

Necessary Spectral Conditions for Coloring Hypergraphs

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Abstract

Hoffman proved that for a simple graph G , the chromatic number $\chi(G)$ obeys $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$ where λ_1 and λ_n are the maximal and minimal eigenvalues of the adjacency matrix of G respectively. Lovász later showed that $\chi(G) \geq 1 - \frac{\lambda_1}{\lambda_n}$ for any (perhaps negatively) weighted adjacency matrix.

In this paper, we give a probabilistic proof of Lovász's theorem, then extend the technique to derive generalizations of Hoffman's theorem when allowed a certain proportion of edge-conflicts. Using this result, we show that if a 3-uniform hypergraph is 2-colorable, then $\bar{d} \leq -\frac{3}{2}\lambda_{\min}$ where \bar{d} is the average degree and λ_{\min} is the minimal eigenvalue of the underlying graph. We generalize this further for k -uniform hypergraphs, for the cases $k = 4$ and 5 , by considering several variants of the underlying graph.

1 Introduction

Spectral graph theory has produced several results in relation to graph coloring. Let λ_{\max} and λ_{\min} denote the maximal and minimal eigenvalues, respectively, of the adjacency matrix of G . In 1967, Wilf [15] showed that $\chi(G) \leq 1 + \lambda_{\max}$, and in 1969, Hoffman showed $\chi(G) \geq 1 - \frac{\lambda_{\max}}{\lambda_{\min}}$ [7]. Hoffman's theorem was improved by Lovász in 1979 with his work on Shannon capacity by considering any (even negative) weighted adjacency matrix of G [11].

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For $k > 2$, the decision problem determining whether or not a graph can be colored with k -colors is NP-hard [1, 14] whereas eigenvalues can be computed accurately in polynomial time. Hence, the results mentioned above give a simple and easily computable approximation to graph coloring. In the case of hypergraphs, for any $k \geq 2$, the problem of determining if a hypergraph is k -colorable is NP-hard [10]. As a result, we aim to adapt these results on graphs to hypergraphs.

There are many different notions of eigenvalues for hypergraphs. These notions generally have one of two approaches. The first is to consider some higher-dimensional analog of a matrix. Friedman and Wigerson used a trilinear form to define the spectral gap of a 3-uniform hypergraph [5], and later, Cooper and Dutle extend this concept to hypermatrices using hyperdeterminants [4]. The other approach is to dissect the hypergraph into several matrices, and consider these matrices jointly. Chung used this approach by considering the homology and cohomology chains of hypergraphs [2]. Later, Rodríguez [13] followed by Lu and Peng [8] considered different walks on the hypergraph with varying degrees of tightness.

The only known result connecting hypergraph coloring to spectra is given by Cooper and Dutle using hypermatrices and hyperdeterminants where they prove an analog of Wilf's theorem for hypergraphs: $\chi(H) \leq \lambda_{\max} + 1$ (see [4]). In this paper, we provide necessary spectral conditions for 2-coloring a hypergraph using the second approach to hypergraph spectra as described above. Specifically, our main strategy will be to consider several different graphs based upon the original hypergraph. A more detailed description is given in the next section. Then, we consider these different graphs and their spectra, jointly, in order to recover information regarding the hypergraph.

This paper is organized as follows: In section 2, we give definitions and preliminaries. In section 3 we state our main results. We prove our results in two sections. In section 4, we prove results for graphs, including a probabilistic proof of the Hoffman-Lovász theorem mentioned above, and in section 5, we adapt these results to various aspects of hypergraphs. Finally, in section 6, we give examples of applications of these results.

2 Preliminaries

A *graph* $G = (V, E)$ is a set of *vertices*, V , and *edges*, E , such that every $e \in E$ is an unordered pair of vertices. If $\{u, v\} \in E$, we say that two

vertices u, v are *adjacent*. In this paper, we consider undirected graphs which allow for multiple edges between two vertices; however, we do not consider graphs with loops (i.e. self-adjoint edges). We define the *degree*, d_u , of a vertex u , to be the number of edges containing u where any repeated edge is counted with multiplicity. We let \bar{d} denote the average degree of G . A graph is *d-regular* if $d_u = d$ for all vertices u .

A *hypergraph*, $H = (V, E)$ is a set of vertices, V , and hyperedges, E , such that every $e \in E$ is a subset of vertices. As with graphs, the degree of a vertex u , denoted d_u , is the number of hyperedges containing u , considering multiplicity, and the average degree will be denoted \bar{d} . We call a hypergraph *r-uniform* if for every edge $e \in E$, $|e| = r$.

We consider traditional vertex coloring of a graph G . A *proper k-coloring* of a graph is a function $g: V \rightarrow \{1, 2, \dots, k\}$ such that $g(v) \neq g(v')$ whenever $\{v, v'\} \in E$. If such a function exists for a particular integer k , we say the graph is *k-colorable*. For a graph G , the chromatic number, $\chi(G)$, is defined to be the least k such that G is *k-colorable*. In this paper, we will consider the case when a coloring is *improper* where $g(v) = g(v')$ even when $\{v, v'\} \in E$; in which case we call the edge $\{v, v'\}$ *monochromatic*.

In addition, we consider the *weak-coloring* of a hypergraph. A (weak) coloring of a hypergraph is a function $h: V \rightarrow \{1, 2, \dots, k\}$ such that for every edge $e \in E(H)$, the function h is not constant on e . As with graphs, the chromatic number of a hypergraph, $\chi(H)$, is defined to be the least k such that H is *k-colorable*.

A hypergraph, H has an *underlying graph* denoted $G(H)$ which has the same vertex set as H , and e is an edge of $G(H)$ if $e \subset f$ for some $f \in E(H)$. For our purposes, we allow for each edge in $G(H)$ to occur once for each hyperedge containing e . For example, $G(K_4^3)$, the underlying graph of the complete 3-uniform hypergraph on 4 vertices, is the complete graph on 3 vertices with each edge occurring twice. More generally, the *s-set graph* of a hypergraph H , denoted $G^{(s)}(H)$ has the vertex set $\binom{V}{s}$, the set of subsets of V with size s , where $\{\{a_1, \dots, a_s\}, \{b_1 \dots b_s\}\}$ is an edge of $G^{(s)}(H)$ if and only if $\{a_1, \dots, a_s\}$ and $\{b_1 \dots b_s\}$ are disjoint, and $\{a_1, \dots, a_s\} \cup \{b_1 \dots b_s\} \subset f$ for some $f \in E(H)$.

For a graph G and hypergraph H , we are concerned with relating $\chi(G)$ and $\chi(H)$ with the spectrum of the adjacency matrix (or, as described in the case of H , several different adjacency matrices). For a graph G , the *adjacency matrix*, A , is the $|V(G)| \times |V(G)|$ matrix where the entry A_{ij} is the number of edges between i and j . A weighted adjacency matrix of a

graph, denoted by \mathbf{W} , is a matrix that satisfies $\mathbf{W}_{ij} = 0$ whenever $\{i, j\} \notin E(G)$. For our purposes, we allow for the possibility of negative weights. We denote the largest and smallest eigenvalues of a (real symmetric) matrix \mathbf{M} , respectively as $\lambda_{\max}(\mathbf{M})$ and $\lambda_{\min}(\mathbf{M})$. In the context of hypergraphs, we will let λ_{\min} denote the minimum eigenvalue of the underlying graph as described above, and we will let $\lambda_{\min}^{(s)}$ denote the minimum eigenvalue of the adjacency matrix for the s -subset graph.

Some of our proofs are probabilistic in nature. The key fact we use is that if a real-valued random variable X obeys $m \leq X \leq M$ almost surely for some constants m, M , then $m \leq \mathbb{E}[X] \leq M$ where $\mathbb{E}[X]$ denotes the expectation of X . In particular, we combine matrix theory and probability in our application of the Courant-Fischer Theorem [9] in the following way:

If $\mathbf{x} \in \mathbb{C}^n$ is a non-zero random vector, and \mathbf{A} is a real-symmetric $n \times n$ matrix, then $\lambda_{\min} \leq \frac{\mathbb{E}[\mathbf{x}^H \mathbf{A} \mathbf{x}]}{\mathbf{x}^H \mathbf{x}} \leq \lambda_{\max}$.

3 Main results

We prove the following theorems.

Using a probabilistic proof, we will prove Lovász's theorem:

Theorem 1. (Hoffman-Lovász 1979) [11] *For any weighted adjacency matrix (even with negative weights), \mathbf{W} , of a graph G*

$$\chi(G) \geq 1 - \frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})}$$

Specifically,

$$\chi(G) \geq \max_{\mathbf{W}} \left(1 - \frac{\lambda_{\max}(\mathbf{W})}{\lambda_{\min}(\mathbf{W})} \right)$$

where the maximum is taken over all weighted adjacency matrices of G

Notice that this is one side of the "sandwich theorem" in [11] with regard to the Lovász Theta Function.

We then adapt the method to prove an analogous result if we allow a certain proportion of monochromatic edges:

Lemma 1. *Let G be a graph (perhaps with multiedges) with average degree \bar{d} . For $0 \leq p \leq 1$, if the vertices of G can be (improperly) colored with k -colors such that number monochromatic edges is at most $p|E|$, then,*

$$k \geq \frac{1 - \frac{\lambda_{\min}(\mathbf{A})}{\bar{d}}}{p - \frac{\lambda_{\min}(\mathbf{A})}{\bar{d}}} \text{ or more simply, } p \geq \frac{\bar{d} + k\lambda_{\min}(\mathbf{A}) - \lambda_{\min}(\mathbf{A})}{\bar{d}k}$$

Where $\lambda_{\min}(\mathbf{A})$ is the smallest eigenvalue of the adjacency matrix of G .

We remark that the first inequality above, while more complex, explicitly describes the role of p in the result. In particular, if G is regular, Hoffman's theorem results with $p = 0$. Further, if $p = 1$, then the inequality allows for the graph to be colored with 1 color.

Finally, we apply the previous lemma to produce several results for hypergraphs.

Theorem 2. *Let H be a 2-colorable 3-uniform hypergraph with average degree \bar{d} . Then,*

$$\bar{d} \leq -\frac{3}{2}\lambda_{\min}$$

Theorem 3. *Let H be a 2-colorable 4-uniform hypergraph on n vertices with average degree \bar{d} . Then,*

$$\bar{d} \leq -2\lambda_{\min} - \frac{n-1}{3}\lambda_{\min}^{(2)}$$

Theorem 4. *Let H be a 2-colorable 5-uniform hypergraph on n vertices with average degree \bar{d} . Then,*

$$\bar{d} \leq -\frac{5}{2}\lambda_{\min} - \frac{5(n-1)}{12}\lambda_{\min}^{(2)}$$

4 Proofs of Theorem 1 and Lemma 1

We begin with a proof of Lovász's Theorem in order to demonstrate our technique. We remark that a derandomized variant of this proof can be found in [12]. However, we build upon this randomized proof later.

Proof of Theorem 1. For any simple graph G , let \mathbf{W} denote any weighted adjacency matrix for G . Suppose G is k -colorable. We will denote the maximum and minimum eigenvalues of \mathbf{W} as λ_{\max} and λ_{\min} respectively. Let \mathbf{z} denote a real unit eigenvector corresponding to λ_{\max} . We let $g: V \rightarrow \{1, 2, \dots, k\}$ be a k -coloring and $\rho: \{1, 2, \dots, k\} \rightarrow \mathbb{C}$ be a function assigning each number $0, \dots, k-1$ a unique k -th root of unity. If ρ is chosen randomly and uniformly from all permutations, we may let \mathbf{x} denote a random vector indexed by the colors of the vertices G defined by $\mathbf{x}_j = \mathbf{z}_j \cdot (\rho \circ g)(j)$. For any \mathbf{x} we have:

$$\lambda_{\min} \leq \frac{\mathbf{x}^H \mathbf{W} \mathbf{x}}{\mathbf{x}^H \mathbf{x}} = \mathbf{x}^H \mathbf{W} \mathbf{x}$$

So we have,

$$\begin{aligned} \lambda_{\min} &\leq \mathbb{E}[\mathbf{x}^H \mathbf{W} \mathbf{x}] \\ &= \mathbb{E} \left[\sum_{\{u,v\} \in E} \mathbf{W}_{uv} (\overline{\mathbf{x}_u} \mathbf{x}_v + \overline{\mathbf{x}_v} \mathbf{x}_u) \right] \end{aligned}$$

By linearity of expectation, we get

$$= \sum_{\{u,v\} \in E} \mathbf{W}_{uv} \mathbb{E}[\overline{\mathbf{x}_u} \mathbf{x}_v + \overline{\mathbf{x}_v} \mathbf{x}_u]$$

where the sum is over all unordered pairs of vertices that form an edge. Thus, we have,

$$= \sum_{\{u,v\} \in E} \mathbf{W}_{uv} \mathbb{E}[(\mathbf{z}_u \mathbf{z}_v) ((\rho \circ g)(u) \overline{(\rho \circ g)(v)} + (\rho \circ g)(v) \overline{(\rho \circ g)(u)})]$$

Since v is deterministic,

$$= \sum_{\{u,v\} \in E} (\mathbf{z}_i \mathbf{z}_j \mathbf{W}_{uv}) \mathbb{E}[(\rho \circ g)(u) \overline{(\rho \circ g)(v)} + (\rho \circ g)(v) \overline{(\rho \circ g)(u)}]$$

Observe that for any coloring, g , any permutation of the colors is also a coloring. Hence, the random quantity $((\rho \circ g)(u) \overline{(\rho \circ g)(v)} + (\rho \circ g)(v) \overline{(\rho \circ g)(u)})$ has an equal probability for taking on each of the possible values. Hence, $\mathbb{E}[(\rho \circ g)(u) \overline{(\rho \circ g)(v)} + (\rho \circ g)(v) \overline{(\rho \circ g)(u)}] = 2 \sum_{j=1}^{k-1} \exp(2\pi i j/k) = \frac{-2}{k-1}$. Hence, for the last sum above we get:

$$\begin{aligned} &= \frac{-1}{k-1} \sum_{\{u,v\} \in E} 2 \mathbf{z}_u \mathbf{z}_v \mathbf{W}_{uv} = \frac{-1}{k-1} \sum_{\{u,v\} \in E} \overline{\mathbf{z}_u} \mathbf{W}_{uv} \mathbf{z}_v + \overline{\mathbf{z}_v} \mathbf{W}_{uv} \mathbf{z}_u \\ &= \frac{-1}{k-1} \mathbf{z}^H \mathbf{W} \mathbf{z} = \frac{-1}{k-1} \lambda_{\max} \end{aligned}$$

Altogether we have:

$$\lambda_{\min} \leq \frac{-1}{k-1} \lambda_{\max}$$

And solving for k we get:

$$1 - \frac{\lambda_{\max}}{\lambda_{\min}} \leq k$$

Choosing $k = \chi(G)$ proves the theorem. \square

Many of the previous proofs and the Hoffman-Lovász theorems use discrete methods such as matrix partitioning and covers [7] whereas the proof above is analytic which allows for much easier generalizations such as Lemma 1.

Proof of Lemma 1. By hypothesis, suppose there is an improper k -coloring, $g: G \rightarrow \{1, 2, \dots, k\}$ with at most $p|E|$ monochromatic edges. As in the previous proof, let $\rho: \{1, 2, \dots, k\} \rightarrow \mathbb{C}$ be a function mapping each number $1, \dots, k$ to a unique k -th root of unity. We vary ρ randomly where the function ρ is chosen uniformly from all possible permutations. Let \mathbf{x} denote a random vector determined by the colors of the vertices of G defined by $\mathbf{x}_j = (\rho \circ g)(j)$. Let n denote the number of vertices of G . For any \mathbf{x} we have:

$$\lambda_{\min}(G) \leq \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{\mathbf{x}^H \mathbf{x}} = \frac{\mathbf{x}^H \mathbf{A} \mathbf{x}}{n}$$

So we have,

$$\begin{aligned} n\lambda_{\min} &\leq \mathbb{E}[\mathbf{x}^H \mathbf{A} \mathbf{x}] \\ &= \sum_{\{u,v\} \in E} \mathbf{A}_{uv} \mathbb{E}[\overline{\mathbf{x}}_u \mathbf{x}_v + \overline{\mathbf{x}}_v \mathbf{x}_u] \end{aligned}$$

where the sum goes over all unordered pairs of vertices.

The monochromatic edges contribute 1 to the sum whereas the other edges contribute on average $\frac{-1}{k-1}$. Since the proportion of edges which are monochromatic is at most p ,

$$\sum_{\{u,v\} \in E} \mathbb{E}[\overline{\mathbf{x}}_u \mathbf{x}_v + \overline{\mathbf{x}}_v \mathbf{x}_u] \leq \sum_{\{u,v\} \in E} 2 \frac{pk-1}{k-1}$$

Hence,

$$n\lambda_{\min} \leq \sum_{\{u,v\} \in E} 2A_{uv} \frac{pk-1}{k-1} = \frac{pk-1}{k-1} \sum_{\{u,v\} \in E} 2A_{uv}$$

Since $\sum_{\{u,v\} \in E} 2A_{uv} = n\bar{d}$, we have:

$$\lambda_{\min} \leq \frac{\bar{d}(pk-1)}{k-1}$$

Solving for k (keeping in mind that λ_{\min} is negative) yields the result. \square

5 Proofs of the remaining results

Proof of Theorem 2. Suppose H is a 2-colorable 3-uniform hypergraph with average degree \bar{d} . If we consider a specific 2-coloring of H , observe that for every edge in H , there must be 2 vertices of one color and 1 vertex of the other. Hence, the underlying graph of H can be 2-colored if we allow $1/3$ of the edges to be monochromatic. By applying lemma 1, to the underlying graph which has average degree $2\bar{d}$:

$$2 \geq \frac{1 - \frac{\lambda_{\min}}{2\bar{d}}}{1/3 - \frac{\lambda_{\min}}{2\bar{d}}}$$

Solving for \bar{d} yields the result. \square

Proof of Theorem 3. Suppose H is a 2-colorable 4-uniform hypergraph on n vertices. Let $g : V(H) \rightarrow \{0, 1\}$ denote a specific 2-coloring of H . Observe that for every edge in H , there are two cases: 2 vertices of each color; or 3 vertices of one color and 1 of the other. Let p denote the proportion of edges with 3 of one color and 1 of the other. Hence, the underlying graph of H can be 2-colored using g if we allow $1/3 + p/6$ of the edges to be monochromatic.

$$2 \geq \frac{1 - \frac{\lambda_{\min}}{3\bar{d}}}{1/3 + p/6 - \frac{\lambda_{\min}}{3\bar{d}}}$$

This reduces to $p \leq 1 + \frac{\lambda_{\min}}{\bar{d}}$.

Next, we consider the 2-subset graph of H , $G^{(2)}(H)$. Note that the average degree of $G^{(2)}(H)$ is $\frac{3\bar{d}}{n-1}$. Let $h : G^{(2)}(H) \rightarrow \{0, 1\}$ be an (improper) 2-coloring of $G^{(2)}(H)$ as follows: If $\{a, b\} \in V(G^{(2)}(H))$, let $h(\{a, b\}) = 0$

if $g(a) = g(b)$, and let $h(\{a, b\}) = 1$ otherwise. For this 2-coloring, the proportion of monochromatic edges is $1 - p$. Hence by applying lemma 1,

$$2 \geq \frac{1 - \frac{(n-1)\lambda_{\min}^{(2)}}{24d}}{1 - p - \frac{(n-1)\lambda_{\min}^{(2)}}{24d}}$$

This reduces to

$$\frac{1}{2} - \frac{(n-1)\lambda_{\min}^{(2)}}{6\bar{d}} \leq p$$

Altogether, we have:

$$\frac{1}{2} - \frac{(n-1)\lambda_{\min}^{(2)}}{6\bar{d}} \leq p \leq 1 + \frac{\lambda_{\min}}{\bar{d}}$$

Eliminating the intermediary p , and solving for \bar{d} yields the result. \square

Proof of Theorem 4. The proof follows with exact the same method as the proof for Theorem 3. Let p be the number of hyperedges with 3 of one color and 2 of another. Then, the underlying graph can be 2-colored with $3/5 - p/5$ of the edges as monochromatic, and the 2-subset graph can be 2-colored with at most $1/5 + 2p/5$ monochromatic edges. The average degree of the underlying graph is $4\bar{d}$, and the average degree of the 2-subset graph is $20\bar{d}/(n-1)$. The remainder follows just as the previous proof and is omitted. \square

6 Examples

The main goal of this section is to show that Theorems 2 and 3 are applicable. Specifically, we show there exists graphs which Theorems 2 and 3 indicate are not 2-colorable.

First, Theorem 2 is tight. The complete 3-uniform hypergraph on 4 vertices, K_4^3 has average degree 3, and the underlying graph is the complete on 4 vertices where edge edge has weight 2, so $\lambda_{\min} = -2$. $\frac{3}{2}\lambda_{\min} = \bar{d}$.

To show Theorem 2 is applicable, consider the complete 3-uniform graph on 5 vertices K_5^3 . For K_5^3 , the average degree $d = 6$. Also, the corresponding underlying graph is the complete graph where each edge has weight 3, so $\lambda_{\min} = -3$. Hence, $\frac{3}{2}\lambda_{\min} < \bar{d}$, and so theorem 2 indicates that K_5^3 is not 2-colorable.

We now show Theorem 3 is applicable. Let K_n^4 be the complete 4-uniform graph on n vertices. Clearly, K_n^4 is not 2-colorable for $n \geq 7$. Note that the average degree of K_n^4 is $\Theta(n^3)$. Observe that the underlying graph of K_n^4 is the complete graph on n vertices where each edge is duplicated $\binom{n-2}{2}$ times. Hence, $\lambda_{\min} = -\Theta(n^2)$. Likewise, observe that the 2-subset graph of H is a strongly regular graph on $\binom{n}{2}$ vertices where each vertex has degree $\binom{n-2}{2}$. Also two disjoint (and hence, adjacent) vertices have $\binom{n-4}{2}$ common neighbors and non-adjacent vertices have $\binom{n-3}{2}$ common neighbors. Hence, using the formula for strongly-regular graphs [6], $\lambda_{\min}^{(2)} = 6 - 2n = -\Theta(n)$. If H_n were 2-colorable for large n , then by Theorem 3, $\bar{d} = \Theta(n^3) \leq \Theta(n^2)$ which poses a contradiction. Therefore, Theorem 3 successfully excludes K_n^4 graphs from being 2-colorable for large enough n .

For a less trivial example, consider $H = (V, E)$ a 4-uniform hypergraph on 18 vertices with where $V = \{1, \dots, 18\}$, and $E = \{\{a, b, c, d\} \subset V : a + b + c + d \not\equiv 0 \pmod{3}\}$. A simple calculation in *MATLAB* shows that $\lambda_{\min}^{(2)} \approx -39.4609$ and $\lambda_{\min} = -85.0346$. Hence if H is 2-colorable, Theorem 3 requires $\bar{d} = \frac{1360}{3} \leq -2\lambda_{\min} - \frac{17}{3}\lambda_{\min}^{(2)} \approx 393.681$. Therefore, H is not 2-colorable.

7 Conclusions and Remarks

In this paper, we have established a connection between hypergraph coloring and spectra using the powerful technique of generating several graphs from one hypergraph and considering them jointly.

We remark that the technique used in Theorems 2, 3 and 4 can be used to derive a necessarily spectral condition for non-uniform hypergraphs such that $|e| \leq 5$ for any $e \in E$. The condition varies depending upon proportions of the size of the hyperedges.

There are several avenues for future work. The first would would explore q -colorings for $q > 2$, and another is to consider r -uniform graphs for $r > 5$. However, these cases present additional challenges. First, the

technique in Theorem 2 yields less and less information as q or r gets large. In fact, on its own, the technique yields a vacuous condition for $q \geq 4$ or $r \geq 5$. Second, for $q \geq 3$ or $r \geq 6$, there are more than two possibilities for the color combinations of on the hyperedges. Hence, in order to follow the technique in Theorems 3 and 4, one must balance several proportions. For example, for the case $q = 2$ and $r = 6$, one must consider the proportion of hyperedges with 1 vertex of one color and 5 of the other, 2 vertices of one color and 4 of the other, and 3 and 3. Lastly, the techniques in this paper may lend itself well toward a spectral approximation algorithm for 2-coloring 4- or 5-uniform hypergraphs.

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