PALINDROMIC AND SŪDOKU QUASIGROUPS

JONATHAN D. H. SMITH

Department of Mathematics
Iowa State University
Ames, Iowa 50011, U.S.A.
email: jdhsmith@iastate.edu

ABSTRACT. Two quasigroup identities of importance in combinatorics, Schröder's Second Law and Stein's Third Law, share many common features that are incorporated under the guise of palindromic quasigroups. A graph-theoretical technique yields a topological proof for the congruence restrictions on the spectrum of Schröder or outer palindromic quasigroups. The potential for a comparable proof applicable to Stein or inner palindromic quasigroups raises open graph-theoretical and combinatorial problems. Imposition of extra sūdoku-like conditions on Latin squares of square order, based on the coloring of so-called sūdoku graphs, leads to the concept of a sūdoku quasigroup. It is shown that the spectrum of inner palindromic sūdoku quasigroups comprises every perfect square, thereby identifying the chromatic number of each sūdoku graph.

1. INTRODUCTION

Two classical quasigroup identities, each of considerable importance in combinatorics, are

$$(xy)(yx) = x$$

— *Schröder's* (*Second*) *Law* [1, 3, 10], and

$$(xy)(yx) = y$$

— *Stein's Third Law* [1, 3, 9]. The first of the two main themes of this paper concerns the relationships between the two identities. Since the traditional eponymic names are merely confusing in the current context, it will be helpful to rename the two identities generically as *palindromic identities*, with reference to the palindromic form of their common left-hand sides. Since the single variable on the right-hand side of Schröder's Second Law is the outer variable in the palindrome, that law will be called the *outer*

2010 *Mathematics Subject Classification.* 20N05, 05B15, 05C25.

Key words and phrases. Latin square, quasigroup, Schroeder's Second Law, Stein's Third Law, palindromic quasigroup, sudoku Latin square, sudoku graph.

palindromic identity. Similarly, Stein's Third Law will be called the *inner palindromic* identity, as the single variable on its right-hand side is the inner variable in the palindrome. (As an additional mnemonic, one might note that the word "outer" begins with the vowel "o" appearing in the name Schröder, while the word "inner" begins with the vowel "i" appearing in Stein's name.)

The congruence condition

$$n \equiv 0, 1 \mod 4 \tag{1.1}$$

on the spectrum (set of orders of finite models) of outer palindromic quasigroups was obtained by Lindner *et al.* using a combinatorial analysis [10, Theorem 2]. Recently, B. Kerby and the author gave a direct topological proof for this condition [8, Corollary 7.5], a graph-theoretical variant of the "alternate proof" proposed by Norton and Stein for the idempotent case [12, Theorem 4.2]. Baker [1] proved the necessity of the congruence condition (1.1) for idempotent outer palindromic quasigroups by associating a balanced incomplete block design with parameters $k = 4$ and $\lambda = 3$ to each such quasigroup, and then quoting known restrictions for these BIBDs [5]. By similar means, he showed that idempotent, inner palindromic quasigroups are also subject to the restriction (1.1). Later, Lindner *et al.* used a combinatorial method [9, §8] to extend the restriction to arbitrary inner palindromic quasigroups. Thus the spectral condition (1.1) may be seen as a general feature of palindromic quasigroups.

Section 2 presents the graph-theoretical background to the topological approach used in [8]. A *cycle graph* C_Q, consisting entirely of disjoint cycles in the finite case, is associated with each quasigroup Q. Among all finite quasigroups of a given order $n \equiv 0, 1 \mod 4$, the outer palindromic quasigroups are characterized as those that maximize the number of cycles in the cycle graph (Theorem 2.1). Problem 2.2 raises the open question of what happens when $n \equiv 2, 3 \mod 4$. Section 3 then shows how the topological method may be used to derive the congruence restriction (1.1) on the spectrum for outer palindromic quasigroups, and formulates open problems in connection with its possible extension to inner palindromic quasigroups.

Sūdoku quasigroups, whose multiplication tables are Latin squares that satisfy additional constraints familiar from sūdoku puzzles, form the second main theme of this paper. In analogy with the graph-theoretical definition of an $n \times n$ Latin square as an n-coloring of the Cartesian product $K_n \square K_n$, Section 4 defines a *sūdoku graph* SD_r on r^4 vertices, and then specifies a sūdoku Latin square as an r^2-coloring of SD_r. The main new result of the paper, Theorem 5.6, shows that the spectrum of inner palindromic sūdoku quasigroups comprises every perfect square, and thereby identifies the chromatic number of each sūdoku graph.

2. THE CYCLE GRAPH OF A QUASIGROUP

A quasigroup $(Q, \cdot)$ is defined as a set Q that is equipped with a binary *multiplication* $x \cdot y$ or xy (which binds more strongly than $x \cdot y$), where in $x \cdot y = z$, any two of x, y, z specify the third uniquely. This unique specification may be formulated as

$$x \cdot y = z \quad \Leftrightarrow \quad x = z/y \quad \Leftrightarrow \quad y = x\backslash z$$

using supplementary binary operations z/y of *right division* and $x\backslash z$ of *left division*.[1] Since a quasigroup multiplication is not necessarily assumed to be associative, brackets proliferate. However, the stronger binding of juxtaposition may be used to reduce their number. For example, the outer and inner palindromic identities may be written as $xy{\cdot}yx = x$ and $xy{\cdot}yx = y$ respectively. The body of the multiplication table of a quasigroup is a Latin square. Conversely, each Latin square may be given row and column labels to make it the multiplication table of a quasigroup. For a recent treatment of quasigroups, see [13].

Consider the group $G = \langle t_1, t_2, t_3 \mid t_1^2 = t_2^2 = t_3^2 = 1 \rangle$, the free product of three cyclic groups of order 2. It may be implemented as the set of words in the alphabet $\{t_1, t_2, t_3\}$ without repeated letters. The identity element is the empty word. Inversion of a word reverses it. The product of two words is their juxtaposition, with cancelation of any repeated letters, e.g. $t_1t_2t_1 \cdot t_1t_3 = t_1t_2t_3$. The group G has a right action on the index set $I = \{1, 2, 3\}$ by the transpositions $t_1 = (2\ 3), t_2 = (3\ 1), t_3 = (1\ 2)$ — so that t_i fixes i for $1 \leq i \leq 3$.

The *marked multiplication table* M_Q of a quasigroup Q is defined to be the set $M_Q = \{(x, y, z, i) \in Q^3 \times I \mid xy = z\}$, of size $3|Q|^2$. The group G has a right action on M_Q with:

$$\begin{aligned}(x, y, z, i)t_1 &= (y/z, z, y, it_1)\,;\\ (x, y, z, i)t_2 &= (z, z\backslash x, x, it_2)\,;\\ (x, y, z, i)t_3 &= (y, x, y \cdot x, it_3).\end{aligned}$$

The undirected Cayley graph of this action is denoted by Γ_Q. A *stabilizing edge* in Γ_Q is an edge of the form

$$(\ ,\ ,i) \xrightarrow{\ t_i\ } (\ ,\ ,i)$$

for $1 \leq i \leq 3$. Then the *cycle graph* C_Q of Q is the subgraph of Γ_Q obtained by removing all the stabilizing edges. If Q is finite, the cycle graph is a union of disjoint cycles

$$\cdots\ (\ ,\ ,2) \xrightarrow{t_1} (\ ,\ ,3) \xrightarrow{t_2} (\ ,\ ,1) \xrightarrow{t_3} (\ ,\ ,2)\ \cdots$$

[1]The notation for the divisions should be familiar to users of MATLAB®.

[8, Proposition 3.1]. The *cycle number* $\sigma(C_Q)$ or $\sigma(Q)$ of the quasigroup Q is defined as the number of (connected) components (cycles for finite Q) in the cycle graph C_Q. The following result shows how outer palindromic quasigroups maximize the cycle number.

Theorem 2.1. [8, Theorem 4.1] *Let Q be a quasigroup of finite order n.*

(a) *The cycle number of Q satisfies the inequality*

$$\sigma(C_Q) \leq n^2 . \tag{2.1}$$

(b) *Equality obtains in* (2.1) *iff Q is outer palindromic.*

Since equality in (2.1) is possible iff n lies in the spectrum of outer palindromic quasigroups, the following problem is raised.

Problem 2.2. For $n \equiv 2, 3 \mod 4$:

(a) Determine the maximum possible cycle number for a quasigroup of order n;
(b) Characterize those quasigroups which achieve the maximum.

Remark 2.3. By [8, Proposition 4.7], the unique (quasi)group of order 2 has cycle number 3.

3. The dual complex

The cycle graph of a quasigroup Q yields a two-dimensional cell complex which is the basis for the combinatorial topology used in [8] to derive the congruence restriction (1.1). The set of 0-dimensional cells ("vertices") is the *unmarked multiplication table*

$$V = \{(x, y, xy) \mid x, y \in Q\} .$$

From now on, assume that Q has finite order n. Then $|V| = n^2$.

The *unmarking projection* is defined as

$$M_Q \to V; (x, y, z, i) \mapsto (x, y, z) .$$

Consider a cycle of C_Q. Its vertices are certain marked triples. Unmarking these vertices induces a quotient graph with loops. Deletion of the loops leaves a cycle, known as a *collapsed cycle.* Define the set F of 2-dimensional cells ("faces") as the set of all collapsed cycles (including mere points arising from idempotents of Q). Then $|F| = \sigma(Q)$.

Altogether, the unmarking projection induces a quotient graph of C_Q on the vertex set V. Delete all loops from this quotient graph (as discussed above for the individual cycles), and let E denote the set of remaining edges. This is the set of 1-cells in the complex. Then $|E| = 3n(n-1)/2$ [8, Proposition 5.4(b)]. Boundaries and an orientation are defined in [8, §6] to make a complex $F \to E \to V$, known as the *dual complex* since it is dual to the complex originally constructed in the idempotent case by Norton and Stein [11], and more recently extended to the general case [7].

For each natural number n, the *triangular number* $T(n) = n(n+1)/2$ is the number of elements in a triangle with n in the base layer, $n-1$ in the next layer, ..., up to 1 at the apex.

Theorem 3.1. [8, Theorem 7.3] *For a quasigroup Q of finite order n, the cycle number $\sigma(C_Q)$ of Q is congruent to $T(n)$ modulo 2.*

Proof. Since the dual complex is oriented, it has even Euler characteristic $|F|-|V|+|E| = \sigma(Q) - 3n(n-1)/2 + n^2$, which is congruent modulo 2 to $\sigma(Q) - T(n)$. □

Corollary 3.2. [8, Corollary 7.5] *If Q is an outer palindromic quasigroup of finite order n, then n is congruent to 0 or 1 modulo 4.*

Proof. By Theorem 2.1, $\sigma(Q) = n^2$, while by Theorem 3.1, one has $\sigma(C_Q) \equiv n(n+1)/2 \mod 2$. However, $n^2 \equiv n(n+1)/2 \mod 2$ if and only if n is congruent to 0 or 1 modulo 4. □

The topological proof of Corollary 3.2, extending the "alternate proof" proposed by Norton and Stein for the idempotent case [12, Theorem 4.2], contrasts with the combinatorial analysis that was given by Lindner *et al.* [10, Theorem 2], and raises the following question.

Problem 3.3. Can a topological/graph-theoretical proof be found for the restriction $n \equiv 0, 1 \mod 4$ on the spectrum of inner palindromic quasigroups?

Preliminary investigations have hitherto revealed no special features of cycle graphs of inner palindromic quasigroups that might be exploited for a solution to Problem 3.3. However, it is conceivable that a different graph might form the basis for a comparable approach.

While the precise spectrum for inner palindromic quasigroups comprises all positive integers congruent to 0 or 1 modulo 4, the spectrum for outer palindromic quasigroups excludes 5 from that set [2, Table III.2.42]. There is a better match in the spectra of idempotent palindromic quasigroups: the inner case excludes 4 and 8, while the outer case excludes 5 and 9 [2, Table III.2.43]. This correspondence suggests the following problem, where a positive solution might look like an elaborate analogue of the well-known relationship between idempotent quasigroups of some finite order h and unipotent loops of order $n+1$ [13, Proposition 1.5].

Problem 3.4. Is the existence of an inner palindromic quasigroup having order congruent to 0 modulo 4 directly related to the existence of an outer palindromic quasigroup of order congruent to 1 modulo 4?

4. SŪDOKU GRAPHS

For each element n of the set $\mathbb{N}$ of natural numbers (which is taken to include 0), define $\underline{n} = \{i \in \mathbb{N} \mid i < n\}$ as the model n-element set. For $a < b \in \mathbb{N}$, define the *half-open interval* $[a, b) = \{i \in \mathbb{N} \mid a \leq i < b\}$.

For a vertex set V and a subset S of the power set 2^V, a graph V_S is determined by the specification that a clique is induced on each element of S, and that there are no further edges. For example, on the vertex set $V = \underline{n} \times \underline{n}$ with a positive integer n, the subset

$$S = \big\{\{i\} \times \underline{n},\ \underline{n} \times \{j\} \bigm| i, j < n\big\} \tag{4.1}$$

of 2^V yields the Cartesian product $K_n \square K_n$ as V_S. In order to relate to Latin square language for n-colorings of $K_n \square K_n$, the subsets $\{i\} \times \underline{n}$ of V appearing in (4.1) are called *rows*, while the subsets $\underline{n} \times \{j\}$ are called *columns*.

Now consider a positive integer r. Set $n = r^2$, and take $V = \underline{n} \times \underline{n}$. Define the subset $S =$

$$\Big\{\{i\} \times \underline{n},\ \underline{n} \times \{j\},\ [rk, rk + r) \times [rl, rl + r) \Bigm| i, j < n;\ k, l < r\Big\} \tag{4.2}$$

of 2^V. Then the *sūdoku graph* SD_r is defined as V_S in this case. As in the previous example, the subsets $\{i\} \times \underline{n}$ of V appearing in (4.2) are called *rows*, while the subsets $\underline{n} \times \{j\}$ are called *columns*. The subsets $[rk, rk+r) \times [rl, rl+r)$ are called *regions*, while their intersections with the rows and columns are known respectively as *subrows* and *subcolumns*.

Proposition 4.1. *For a positive integer r, consider the sūdoku graph SD_r on r^4 vertices.*

(a) *The graph SD_r is regular, of valency $(3r + 1)(r - 1)$.*
(b) *The graph SD_r contains $\frac{1}{2}r^4(3r + 1)(r - 1)$ edges.*
(c) *The chromatic number $\chi(SD_r)$ of SD_r is at least r^2.*

Proof. The sūdoku graph SD_r consists of r^2 cliques K_{r^2} for the rows, r^2 cliques for the columns, and a further clique for each of the r^2 regions. In total, this would give $3r^2T(r^2 - 1)$ edges. However, each region contains r subrows and r subcolumns. Thus the edges of $2r$ cliques K_r are counted twice in each of the r^2 regions. In other words, $2r^3T(r - 1)$ edges have been counted twice. Thus the exact number of edges is

$$\begin{aligned} 3r^2\binom{r^2}{2} - 2r^3\binom{r}{2} &= \frac{r^4}{2}\left(3(r^2 - 1) - 2(r - 1)\right) \\ &= \frac{r^4}{2}(3r^2 - 2r - 1), \end{aligned}$$

as required for (b). Since the edges are distributed evenly over the r^4 vertices, (a) follows. Finally, the cliques K_{r^2} in SD_r witness (c). $\square$

Remark 4.2. In Corollary 5.7 below, it will be noted that $\chi(SD_r) = r^2$. Compare [6, Theorem 3].

Definition 4.3. (a) A *sūdoku Latin square* is defined as an r^2-coloring of SD_r, for some positive integer r known as the *radix*. (As usual, the square r^2 is described as the *order* or *size*.) The colored rows, columns, and regions of SD_r are called the respective *rows*, *columns*, and *regions* of the sūdoku Latin square.
(b) A sūdoku Latin square of order $n = r^2$ is described as *standard* if the coloring set is $\underline{n}$.

As with conventional Latin squares, one may interpret a sūdoku Latin square as an actual square matrix, wherein the regions correspond to submatrices (compare [4, §1.9]).

Example 4.4. The body of Table 4.1 displays a standard sūdoku Latin square of radix 2, with four regions corresponding to the marked 2×2 squares. This sūdoku Latin square will be denoted by T.

T	0	1	2	3
0	0	1	3	2
1	2	3	1	0
2	1	0	2	3
3	3	2	0	1

TABLE 4.1. A sūdoku Latin square and quasigroup.

5. SŪDOKU QUASIGROUPS

Definition 5.1. (a) A *standard sūdoku quasigroup* is a quasigroup where:

(1) The body of the multiplication table is a standard sūdoku Latin square;
(2) The row $\{i\} \times \underline{n}$ is labeled by i, for each natural number $i < n$, and
(3) The column $\underline{n} \times \{j\}$ is labeled by j, for each natural number $j < n$.

(b) A *sūdoku quasigroup* is a quasigroup that is isomorphic to a standard sūdoku quasigroup.

Example 5.2. Table 4.1 portrays a standard sūdoku quasigroup of order 2^2. In accord with Example 4.4, this sūdoku quasigroup is also denoted by

T. As the following calculations show:

$$\begin{aligned} 10 \cdot 01 = 21 = 0 = 13 = 20 \cdot 02\,; &\quad 20 \cdot 02 = 13 = 0 = 00 = 00 \cdot 00\,; \\ 21 \cdot 12 = 01 = 1 = 20 = 31 \cdot 13\,; &\quad 31 \cdot 13 = 20 = 1 = 33 = 11 \cdot 11\,; \\ 32 \cdot 23 = 03 = 2 = 31 = 02 \cdot 20\,; &\quad 02 \cdot 20 = 31 = 2 = 22 = 22 \cdot 22\,; \\ 03 \cdot 30 = 23 = 3 = 02 = 13 \cdot 31\,; &\quad 13 \cdot 31 = 02 = 3 = 11 = 33 \cdot 33\,, \end{aligned}$$

the quasigroup T is inner palindromic (satisfies Stein's Third Law).

Proposition 5.3. *Suppose that L and R are standard sūdoku quasigroups, with respective orders $m = r^2$ and $n = s^2$. Then the direct product $L \times R$ is a sūdoku quasigroup.*

Proof. As a direct product of two quasigroups, $L \times R$ is certainly a quasigroup. Consider the regions $L_{ij} = [ri, ri+r) \times [rj, rj+r)$ of L for $i, j < r$ and $R_{kl} = [sk, sk+s) \times [sl, sl+s)$ of R for $k, l < s$. Since L is a sūdoku quasigroup, the set $[ri, ri+r) \cdot [rj, rj+r)$ of products in each region L_{ij} contains all $m = r^2$ elements of the set L. Similarly, the set $[sk, sk+s) \cdot [sl, sl+s)$ of products in each region R_{kl} contains all $n = s^2$ elements of the set R. Take regions

$$\big([ri, ri+r) \times [sk, sk+s)\big) \times \big([rj, rj+r) \times [sl, sl+s)\big)$$

of $L \times R$ for $i, j < r$ and $k, l < s$. Since the set

$$\big([ri, ri+r) \times [sk, sk+s)\big) \cdot \big([rj, rj+r) \times [sl, sl+s)\big)$$

of products in such a region contains

$$\begin{aligned} &\Big|\big([ri, ri+r) \times [sk, sk+s)\big) \cdot \big([rj, rj+r) \times [sl, sl+s)\big)\Big| \\ &= \big|[ri, ri+r) \cdot [rj, rj+r)\big| \cdot \big|[sk, sk+s) \cdot [sl, sl+s)\big| \\ &= mn = |L \times R| \end{aligned}$$

elements, it follows that $L \times R$ is a sūdoku quasigroup. □

Suppose that $(A, +, 0)$ is an additive abelian group of finite odd order $r = 2q + 1$. For each element a of A, define $\frac{1}{2}a = (q+1)a$, so that $\frac{1}{2}a + \frac{1}{2}a = (2q+2)a = (2q+1)a + a = a$. Similarly, write $\frac{1}{4}a = \frac{1}{2}\frac{1}{2}a$, so that $a = \frac{1}{2}2a = \frac{1}{4}4a$ for each a in A.

Proposition 5.4. *Define a product*

$$(a, b) \cdot (c, d) = \frac{1}{2}\,(a + b + c - d, -a + b + c + d)$$

on the direct square of an additive abelian group A of finite odd order. Then $(A^2, \cdot)$ is an inner palindromic sūdoku quasigroup, with a region

$$A^2_{(a,c)} = \big\{(a, b) \cdot (c, d) \bigm| b, d \in A\big\} \tag{5.1}$$

for each element (a, c) of A^2.

Proof. For elements (a, b), (c, d), and (e, f) of A^2, one has

$$(a,b) \cdot (c,d) = (e,f)$$

$$\Leftrightarrow \quad (a,b) = (d+e-f, -c+e+f) =: (e,f)/(c,d) \tag{5.2}$$

$$\Leftrightarrow \quad (c,d) = (-b+e+f, a-e+f) =: (a,b)\backslash(e,f)\,, \tag{5.3}$$

so that $(A^2, \cdot)$ is a quasigroup $(A^2, \cdot, /, \backslash)$ with divisions defined by (5.2) and (5.3). Furthermore,

$$\begin{aligned}
&(a,b)(c,d) \cdot (c,d)(a,b) \\
&= \frac{1}{2}((a+b+c-d, -a+b+c+d) \cdot (c+d+a-b, -c+d+a+b)) \\
&= \frac{1}{4}(4c, 4d) = (c,d)\,,
\end{aligned}$$

so that $(A^2, \cdot)$ is inner palindromic. Finally, each element (e, f) of A^2 appears in the region $A^2_{(a,c)}$ of (5.1) as $(a,b) \cdot (c,d)$ with $b = -c + e + f$ (comparing the second components in (5.2) above) and $d = a - e + f$ (comparing the second components in (5.3) above). Thus $(A^2, \cdot)$ is a sūdoku quasigroup. □

Example 5.5. Consider the additive group $A = (\mathbb{Z}/_3, +, 0)$ of integers modulo 3. By Proposition 5.4, there is an inner palindromic sūdoku quasigroup $(A^2, \cdot)$ of order 9, whose multiplication table is displayed in Table 5.1. The map $A^2 \to \underline{9}; (a,b) \mapsto 3a + b$ has been applied to "standardize" the quasigroup.

	0	1	2	3	4	5	6	7	8
0	0	5	7	8	1	3	4	6	2
1	8	1	3	4	6	2	0	5	7
2	4	6	2	0	5	7	8	1	3
3	7	0	5	3	8	1	2	4	6
4	3	8	1	2	4	6	7	0	5
5	2	4	6	7	0	5	3	8	1
6	5	7	0	1	3	8	6	2	4
7	1	3	8	6	2	4	5	7	0
8	6	2	4	5	7	0	1	3	8

TABLE 5.1. An order 9 inner palindromic sūdoku quasigroup.

Theorem 5.6. *For each positive integer r as radix, there is an inner palindromic sūdoku quasigroup of order r^2.*

Proof. Suppose $r = 2^e d$ for an odd integer d. Consider the additive group $A = (\mathbb{Z}/_d, +, 0)$ of integers modulo d. By Proposition 5.4, there is an inner palindromic sūdoku quasigroup $(A^2, \cdot)$ of order d^2. Now Example 5.2 and inductive application of Proposition 5.3 yield an inner palindromic sūdoku quasigroup T^e of order 2^{2e}. By Proposition 5.3 again, $T^e \times (A^2, \cdot)$ is an inner palindromic sūdoku quasigroup of order $2^{2e}d^2 = r^2$. □

Corollary 5.7. *For each positive integer r, the chromatic number $\chi(SD_r)$ of the sūdoku graph SD_r is r^2.*

Proof. By Theorem 5.6, $\chi(SD_r) \leq r^2$. On the other hand, it was noted in Proposition 4.1(c) that $\chi(SD_r) \geq r^2$. □

References

[1] R.D. Baker, *Quasigroups and tactical systems*, Aequationes Math. **18** (1978), 296–303.

[2] F.E. Bennett and C.C. Lindner, *Quasigroups*, pp. 152–160 in "Handbook of Combinatorial Designs, Second Edition" (C.J. Colbourn and J.H. Dinitz, eds.), Chapman and Hall/CRC, Boca Raton, FL, 2007.

[3] F.E. Bennett and L. Zhu, *Conjugate-orthogonal Latin squares and related structures*, pp. 41–96 in "Contemporary Design Theory" (J.H. Dinitz and D.R. Stinson, eds.), Wiley, New York, NY, 1992.

[4] C.J. Colbourn, J.H. Dinitz and I.M. Wanless, *Latin squares*, pp. 135–152 in "Handbook of Combinatorial Designs, Second Edition" (C.J. Colbourn and J.H. Dinitz, eds.), Chapman and Hall/CRC, Boca Raton, FL, 2007.

[5] H. Hanani, *The existence and construction of balanced incomplete block designs*, Ann. Math. Statist. **32** (1961), 361–385.

[6] A.M. Herzberg and M.R. Murty, *Sudoku squares and chromatic polynomials*, Notices of the Amer. Math. Soc. **6** (2007), 708–717.

[7] B. Kerby and J.D.H. Smith, *Quasigroup automorphisms and the Norton-Stein complex*, Proc. Amer. Math. Soc. **138** (2010), 3079–3088.

[8] B. Kerby and J.D.H. Smith, *A graph-theoretic approach to quasigroup cycle numbers*, J. Combin. Theory Ser. A **118** (2011), 2232-2245.

[9] C.C. Lindner, R.C. Mullin and D.G. Hoffman, *The spectra for the conjugate invariant subgroups of $n^2 \times 4$ orthogonal arrays*, Can. J. Math. **32** (1980), 1126–1129.

[10] C.C. Lindner, N.S. Mendelsohn and S.R. Sun, *On the construction of Schroeder quasigroups*, Discrete Math. 32 (1980), 271–280.

[11] D.A. Norton and S.K. Stein, *An integer associated with Latin squares*, Proc. Amer. Math. Soc. **7** (1956), 331–334.

[12] D.A. Norton and S.K. Stein, *Cycles in algebraic systems*, Proc. Amer. Math. Soc. **7** (1956), 999–1004.

[13] J.D.H. Smith, *An Introduction to Quasigroups and Their Representations*, Chapman and Hall/CRC, Boca Raton, FL, 2007.