Unit Stack Visibility Graphs

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Abstract

Bar visibility graphs (BVG) are graphs whose vertices can be assigned disjoint horizontal line segments in the plane so that adjacent vertices correspond to pairs of bars that are visible to each other via an unobstructed, vertical band of visibility. A k-stack layout of a graph is a linear vertex ordering and a k-edge coloring such that each color class avoids crossing edges with respect to the linear order. BVG's and stack layouts were introduced separately in the 1970's and have many applications including testing circuit boards, VLSI design, and graph drawing. Motivated by applications to carousel navigation design, we introduce a hybrid class of graphs called unit stack visibility graphs and give a combinatorial characterization of these graphs. We leave open the problem of determining whether a polynomial-time algorithm exists to recognize unit stack visibility graphs.

1 Introduction

We consider finite simple graphs that can be represented by vertical visibilities between disjoint horizontal line segments in the plane. More precisely, a graph is a *bar-visibility graph*, or BVG for short, if each of its vertices can be assigned to a distinct horizontal line segment in the plane — disjoint from other such line segments — so that adjacent vertices of the graph correspond to pairs of line segments with a non-degenerate, vertical rectangle of visibility unobstructed by line segments representing other vertices. Bar-visibility graphs were independently characterized by Wismath [16] and Tommassia and Tollis [11] as those graphs that have a planar embedding in which all cut vertices lie on the same face.

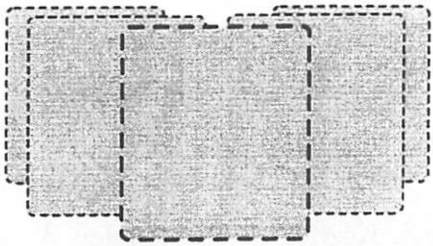

Figure 1: A typical carousel navigation layout.

Visibility representations of graphs have been extensively studied because of their many applications to testing circuit boards, VLSI design, hidden-surface elimination problems, data layout diagrams, customer visibilities/store layout, and graph drawing. Ian Stewart's article [10] gives an easy and well-motivated application inspired by Garey, Johnson, and So's paper [6]. An application to floor plans, for example, can be found in the article by Wimer, Koren, and Cederbaum [14].

We are motivated by applications to carousel navigation design. Currently popular touch-screen devices ask users to navigate through a list of selections and present this list using a carousel navigation layout (see Figure 1 for a typical carousel layout). These layouts can be modeled using bar visibility representations of unit stack visibility graphs that we introduce in this paper. A fundamental problem in this area is to determine which binary relations between navigation choices can be effectively presented using this type of layout; that is, which graphs are unit stack visibility graphs. More generally, the efficient design of human-friendly database navigation tools poses a rich and challenging area that will remain a core issue in the production of popular electronic devices. An important role in developing this area will be played by visibility representations, specifically bar representations of graphs.

Bar representations of graphs have been generalized in many directions including higher dimensions [2], other surfaces [12], other types of visibilities [3, 5], vertices represented by objects other than bars [7], and restricted bar lengths or fixed endpoints [2]. As an example of restricted bar lengths consider the following family of graphs introduced by Dean and Veytsel [4]. A unit bar visibility graph is a BVG in which all bars have unit length. Dean and Veytsel argue that this is a more realistic model than general bar visibility graphs because unrestricted bar representations may use bars of arbitrary, and thus impractical, size. Applications often require a representation using bars of similar size (e.g. circuit board and carousel navigation design). Unit bar visibility graphs have since been studied well, but a polynomial-time recognition algorithm for them remains elusive (see [13] for more references). In Section 4 we adopt a unit-bar model as part of the hybrid class of graphs we call 'unit stack visibility graphs' that also blend ideas that arise from stacks.

A k-stack layout of a graph is a linear vertex ordering and a k-edge coloring such that each color class avoids crossing edges with respect to the linear order. Graphs with a k-stack layout are called k-stack graphs. These graphs were introduced by Ollman [9] and have been studied extensively because of their many applications to sorting permutations, VLSI design, compact routing tables and graph drawing. Analogues to queues and arches have also been introduced and studied. Bernhart and Kainen [1] proved that the 1-stack graphs are precisely the outerplanar graphs, which can be recognized in polynomial-time via Mitchell's algorithm [8]. Bernhart and Kainen [1] also characterized the 2-stack graphs as the subgraphs of planar Hamiltonian graphs, which, by a result of Wigderson [15], are NP-complete to recognize.

In Section 3 we introduce and characterize 'one-stack visibility' graphs. These graphs have been studied before, but their analysis here is included to emphasize the connection between bar representations and stacks. We quickly consider the next logical generalization, the 'stack visibility graphs', but prove that they are essentially the 2-stack graphs, which we noted above are NP-complete to recognize. In an attempt to mitigate this computational complexity, Section 4 introduces 'unit stack visibility' graphs, a hybrid family of unit bar visibility and stack graphs. We prove a combinatorial characterization of these graphs that reveals the intertwined pair of stacks at their core. However, the characterization leaves open the problem of determining whether a polynomial-time algorithm exists to recognize unit stack visibility graphs.

2 Preliminaries

This section summarizes common definitions for the reader's reference. The *neighborhood* of a vertex v in the graph G is the set of neighbors v; it is denoted $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. If P is a linear ordering of the vertices of G, then $u <_P v$ signifies that vertex u precedes v in P; as usual, $u \leq_P v$ is short for $u <_P v$ or $u = v$. The open interval between vertices u and v is defined as $P(u, v) = \{w \in V(G) : u <_P w <_P v\}$; $P(u, u)$ is the empty set. The halfopen intervals $P(u,v]$, $P[u,v)$ and the closed interval $P[u,v]$ are defined in a like manner. It is convenient to define $P^+(v) = \{w \in V(G) : v <_P w\}$. Similarly define $P^-(v), P^+[v], P^-[v]$. The set of forward neighbors of v is defined as $N_P^+(v) = N_G(v) \cap P^+(v)$. The set of backwards neighbors of v is $N_P^-(v) = N_G(v) \cap P^-(v)$ with appropriate modifications of these for their closed counterparts. Set $E_P^+(v) = \{e \in E(G) \setminus E(P) : e = vu \text{ and } v <_P u\}$. Similarly define $E_P^-(v)$. Observe that $E_P^+(v)$ does not contain edges of P. Subscripts on all of these notations may be omitted if context alleviates any ambiguity.

Given a linear ordering P of $V(G)$ and an edge e, let $\ell(e)$ and $r(e)$ denote, respectively, the left and right vertices of e; so $\ell(e) <_P r(e)$. Two edges $e, f \in E(G)$ *cross* (with respect to P) if $\ell(e) <_P \ell(f) <_P r(e) <_P r(f)$. Two edges that do not cross are *non-crossing* edges. A *stack* is a subset of pairwise non-crossing edges. If $r(e) \leq_P \ell(f)$, then we write $e \leq_P f$. Two edges e and f are *comparable* if $e \leq_P f$ or $f \leq_P e$.

A(n) (outer)plane graph is a graph together with a(n) (outer)planar embedding. A *Hamiltonian path* in a graph is a path spanning all vertices. An *external Hamiltonian path* of a plane graph is a Hamiltonian path in which consecutive vertices of the path determine an edge on the external face of the given planar embedding.

3 One-stack visibility graphs

In this section we introduce and characterize a family of graphs called one-stack visibility graphs. These graphs have been characterized before under other names, for example "representation index $1 + \frac{1}{2}$"-graphs [2] and "semi-bar" graphs [5]. Here we reprove a familiar characterization of these graphs in a style that foreshadows the characterization of unit stack visibility graphs in the next section. Because these graphs exhibit the fundamentally planar- and stack-like properties important in the study of unit stack visibility graphs, these graphs make a natural and important place to begin our study.

A graph is a *one-stack visibility graph* if it is a BVG with a bar visibility representation in which the x-axis projections of all of the bars share a common right-hand endpoint. Figure 2 depicts a bar visibility representation and the corresponding one-stack visibility graph.

An outerplanar graph is *externally traceable* if it admits an outerplanar embedding with an external Hamiltonian path. Note that even though an externally traceable graph is necessarily outerplanar, not all outerplanar graphs are externally traceable. Indeed not all traceable outerplanar graphs

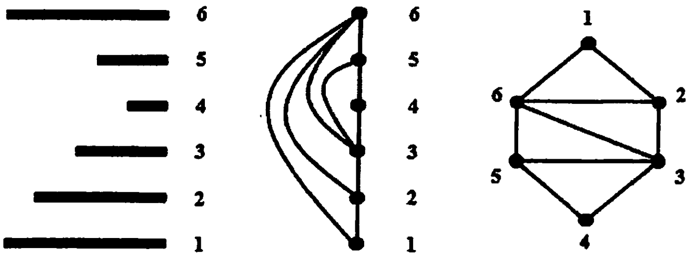

Figure 2: Three representations of the same one-stack visibility graph.

are externally traceable (see Figure 4 for an example).

Theorem 1 *A graph is a one-stack visibility graph if and only if it is an externally traceable outerplanar graph.*

Proof. Suppose that G is a one-stack visibility graph and consider a one-stack visibility representation of G in which all bars share a common right-hand endpoint. Bars of consecutive height are mutually visible and correspond to a Hamiltonian path in G. Now connect each pair of vertically visible bars by an edge within the corresponding visibility rectangle. All of these edges are disjoint and remain internally disjoint after contracting each bar to its right-hand endpoint. Thus these bar contractions produce externally traceable outerplanar embedding of G.

Conversely, suppose that G is a graph with an externally traceable outerplanar embedding. Let $P = v_1 v_2 \ldots v_n$ be an external Hamiltonian path of such an embedding. Because P traces the external face of the planar embedding, edges of G do not cross; that is, the edges of G form a stack with respect to the linear ordering of $V(G)$ determined by P.

We seek to create a one-stack visibility representation of G in which each vertex $v_i \in P$ is assigned a horizontal bar whose vertical coordinate is i and whose projection onto the x-axis is a subinterval of the unit interval $[0, 1]$ with right-hand endpoint 1. It suffices then to define, for each vertex v, the left-hand endpoint, $\lambda(v)$, of the corresponding bar b_v. First define, for each vertex $v \neq v_1$, the first neighbor $f(v)$ of v along P; that is, $f(v) \in N^-(v)$ with the property that $f(v) \leq_P w$, for all $w \in N(v)$. Second define a rank function for the vertices in the forward neighborhood of a vertex u this way: for $v \in N^+(u)$, the rank of v is $\text{rank}_u(v) = |N^+(u) \cap P^+[v]|$. So, for

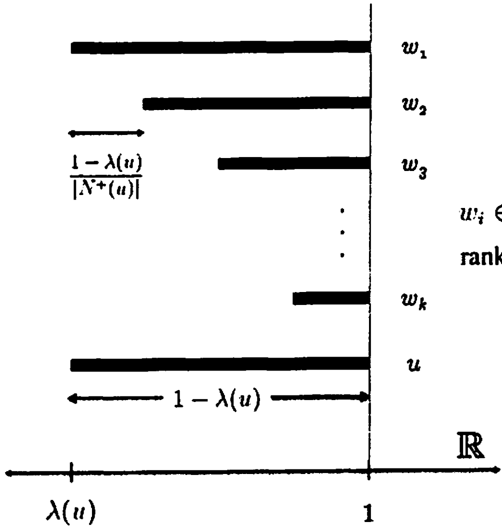

Figure 3: Assignment of bar lengths to $N^+(u)$.

example, $\text{rank}_u(v) = 1$ if and only if v is the last neighbor of u along P. Also $\text{rank}_u(v) = |N^+(u)|$ if and only if v is the successor of u on P. We can now recursively define function λ which determines the left-hand endpoints of the bars. Set $\lambda(v_1) = 0$. If $i > 0$, set $u = f(v_i)$ and define

$$\lambda(v_i) = \lambda(u) + \left(\frac{1-\lambda(u)}{|N^+(u)|}\right)(\text{rank}_u(v_i) - 1). \tag{1}$$

Figure 3 illustrates the assignment of bar lengths to $N^+(u)$ using the definition of λ. Observe that, except in the case when $i = 1$, the value of $\lambda(v_i)$ is determined by the rank of v_i in the forward neighborhood of $u = f(v_i)$; in particular, $\lambda(v_i) \geq \lambda(u)$. Moreover, $\lambda(v_i) < 1$, for all $i = 1, \ldots, n$, because $\text{rank}_u(v_i) \leq |N^+(u)|$, for all $v_i \in N^+(u)$. Also observe that $\lambda(v_i) = \lambda(u)$ if $u = f(v_i)$ and v_i is the last neighbor of u along P. Now assign each bar b_v the left-hand endpoint $\lambda(v)$. We must now show that two vertices are adjacent if and only if their corresponding bars have an unobstructed, non-degenerate, vertical band of visibility between them.

Suppose that uv is an edge of G. We may assume that $u <_P v$ and they are not consecutive on P. We now argue by contradiction that the bars associated with u and v have an unobstructed, non-degenerate, vertical band of visibility between them. Suppose there is a vertex w such that

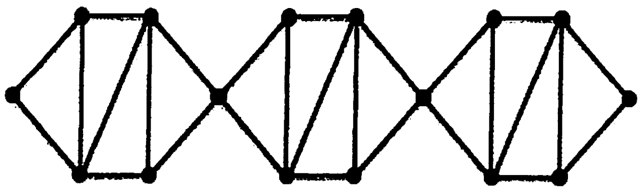

Figure 4: A traceable outerplanar graph that is not a 1-stack visibility graph.

$u <_P w <_P v$ and $\lambda(w) \leq \max\{\lambda(u), \lambda(v)\}$; we may assume that w is chosen to be closest to u with this property. Now $\lambda(w)$ is determined by the rank of w in the forward neighborhood of some vertex $x = f(w)$. Observe that $x \in P[u, w)$ since edges uv and wx do not cross with respect to the stack ordering determined by P. Now (1) implies $\lambda(x) \leq \lambda(w)$, in particular $\lambda(x) \leq \max\{\lambda(u), \lambda(v)\}$. By the choice of w, it follows that $x = u$. Therefore $w, v \in N^+(u)$ and $\text{rank}_u(v) < \text{rank}_u(w)$. So $\lambda(w) \leq \max\{\lambda(u), \lambda(v)\}$ contradicts the definition of $\lambda(w)$ via (1).

Finally we must prove that if two bars are mutually visible via an unobstructed, non-degenerate, vertical band of visibility between them, then their corresponding vertices are adjacent in G. Suppose that b_x and b_y form a pair of mutually visible bars. We may assume that $x <_P y$ and y is not the successor of x along P because such vertices are obviously adjacent in G. Let $z = f(y)$. If $x <_P z <_P y$, then $\lambda(z) \leq \lambda(y)$ would imply bar b_z blocks mutual visibility between b_x and b_y; so we may assume that $z \leq_P x$. If $z <_P x$, then a maximum neighbor of x would have to occur before y along P (because its edge to x could not cross edge zy) and its bar would block visibility between the bars b_x and b_y. Thus $z = x$ and $xy \in E(G)$, as desired. □

As noted by Cobos et al. [2], Mitchell's [8] polynomial-time algorithm to recognize outerplanar graphs can be modified to recognize 1-stack visibility graphs in polynomial time.

A natural generalization of one-stack visibility graphs to consider is the family of graphs with a bar visibility representation in which the x-axis projection of all bars share a common point. Let us call such a representation a *stack visibility representation* and the corresponding graph a *stack visibility* graph. It is easy to see that a stack visibility representation can be partitioned into two one-stack visibility representations (see Figure 5).

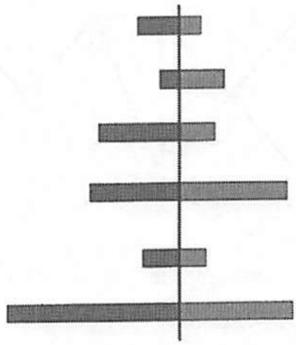

Figure 5: A natural partition into two one-stack visibility representations.

It follows that a graph is a stack visibility graph if and only if it is a 2-stack graph, the latter which have been characterized by Bernart and Kainen [1] as the subgraphs of a planar Hamiltonian graphs. By a result of Wigderson [15], stack visibility graphs are thus NP-complete to recognize. We are thus lead into the next section which introduces an intermediate family of graphs in the hopes of mitigating this computational complexity.

4 Unit stack visibility graphs

In this section we consider a class of graphs inspired by carousel navigation design. Other line-of-sight designs (for example, the old-fashioned linear tabbed Rolodex) may also be modeled by these graphs. The class is a hybrid of stack graphs and unit-bar visibility graphs.

A *unit stack visibility graph* is a unit-bar visibility graph in which all bars have a left-hand x-coordinate in the open interval $(0, 1)$. In particular, this means that 1 is an element of the x-axis projection of each bar, so one may view the bars as forming a single stack of unit-length bars pierced by the vertical line $x = 1$. We shall say a bar is *exposed upward on the left (resp. right)* if the next higher bar in the stack has a greater (resp. smaller) left-hand coordinate; this is to suggest that this bar's visibilities to higher placed bars must occur to the left (resp. right) of $x = 1$. A similar definition applies to exposure downward.

If $P = v_1 v_2 \ldots v_n$ is a linear ordering of the vertices of G, then $v_i <_P v_j$ signifies that vertex v_i precedes v_j along P; that is, $i < j$. As usual, $v_i \leq_P v_j$ means $i \leq j$. For each vertex $v_j \neq v_1$, the vertex $f(j)$ (resp. $F(j)$) denotes the first (resp. final) neighbor of v_j along P. So $f(j) = i$ and $F(j) = k$ means that $v_i, v_k \in N(v_j)$ and $i \leq r \leq k$, for all $v_r \in N(v_j)$. The definitions of $P^+(v)$, $P^+[v]$, $N_P^+(v)$, $E_P^+(v)$, and their "−" counterparts, are as given in earlier sections. Recall that, for $v \in N^+(u)$, the rank of v (with respect to u) is $\text{rank}_u(v) = |N^+(u) \cap P^+[v]|$. If $P = v_1 v_2 \ldots v_n$ is a linear ordering of the vertices of G, then we shall also use the notation $\text{rank}_i(j)$ to represent $\text{rank}_{v_i}(v_j)$.

We will consider partitions of edges of a graph into three sets, one of which is the set of edges of the Hamiltonian path corresponding to the linear order P. For convenience, the other two sets will be referred to as E_L and E_R, left and right, respectively. This is also to suggest the relationship between the edges and the corresponding bar visibilities; so, bar visibilities to the left (resp. right) of $x = 1$ correspond to edges of E_L (resp. E_R). A set of edges is *monochromatic* with respect to this partition if it is a subset of one of these sets; that is, $S \subseteq E(G) \setminus E(P)$ is monochromatic if $S \subseteq E_L$ or $S \subseteq E_R$.

In addition define, for any vertex v, the set of *edges that cross v on the left (resp. right)*, as

$$\text{cross}_L(v) = \{xy \in E_L : x \leq_P v <_P y\}$$
$$\text{cross}_R(v) = \{xy \in E_R : x \leq_P v <_P y\}.$$

The next theorem gives a combinatorial characterization of unit stack graphs. After the theorem we prove that this combinatorial characterization is "sharp" in the sense that the conditions in this characterization are independent of one another.

Theorem 2 *A graph $G(V, E)$ is a unit stack visibility graph if and only if G has a Hamiltonian path $P = v_1 v_2 \ldots v_n$ together with a partition of $E \setminus E(P)$ into two color classes, E_L and E_R, satisfying*

(i) E_L and E_R are stacks with respect to P, and

(ii) $E_P^+(v)$ and $E_P^-(v)$ are monochromatic, for all $v \in V$, and

(iii) for all $v_i \in V$, if $i = f(i+1)$ then $E_P^+(v_i) = \emptyset$ or $\text{cross}_L(v_i) = \emptyset$ or $\text{cross}_R(v_i) = \emptyset$, and

(iv) if $v_i v_j \leq_P v_r v_s$ are comparable edges of one stack, then $F(j-1) > j$ or $f(r+1) < r$.

Proof. "⇒" Consider a unit stack visibility representation of a unit stack visibility graph $G(V, E)$. Linearly order all vertices according to increasing

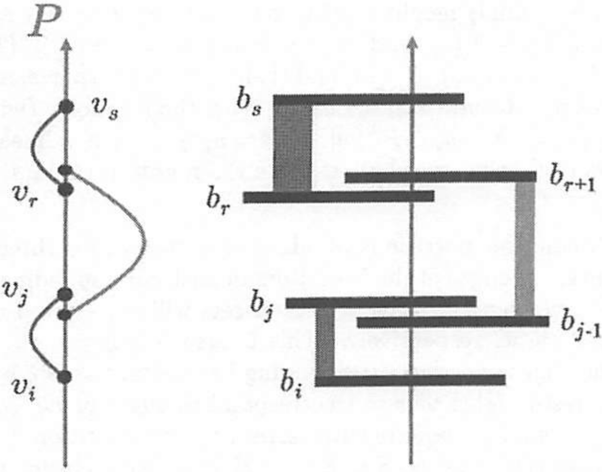

Figure 6: Property (iv): comparable edges in one stack induce a 'crossing edge' in the other.

height of their corresponding bars. Because bars of consecutive height have an unobstructed, non-degenerate, vertical band of visibility between them near x-coordinate 1, this linear ordering produces a Hamiltonian path $P = v_1 v_2 \dots v_n$ of G. Now partition the edges of $E \setminus E(P)$ into two sets E_L and E_R, placing an edge into E_L (resp. E_R) if the visibility rectangle between the corresponding bars occurs to left (resp. right) of the x-coordinate 1. Each of these color classes forms a stack with respect to P since each of them separately with P determines an outerplanar subgraph of G after contracting bars (as in the proof of Theorem 1). To prove property (ii) holds, consider a vertex v_i and its successor v_{i+1} on P. Because all bars have unit length and their x-axis projections must contain the element 1, the bar b_{i+1} blocks all visibility to the bar b_i from above either from the left or right (possibly both), depending upon whether b_{i+1} appears to the left or right (or directly above) the bar b_i. If b_{i+1} blocks all visibility to the bar b_i from above on the right, then all other neighbors of v_i above v_{i+1} must also produce edges in E_L. Thus $E_P^+(v_i) \subset E_L$ in this case. Symmetrically, $E_P^+(v_i) \subset E_R$, if b_{i+1} blocks all visibility to the bar b_i from above on the left. A similar argument applies to $E_P^-(v_i)$, so property (ii) is satisfied.

To establish property (iii), consider a vertex v_i whose successor, v_{i+1}, has v_i as its first neighbor; that is, $f(i+1) = i$. Now the bar b_{i+1} is the next bar above the bar b_i in the unit stack visibility representation of G. It must appear to the left, right, or directly above. If b_{i+1} appears directly above b_i, then b_i's visibility is blocked above so $E_P^+(v_i) = \emptyset$. On the other hand, if b_{i+1} appears to the left of b_i, then b_{i+1} is exposed downward on the left. However since $f(i+1) = i$, no bars below b_i can be exposed upward as far as b_i, so $\text{cross}_L(v_i) = \emptyset$. A symmetric argument show that $\text{cross}_R(v_i) = \emptyset$ if b_{i+1} appears to the right of b_i.

Finally we must establish property (iv) (see Figure 6 for one possible configuration). Consider two comparable edges e, f of one stack. Without loss of generality, $e, f \in E_L$, $e = v_i v_j$, $f = v_r v_s$, with $i < j \leq r < s$. Because $e, f \in E_L$, the right-side of bar b_{j-1} is exposed upward while the right-side b_{r+1} is exposed downward. Either these bars are visible to each other on the right (as shown in Figure 6), or at least one of them is visible to another bar that lies between them. In either case an edge of G in E_R is produced; the edge implies $F(j-1) > j$ or $f(r+1) < r$, as desired.

"$\Leftarrow$" Suppose that G has a Hamiltonian path $P = v_1 v_2 \ldots v_n$ that together with a 2-coloring, E_L and E_R, of $E \setminus E(P)$, satisfies conditions (i)–(iv). We must prove that G has a unit stack visibility representation. To this end, assign vertex $v_i \in P$ a unit-length horizontal bar, b_i, whose y-coordinate is i. Because the bars all have unit length, to complete the representation it suffices to define the left-hand x-coordinate of each bar. Let λ_i be the left-hand x-coordinate of bar b_i. Using the ordering of $P = v_1 v_2 \ldots v_n$, we now recursively define function λ that determines the left-hand endpoints of the bars associated with these vertices. Set $\lambda(1) = \frac{1}{2}$. If $j > 1$ and $i = f(j)$, define

$$\lambda(j) = \begin{cases} \lambda(i) & \text{if } |N^+(v_i)| = 1, j = i+1, E_P^+(v_i) = \emptyset \\ \frac{\lambda(i)}{2} & \text{if } |N^+(v_i)| > 1, j = i+1, \text{cross}_L(v_i) = \emptyset \\ \frac{1+\lambda(i)}{2} & \text{if } |N^+(v_i)| > 1, j = i+1, \text{cross}_R(v_i) = \emptyset \\ & \text{if } |N^+(v_i)| > 1 \text{ and } j > i+1 \text{ then} \\ & \lambda(i) + \left(\frac{\lambda(i+1)-\lambda(i)}{|N^+(v_i)|-1}\right)(\text{rank}_i(j) - 1) \end{cases} \tag{2}$$

Figure 7 illustrates the resulting unit stack visibility representation of a 5-wheel after applying this recursive assignment to the given Hamiltonian path and edge-partition.

Notice that under assignment (2), the value of $\lambda(j)$ is determined by $i = f(j)$, $\text{rank}_i(j)$ and possibly $\lambda(i+1)$, the latter of which may be determined by an earlier value of λ (namely the value at $f(i+1)$). In particular, if $E_P^+(v_i) \subseteq E_L$ (resp. E_R), then $\lambda(i+1) - \lambda(i) > 0$ (resp. < 0) so the unit-length bars associated with the forward neighbors of v_i beyond v_{i+1} move right (resp. left) a distance of $\left(\frac{|\lambda(i+1)-\lambda(i)|}{|N^+(v_i)|-1}\right)$ as their rank in

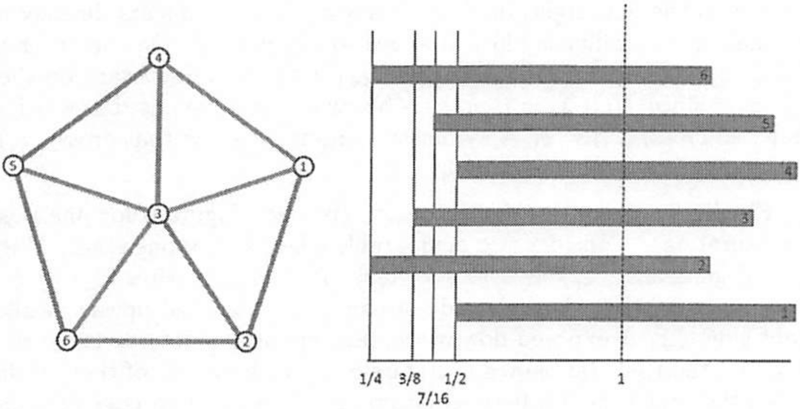

Figure 7: A unit-bar stack representation of a 5-wheel.

$N^+(v_i)$ increases. Consequently, the bars in the forward neighborhood of v_i have x-coordinates between $\lambda(i)$ and $\lambda(i+1)$. To summarize: if $i = f(j)$ and $E_P^+(v_i) \subseteq E_L$ (resp. E_R), then $\lambda(i) \leq \lambda(j) \leq \lambda(i+1)$ (resp. $\lambda(i+1) \leq \lambda(j) \leq \lambda(i)$).

First we prove that the algorithm halts only after assigning a λ-value to every vertex of G. This is done by induction on j; that is, we prove that $\lambda(j)$ is well-defined after $\lambda(1), \ldots, \lambda(j-1)$ have been defined. The basis of the induction, $j = 1$ is clear. Assume $j > 1$ and $i = f(j)$. If $|N^+(v_i)| = 1$ or $((N^+(v_i)| > 1)$ and $j \neq i+1)$, then the value of $\lambda(j)$ is well-defined by (2). So we may assume that $j = i+1$ and $|N^+(v_i)| > 1$; in particular $E^+(v_i) \neq \emptyset$. In this case, property (iii) guarantees that $\text{cross}_L(v_i) = \emptyset$ or $\text{cross}_R(v_i) = \emptyset$. Because $|N^+(v_i)| > 1$, it is not possible that both these occur, so exactly one occurs. Without loss of generality, $\text{cross}_R(v_i) = \emptyset$ so (2) assigns $\lambda(j) = \frac{1+\lambda(i)}{2}$.

Because $\lambda(1) = \frac{1}{2}$ and $\text{rank}_i(v_j) \leq |N^+(v_i)|$, for all $v_j \in N^+(v_i)$, a straight forward induction (which we omit) proves that $0 < \lambda_j < 1$, for all $j = 1, \ldots, n$. Assign each bar b_j the left-hand endpoint $\lambda(j)$. We must now show that two vertices are adjacent if and only if their corresponding bars have an unobstructed, non-degenerate, vertical band of visibility between them.

First we prove that if two bars are mutually visible via an unobstructed, non-degenerate, vertical band of visibility between them, then their corresponding vertices are adjacent in G. We proceed by induction on j, proving at stage j of the induction that bars $b_1, \ldots, b_j$ satisfy the claim. The basis, $j = 1$, is vacuously true. Considering now the inductive step at stage j, suppose that b_i and b_j form a pair of mutually visible bars, $i < j$, and the claim is true for all lexicographically smaller pairs; that is, all pairs b_r, b_s with $1 \leq r < s \leq j$ in which either $r < i$ or $s < j$. By assignment, the height of b_j is greater than the height of b_i. If $j = i+1$, then $v_i v_{i+1} \in E(P)$; so assume $j > i+1$. Because of symmetry it suffices to consider the case in which $\lambda(i) < \lambda(i+1)$; that is, the visibility between b_i and b_j occurs to the left of the vertical line at 1. Let $k = f(j)$. If $k = i$ then $v_i v_j \in E(G)$, as desired. So, two cases remain to be considered: $k < i$ and $i < k < j$.

If $k < i$, then because $\lambda(i) < \lambda(i+1)$ and the visibility between b_i and b_j occurs to the left of the vertical line at 1, the visibility between b_k and b_j also occurs to the left of the vertical line at 1; therefore $v_k v_j \in E_L$. Now consider v_r, the final neighbor of v_i along P. Both $v_i v_r$ and $v_k v_j$ are in E_L so can not cross since E_L is a stack. Therefore $r \leq j$. If $r = j$, then $v_i v_j \in E(G)$, as desired. If $r < j$, then consider $f(r)$. Now $v_i v_r \in E_L$ so $v_{f(r)} v_r \in E_L$ since property (ii) guarantees $E^-(v_r)$ is monochromatic. Because $r \leq j$ and $v_{f(r)} v_r \in E_L$, it follows that $\lambda(r) \leq \lambda(i)$ since otherwise b_i blocks the visibility between $b_{f(r)}$ and b_r. However this implies $i = f(r)$ and, by assignment (2), $\lambda(i) = \lambda(r)$, so b_r blocks the visibility between b_i and b_j, a contradiction.

If $i < k < j$ then, since b_i and b_j are mutually visible, $\lambda(i) < \lambda(i+1)$ implies $\lambda(j) < \lambda(k)$. This forces $E^+(v_k) \subseteq E_R$, since v_k is exposed upward beyond v_r on the right and v_j can not be the final neighbor of v_k since assignment (2) would give the final neighbor of v_k a λ-value that blocks visibility of the bar b_k. So the forward neighbors of v_k have λ-values between $\lambda(k+1)$ and $\lambda(k)$. In particular, $j = k+1$, otherwise b_{k+1} would block the visibility between v_i and v_j. Also $\lambda(i) < \lambda(k)$ because b_k does not block the visibility between b_i and b_j. Now consider v_r, the final neighbor of v_i along P. It follows from the assignment (2) that $\lambda(r) = \lambda(i)$ or $f(r) < i$.

Assume, for the moment, that $\lambda(r) = \lambda(i)$. If $r = j$, then $v_i v_j \in E(G)$, as desired. If $r < j$, then b_r blocks the visibility between b_i and b_j. Hence we may assume that $r > j$; that is, $\text{cross}_L(v_k) \neq \emptyset$. So (2) implies that $\lambda(j) > \lambda(k)$ since $k = j-1 = f(j)$; in particular, $\lambda(v_j) \geq \lambda(v_k)$ contradicting the visibility between b_i and b_j occurs to the left of the vertical line at 1.

So we may assume that $\lambda(r) \neq \lambda(i)$ and $f(r) < i$. Because b_r does not block the visibility between v_i and v_j, it follows that $\lambda(r) > \lambda(i)$. Now $f(r) < i$ means that $\emptyset \neq E^-(v_r) \subseteq E_R$. It follows that $\lambda(r-1) < \lambda(r)$, since b_r must be exposed downward. However, b_{r-1} can not block visibility between b_i and b_r which is a visibility corresponding to an edge in E_L. Consequently, $r - 1 = i$. Consider $\emptyset \neq E^-(v_r) \subseteq E_R$, $\emptyset \neq E^+(v_k) \subseteq E_R$,

$j = k+1$ and $r = i+1$. This means that there are two non-crossing edges in E_R that end in v_r and v_k, respectively. Applying property (iv), it follows that $F(i) > r$ or $f(j) < k$, each of which lead to a contradiction to the definition of either r or k.

Finally we must argue that if two vertices are adjacent then the corresponding bars have an unobstructed, non-degenerate, vertical band of visibility between them. Assume to the contrary that there are vertices, v_i and v_j with $i < j$, violating this statement. Choose such a pair that minimizes i and then minimizes j. Clearly $v_iv_j \notin E(P)$ since bars of consecutive height are mutually visible. Symmetry permits us to assume $v_iv_j \in E_L$. Choose k minimum such that $i < k < j$ and b_k blocks the visibility between b_i and b_j on the left. We may assume that bar b_i and b_k are mutually visible since otherwise a smaller value of k could be chosen, contradicting the choice of k. This means that $v_iv_k \in E(G)$ since we earlier proved that mutually visible bars correspond to edges of G. Now $E^+(v_i)$ is monochromatic by property (ii), so $v_iv_j \in E_L$ implies $v_iv_k \in E_P$ or $v_iv_k \in E_L$.

CASE 1: $v_iv_k \in E_P$

If $\lambda(k) < \lambda(i)$, then $f(k) = i$, since otherwise the edge v_iv_j and $v_{f(k)}v_k$ are crossing edges in the stack E_L. However according to the assignment given by (2), assigning $\lambda(k) < \lambda(i)$ when $f(k) = i$ can only occur when $v_iv_k \in E_R$, a contradiction. Therefore, we may assume $\lambda(k) \geq \lambda(i)$.

It follows that $\lambda(j) \geq \lambda(k)$, otherwise b_k would not block the visibility between b_i and b_j. The choice of i and j guarantees then that $f(j) = i$ since otherwise an earlier bar, $b_{f(j)}$ would be adjacent to j but not visible to b_j on the left. The comments following the assignment (2) show that, if $i = f(j)$, $j \neq i+1$, and $E_P^+(v_i) \subseteq E_L$ (resp. E_R), then $\lambda(i) \leq \lambda(j) < \lambda(i+1)$, contradicting $\lambda(j) \geq \lambda(k)$.

CASE 2: $v_iv_k \in E_L$:

Note that $f(k) = i$ since $f(k) < i$ would imply crossing edges v_iv_j and $v_{f(k)}v_k$ in E_L. Now the v_j and v_k are both neighbors of v_i, $f(k) = i$ and $j > i$ so the assignment (2) ensures that $\lambda(k) > \lambda(i)$ whereas $\lambda(j) < \lambda(k)$. This contradicts that b_k blocks the visibility of v_i and v_j. □

Clearly any unit stack visibility graph is necessarily a unit bar visibility graph. On the other hand, any non-traceable unit bar visibility graph, for example the claw $K_{1,3}$, is not a stack visibility graph. So the set of stack visibility graphs is a proper subset of the set of unit bar visibility graphs.

We now provide several examples to establish that the conditions in Theorem 2 are independent of one another. Any partition of the edges of K_4 into a Hamiltonian path and two sets E_L and E_R satisfying condition

Figure 8: A graph that is not a unit stack visibility graph only because of condition (iv).

(ii) must necessarily violate condition (i) only. Therefore condition (i) is independent of the other conditions. Any partition of the edges of K_4 into a Hamiltonian path and two stacks will result in a violation of condition (ii) only. Therefore condition (ii) is independent of the other conditions. Indeed, it is well known that K_4 is not even a unit bar visibility graph. Similarly, any partition of the edges of $K_{2,3}$ into a Hamiltonian path and two stacks will result in a violation of condition (iii) only. Therefore condition (iii) is independent of the other conditions and $K_{2,3}$ is not a unit stack visibility graph. Indeed, it also is well known that $K_{2,3}$ is not a unit bar visibility graph. The graph shown in Figure 8 has a unique Hamiltonian path because of the vertices of degree two. Any partition of the remaining three edges into two stacks necessarily produces two comparable edges in one stack without a 'crossing edge' in the other stack, a violation only of condition (iv). Thus, this graph is not a unit stack visibility graph and condition (iv) is independent of the other conditions.

Open Problem: Is there a polynomial-time algorithm to recognize unit stack visibility graphs?

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