.2 *The average path length of a nested 's' graph is*

$$L = 1 + \frac{p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1 mp}{\binom{n}{2}}.$$

Proof: Since the diameter of the graph is 2, the only possible paths are those of length 1 and length 2. The total number of paths of length one are the total number of edges in the graph, therefore there are $P_1 = \binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1 m_1 p_1 + m_1 p_1 mp$. By definition of average path length, we found that the total number of paths of length 2 are $P_2 = \binom{n}{2} - P_1$. Average path length, for graphs of diameter 2, is defined as follows

$$L = \frac{P_1 + 2P_2}{\binom{n}{2}}.$$

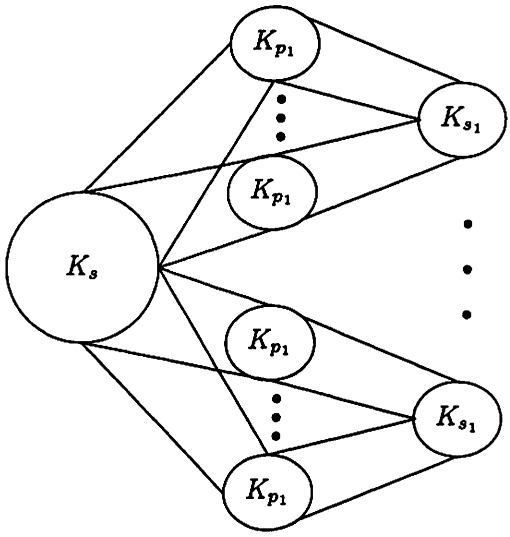

Figure 7: General form of a Nested 'p' graph

By substituting in our P_1 and P_2, we obtain the following

$$\begin{aligned} L &= \frac{\binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1 m_1 p_1 + m_1 p_1 m p}{\binom{n}{2}} \\ &+ \frac{2(\binom{n}{2} - \binom{s_1}{2} + m_1\binom{p_1}{2} + m\binom{p}{2} + s_1 m_1 p_1 + m_1 p_1 m p)}{\binom{n}{2}}. \end{aligned}$$

Which reduces to

$$L = \frac{\binom{n}{2} + p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1 m p}{\binom{n}{2}}.$$

Thus the average path length of a nested 's' is

$$L = 1 + \frac{p^2\binom{m}{2} + p_1^2\binom{m_1}{2} + s_1 m p}{\binom{n}{2}}. \blacksquare$$

Theorem 3.3 *The average path length of a nested 'p' graph is*

$$L = 1 + \frac{\binom{m}{2}(4s_1 m_1 p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}.$$

Proof: The diameter of a nested 'p' graph is 4, and the only possible paths are of length 1, 2, 3, and 4. The total number of paths of length 1 are $P_1 = \binom{s}{2} + m\binom{s_1}{2} + mm_1\binom{p_1}{2} + ms_1m_1p_1 + smm_1p_1$, paths of length 2 $P_2 = sms_1 + p_1^2\binom{mm_1}{2}$, paths of length 3 $P_3 = m^2s_1m_1p_1 - ms_1m_1p_1$ and paths of length 4 $P_4 = s_1^2\binom{m}{2}$. The average path length, for graphs of diameter 4, is defined as follows

$$L = \frac{P_1 + 2P_2 + 3P_3 + 4P_4}{\binom{n}{2}}.$$

By substituting and simplifying we obtain the following

$$L = \frac{\binom{n}{2} + \binom{m}{2}(4s_1m_1p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}.$$

Therefore the average path length of a nested 'p' graph is

$$L = 1 + \frac{\binom{m}{2}(4s_1m_1p_1 + 3s_1^2) + p_1^2\binom{mm_1}{2} + sms_1}{\binom{n}{2}}.\blacksquare$$

3.2 Clustering Coefficient of nested $(2,r)$-regular Graphs

By the structure of the nested 's' graph, there are only three distinct values for the clustering coefficients of the vertices. Let C_u denote the set of vertices whose clustering coefficient is $C_U = \frac{m_1\binom{p_1+s_1-1}{2} - (m_1-1)\binom{s_1-1}{2}}{\binom{s_1-1+m_1p_1}{2}}$, C_v denote the set of vertices whose clustering coefficient is $C_V = \frac{m\binom{p_1-1+p}{2} - (m-1)\binom{p_1-1}{2} + \binom{p_1-1\cdots}{2}}{\binom{p_1-1+s_1+mp}{2}}$ and C_w denote the set of vertices whose clustering coefficient is $C_W = \frac{m_1\binom{p-1+p_1}{2} - (m_1-1)\binom{p-1}{2}}{\binom{p-1+m_1p_1}{2}}$.

Theorem 3.4 *The clustering coefficient of a nested 's' graph is*

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mp(C_W)}{s_1 + m_1p_1 + mp}.$$

Proof: The clustering coefficient of the graph is defined as the sum of the clustering coefficients of the vertices in the graph divided by the total number of vertices in the graph. Therefore, the clustering coefficient of a nested 's' graph is

$$C = \frac{s_1(C_U) + m_1p_1(C_V) + mp(C_W)}{s_1 + m_1p_1 + mp}.\blacksquare$$

By the structure of the nested 'p' graph, there are only three distinct values for the clustering coefficients of the vertices. Let C_u denote the set of vertices whose clustering coefficient is $C_U = \frac{mm_1\binom{s-1p_1}{2} - (mm_1-1)\binom{s-1}{2}}{\binom{s-1+mm_1p_1}{2}}$, C_v denote the set of vertices whose clustering coefficient is $C_V = \frac{\binom{p_1-1+s}{2} + \binom{p_1-1+s_1}{2} - (\cdots}{\binom{p_1-1+s+s_1}{2}}$

and C_w denote the set of vertices whose clustering coefficient is $C_W = \frac{m_1\binom{s_1-1+p_1}{2}-(m_1-1)\binom{s_1-1}{2}}{\binom{s_1-1+m_1p_1}{2}}$.

Theorem 3.5 *The clustering coefficient of a nested 'p' graph is*

$$C = \frac{s(C_U) + mm_1p_1(C_V) + ms_1(C_W)}{s + ms_1 + mm_1p_1}.$$

Proof: The clustering coefficient of the graph is defined as the sum of the clustering coefficients of the graph divided by the total number of vertices in the graph. Therefore, using the counting arguments for C_U, C_V, and C_W defined above, we obtain the clustering coefficient of a nested 'p' graph as

$$C = \frac{s(C_U) + mm_1p_1(C_V) + ms_1(C_W)}{s + ms_1 + mm_1p_1}. \blacksquare$$

4 Spectral Properties of $(2, r)$-regular Graphs and Nested 's' Graphs

The Laplacian Matrix of a graph G is defined as

$$L(G) = D - A$$

where D is the diagonal matrix whose diagonal consists of the degrees of the vertices, and A is the adjacency matrix of G. The Laplacian of a $(2, r)$-regular graph, of the form $K_s + mK_p$, is given by

$$L = \left[\begin{array}{c|cccc} C & & -\mathbf{1}^T & & \\ \hline & P & 0 & \cdots & 0 \\ -\mathbf{1} & 0 & P & \ddots & \vdots \\ & \vdots & \ddots & \ddots & 0 \\ & 0 & \cdots & 0 & P \end{array}\right].$$

where $-\mathbf{1}$ represents the $mp \times s$ matrix in which every entry is -1 and the matrices C and P are of the form

$$C = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & n-1 \end{bmatrix}, \quad P = \begin{bmatrix} q & -1 & \cdots & -1 \\ -1 & q & \ddots & \vdots \\ \vdots & \ddots & \ddots & -1 \\ -1 & \cdots & -1 & q \end{bmatrix},$$

with $n = s + mp$ and $q = p + s - 1$. The eigenvalues and eigenvectors of L have a distinctive pattern derived in large part from the fact that $K_s + mK_p$ is the *join* of K_s with the disjoint union of m copies of K_p. In particular, the eigenvalues of L follow immediately from the following [14, 15]:

Theorem 4.1 *Let $G = G_1 + G_2$ be the join of two graphs G_1 and G_2. If n_j is the number of vertices of $G_j, j = 1, 2$, then the eigenvalues $\lambda_i(G), i = 1, \ldots, |V(G)|$ of $L(G)$ are 0; $n_1 + n_2$; $n_2 + \lambda_i((G_1)$, $1 \leq i < n_1$; and $n_1 + \lambda_i(G_2), 1 \leq i < n_2$.*

The eigenvectors, similarly, follow from a close analogue of the theorem above. However, before doing so, we need to clarify what we mean by the Laplacian of a subgraph.

Definition 4.2 *If S is a subgraph of a graph G, then $L_e(S)$ is the trivial extension of the Laplacian of S to the same order as $L(G)$. Namely, $L_e(S)$ is the Laplacian of S union $|G| - |S|$ isolated vertices of degree 0.*

This leads to the following lemma:

Lemma 4.3 *Let G be either the disjoint union or the join of two graphs G_1 and G_2. Then $L(G)$ commutes with $L_e(G_j), j = 1, 2$, and as a result, $\mathbf{v}$ is an eigenvector of $L(G)$ only if $\mathbf{v}$ is an eigenvector of either $L_e(G_1)$ or $L_e(G_2)$. Moreover, eigenvectors of $L_e(G_1)$ are orthogonal to eigenvectors of $L_e(G_2)$.*

Proof: The graphs $L_e(G_i), j = 1, 2$ are of the form

$$L_e(G_1) = \begin{bmatrix} L(G_1) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \quad L_e(G_2) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L(G_2) \end{bmatrix}$$

Commutativity follows by direct calculation, and the eigenvector properties are well-known consequences of commutativity. ∎

In part, the lemma is based on the idea that if two symmetric matrices commute, then they are simultaneously diagonalized. Since $K_s + mK_p$ is the join of a complete graph with a disjoint union of complete graphs, the eigenvalue problem is reduced to that of finding eigenvalues and eigenvectors of $L_e(K_s)$ and $L_e(K_m)$, respectively. However, the eigenspace decomposition of $L(K_n)$ for any $n \in \mathbb{Z}^+$ is characterized by a 1-dimensional eigenspace of the eigenvalue 0 and an $n-1$ dimensional eigenspace of the eigenvalue n. This leads to the following theorem.

Theorem 4.4 *The eigenvalues of the Laplacian L of a $K_s + mK_p$ are*

- $\lambda_1 = 0$ *with multiplicity* 1
- $\lambda_2 = s$ *with multiplicity* $m - 1$
- $\lambda_3 = p + s$ *with multiplicity* $m(p-1)$
- $\lambda_4 = n$ *with multiplicity* s.

Proof: The eigenvalues are a direct consequence of the theorem in [14]. The multiplicities of the eigenvalues are a direct consequence of the lemma and the observations immediately thereafter. ∎

The eigenvectors and eigenspace decomposition of L are no less interesting. In particular, L is diagonalized by the direct sum of the mutually orthogonal eigenspaces of the complete graphs used to define it. Moreover, the eigenspaces of non-zero eigenvalues of the Laplacian of a complete graph are spanned by vectors in which the sum of the coefficients is 0.

5 Conclusion

In this work we develop algorithms to reduce the number of edges in a $(2, r)$-regular graph using a nested graph approach and calculate the clustering coefficients and average path lengths of these graphs. We then investigate the Laplacian Matrices of the $(2, r)$-regular graphs. We were successful in finding a general form of the eigenvalues of canonical $(2, r)$-regular graphs. We have left the door open for further research into spectral properties of nested graphs. Some open problems which came out of this research are:

- Determine the Laplacian of the two remaining nested graphs.
- Find the general form of the set of eigenvalues of each of the nested graphs.
- Define an algorithm to grow a nested graph and examine the scale-free properties of the resulting graphs.

We have introduced a novel method to construct graphs with desirable network properties. In keeping with the Open Problems theme of the 25th Cumberland Conference on Combinatorics, Graph Theory and Computing, we also provide a number of open problems with respect to this method. We expect that these graphs will be of interest to those working in the field of networks and spectral graph theory.

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