Some direct constructions of cyclic $(3, \lambda)$-GDD of type g^v having prescribed number of short orbits

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1 Introduction

A (k, λ)-*GDD* of type g^v is an ordered triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of size gv, $\mathcal{G}$ a partition of X into groups of size g, and $\mathcal{B}$ a set of k-subsets of X (called blocks), such that each pair of elements from different groups appears in λ blocks and no block contains two elements from a common group. A GDD is *cyclic* if it admits a cyclic automorphism group G acting sharply transitively on X.

For a cyclic (k, λ)-*GDD* of type g^v, we may assume that $X = Z_{gv}$. Let $B = \{b_1, b_2, \dots, b_k\}$ be a block of a cyclic (k, λ)-GDD of type g^v. The block orbit generated by B is defined as the set of distinct blocks $B + i = \{b_1 + i, b_2 + i, \dots, b_k + i\} \pmod{gv}$ for $i \in Z_{gv}$. If a block orbit has gv blocks, then the block orbit is said to be *full*, otherwise *short*.

A difference family of an abelian group G is a collection $\{B_1, B_2, \dots, B_t\}$ of k-subsets (called *base blocks*) of G satisfying certain properties. For any base block B of a difference family over an abelian group G, the subgroup

$$\{z \in G : B + z = B\}$$

is called the *stabilizer* of B in G. A base block B is called *full* if its stabilizer is trivial, otherwise it is called *short*. The stabilizer of B is denoted as S_B.

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Let H be a subgroup of order h of an abelian group G of order u. A collection $\{B_1, B_2, \ldots, B_t\}$ of k-subsets (called *base blocks*) of G forms a (u, h, k, λ) *difference family over G and relative to H with α short blocks* if $\bigcup_{i=1}^{t} \partial B_i$ covers each elements of $G - H$ exactly λ times but no element in H, and there are exactly α short base blocks, where $\partial B = \frac{1}{|S_B|}\{a - b : a, b \in B, a \neq b\}$. We denote such a design as $(u, h, k, \lambda)_\alpha$-DF. When the value of short base blocks is not specified, the design is denoted as (u, h, k, λ)-DF. Observe that if k is a prime and G is cyclic, then we could have short base block only when k is a divisor of u but not of h. For simplicity, our definition is just a special case of difference families. For general information of difference families, the readers refereed to [1, 2]. Note that the base blocks of a $(u, \{h, 3_\alpha\}, 3, \lambda)$-DF defined in [4] together with exactly α short base blocks $\{0, u/3, 2u/3\}$ forms a $(u, h, 3, \lambda)_\alpha$-DF, and $(u, h, k, 1)_1$-DF is denoted as $(u, \{h, k\}, k, 1)$-DF in [3].

It is not difficult to see that the existence of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv} is equivalent to the existence of a cyclic $(3, \lambda)$-GDD of type g^v with α short block orbits. For a cyclic $(3, \lambda)$-GDD of type g^v, the possible short orbit must be generated by $\{0, gv/3, 2gv/3\}$. Therefore in what follows, we only display the full base blocks for a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv}.

In [4], the necessary and sufficient conditions have been determined for the existence of a cyclic $(3, \lambda)$-GDD of type g^v. In [5], we further investigated the existence spectrum of a cyclic $(3, \lambda)$-GDD of type g^v with exactly α short orbits, where α can be any possible value.

The main purpose of this note is to give some direct constructions of $(u, h, k, \lambda)_\alpha$-DF used in [5], which gives a complete solution for the necessary and sufficient conditions of the existence of a cyclic $(3, \lambda)$-GDD of type g^v with exactly α short block orbits for all possible parameters λ, g, v and α. The following theorem is proved in [5].

Theorem 1.1 *A cyclic $(3, \lambda)$-GDD of type g^v having α short orbits exists if and only if*

(1) $\lambda g(v-1) - 2\alpha \equiv 0 \pmod{6}$, $\alpha \leq \lambda$, $v \geq 3$;

(2) $v \not\equiv 2, 3 \pmod{4}$ *when* $g \equiv 2 \pmod{4}$ *and* $\lambda \equiv 1 \pmod{2}$;

(3) $v \not\equiv 2 \pmod{4}$ *when* $g \equiv 1 \pmod{2}$ *and* $\lambda \equiv 2 \pmod{4}$;

(4) $g \not\equiv 0 \pmod{3}$ *and* $v \equiv 0 \pmod{3}$ *when* $\alpha \neq 0$;

(5) $\lambda(3g - 1) - 2\alpha g \equiv 0 \pmod{6}$ *when* $v = 3$;

(6) $\lambda = \alpha$ *when* $(g, v) = (1, 3)$, $\lambda = 2\alpha$ *when* $(g, v) = (2, 3)$, $\lambda = 4\alpha$ *when* $(g, v) = (1, 6)$, $\lambda \geq 2\alpha$ *when* $(g, v) = (2, 6)$, $\lambda \equiv 0 \pmod{3}$ *when*

$(g, v) = (1, 9)$ *and* $\lambda = \alpha$.

Several recursive constructions are applied to prove the above theorem. To use the recursive constructions, many designs should be directly constructed. This note aims to provide infinite classes of difference families with certain number of short orbits to support the proofs in [5]. The chosen of the parameters for the difference families in this note is based on the requirement of the proofs of Theorem 1.1. The outline of the proofs of Theorem 1.1 in [5] are as follows. Firstly, all of the $(gv, g, 3, \lambda)_0$-DFs are constructed. Secondly, the $(gv, g, 3, 3)_3$-DFs are constructed. Then by using recursive constructions, all of the designs are constructed. The rest of the paper is arranged as follows. In Section 2, we shall establish some DFs without short orbit, which are used to construct all $(gv, g, 3, \lambda)_0$-DFs in [5]. In Section 3 we present some DFs with prescribed number of short orbits, which are used to construct all $(gv, g, 3, 3)_3$-DFs in [5]. Section 4 provides some other direct constructed DFs used for the final recursive constructions of [5].

2 Constructions of some DFs without short orbit

In this note, $[a, b]$ denotes the set of integers n such that $a \leq n \leq b$ and λS denotes the multiset containing each element of S exactly λ times for a set S.

This section provides designs for the constructions of $(gv, g, 3, \lambda)_0$-DFs in [5]. A $(3g, g, 3, 3)_0$-DF over Z_{3g} from Construction 2.1 and a $(v, 1, 3, 3)_0$-DF over Z_v from Construction 2.2 are used to produce a $(gv, g, 3, 3)_0$-DF over Z_{gv} for $g \equiv 1, 5 \pmod 6$, $v \equiv 3 \pmod 6$ and $(g, v) \neq (1, 3)$, which appeared as Lemma 3.4(1) in [5]. Applying Theorem 2.3 in [5] with a $(4v, 4, 3, 3)$-PDF over Z_{4v} from Construction 2.3, to get a $(gv, g, 3, 3)_0$-DF over Z_{gv} for $g \equiv 4, 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$, that was stated as Lemma 3.4(3) in [5]. A $(v, 1, 3, 6)_0$-DF over Z_v from Construction 2.4 is used to get a $(gv, g, 3, 6)_0$-DF over Z_{gv} for $g \equiv 1, 5 \pmod 6$, $v \equiv 0 \pmod{12}$. A $(v, 1, 3, 12)_0$-DF over Z_v from Construction 2.5 and a $(6g, 6, 3, 12)_0$-DF over Z_{6g} from Construction 2.7 are used to get a $(gv, g, 3, 12)_0$-DF over Z_{gv} for $g \equiv 1, 5 \pmod 6$, $v \equiv 6 \pmod{12}$ and $(g, v) \neq (1, 6)$. The details appeared in Case 1 of Lemma 3.5 in [5]. A $(2v, 2, 3, 6)_0$-DF over Z_{2v} from Construction 2.3, together with Construction 2.6 are applied to obtain a $(gv, g, 3, 6)_0$-DF for $g \equiv 2, 10 \pmod{12}$, $v \equiv 0 \pmod 3$ and $(g, v) \neq (2, 3)$, which appeared in Case 2 of Lemma 3.5 in [5]. Also Construction 2.6 was used in Case 3 of Lemma 3.5 in [5] when $g \equiv 4, 8 \pmod{12}$.

Construction 2.1 *For* $g \equiv 1, 5 \pmod 6$ *and* $g > 1$, *there exists a* $(3g, g, 3, 3)_0$-DF *over* Z_{3g}.

Proof When $g \equiv 1 \pmod 6$ and $g > 1$, let $g = 6t + 1$ where $t \geq 1$. The base blocks are

$2\{0, 3t-2, 6t-1\}$, $\{0, 3t-2, 6t+2\}$, $\{0, 3t-1, 6t+1\}$, $2\{0, 3t-1, 9t+1\}$, $\{0, 3t+1, 9t+2\}$,

and when $t \geq 2$, the following base blocks are also used:

$$2\{0, 3t-8-6r, 6t-4-3r\}, \quad 3\{0, 3t-7-6r, 9t-2-3r\},$$
$$\{0, 3t-8-6r, 6t-1-3r\},$$

for $r \in [0, t/2-2]$ if t is even ($r \in \emptyset$ when $t = 2$), and $r \in [0, (t-3)/2]$ if t is odd,

$$2\{0, 3t-4-6r, 6t-2-3r\}, \quad 3\{0, 3t-5-6r, 9t-1-3r\},$$
$$\{0, 3t-4-6r, 6t+1-3r\},$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd.

When $g \equiv 5 \pmod 6$, let $g = 6t + 5$ where $t \geq 0$. The base blocks are

$2\{0, 3t+1, 6t+5\}$, $\{0, 3t+2, 6t+4\}$, $\{0, 3t+2, 9t+7\}$, $\{0, 3t+1, 9t+8\}$,

and when $t \geq 1$, the following base blocks are also used:

$$2\{0, 3t-5-6r, 6t+2-3r\}, \quad 2\{0, 3t-4-6r, 9t+4-3r\},$$
$$\{0, 3t-4-6r, 6t+1-3r\}, \quad \{0, 3t-5-6r, 9t+5-3r\},$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd ($r \in \emptyset$ when $t = 1$),

$$2\{0, 3t-1-6r, 6t+4-3r\}, \quad 2\{0, 3t-2-6r, 9t+5-3r\},$$
$$\{0, 3t-2-6r, 6t+2-3r\}, \quad \{0, 3t-1-6r, 9t+7-3r\},$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-1)/2]$ if t is odd. □

Construction 2.2 *A* $(v, 1, 3, 3)_0$-DF *over* Z_v *exists for* $v \equiv 3 \pmod 6$ *and* $v > 3$.

Proof The base blocks are

- $v \equiv 3 \pmod{24}$ and $v > 3$: $\{0, 1, (v+5)/4\}$, $\{0, (v-3)/8, (v+1)/2\}$,

$\{0, 1+2j, (v+5)/4+j\}$, $j \in [0, (v-7)/4]$,

$\{0, 2+2j, (v+5)/4+j\}$, $j \in [0, (v-7)/4] \setminus \{(v-3)/8\}$.

- $v \equiv 9 \pmod{24}$: $\{0,(v-1)/8,(v-1)/2\}$, $\{0,(v-1)/4,(v-1)/2\}$,

$\{0,1+2j,(v+3)/4+j\}$, $j \in [0,(v-5)/4]$,

$\{0,2+2j,(v+3)/4+j\}$, $j \in [0,(v-9)/4]\backslash\{(v-9)/8\}$.

- $v \equiv 15 \pmod{24}$: $\{0,(v+1)/8,(v-1)/2\}$, $\{0,(v-3)/4,(v-1)/2\}$,

$\{0,1+2j,(v+1)/4+j\}$, $j \in [0,(v-7)/4]\backslash\{(v-7)/8\}$,

$\{0,2+2j,(v+5)/4+j\}$, $j \in [0,(v-7)/4]$.

- $v \equiv 21 \pmod{24}$: $\{0,1,(v+3)/4\}$, $\{0,(v+3)/8,(v+1)/2\}$,

$\{0,1+2j,(v+3)/4+j\}$, $j \in [0,(v-5)/4]\backslash\{(v-5)/8\}$,

$\{0,2+2j,(v+7)/4+j\}$, $j \in [0,(v-9)/4]$. □

Let g be a divisor of v such that $v = gv_0$. Suppose that $\mathscr{F} = \{B_i : i = 1,2,\ldots,t\}$ is the family of base blocks of a $(hv,hg,k,\lambda)_0$-DF over Z_{hv} where $B_i = \{0,b_{1i},b_{2i},\ldots,b_{k-1,i}\}$ for $i = 1,2,\ldots,t$. Define $ele(\mathscr{F}) = \cup_{i=1}^{t}\{b_{1i},b_{2i},\ldots,b_{k-1,i}\}$. The $(hv,hg,k,\lambda)_0$-DF over Z_{hv} is said to be *h-perfect*, denoted by (hv,hg,k,λ)-h-PDF over Z_{hv}, if $ele(\mathscr{F}) \subseteq \{a+bv : 0 \le a \le \lfloor \frac{v}{2} \rfloor, a \ne 0, v_0, 2v_0, \ldots, (g-1)v_0, b = 0,1,\ldots,h-1\}$, where all the calculations are modulo hv. When $h = 1$, write (hv,hg,k,λ)-1-PDF over Z_{hv} briefly as (v,g,k,λ)-PDF over Z_v. The following construction gives some PDFs that are also useful in [5].

Construction 2.3 *There exists a* $(4v,4,3,3)$-PDF *over* Z_{4v} *which is also a* $(2v,2,3,6)_0$-DF *over* Z_{2v} *for* $v \equiv 0 \pmod 3$ *and* $v > 3$.

Proof The base blocks are

- $v \equiv 3 \pmod 6$ and $v > 3$:

$\{0,1,v-1\}$, $\{0,1,2v-1\}$, $\{0,(v-1)/2,(3v+1)/2\}$, $\{0,2,6\}$,
$\{0,1,v+3\}$, $\{0,2,v+4\}$, $\{0,(v+1)/2,(3v+3)/2\}$,

$\{0,3+2j,v+2+j\}$, $j \in [0,v-3]\backslash\{(v-3)/2,(v-1)/2\}$,

$\{0,8+2j,v+5+j\}$, $j \in [0,v-6]$.

- $v \equiv 0 \pmod 6$ and $v \ge 6$:

$\{0,1,2\}$, $\{0,1,2v-1\}$, $\{0,2,v+3\}$, $\{0,v/2-1,3v/2+1\}$,
$\{0,v/2+1,3v/2+2\}$,

$\{0,3+2j,v+2+j\}$, $j \in [0,v-3]\backslash\{v/2\}$,

$\{0,4+2j,v+3+j\}$, $j \in [0,v-4]\backslash\{v/2-2\}$. □

Construction 2.4 *A* $(v,1,3,6)_0$-DF *over* Z_v *exists for* $v \equiv 0 \pmod{12}$.

Proof The base blocks are $\{0,1,v/4+1\}$, $\{0,1,v-1\}$, $\{0,2,v/2+1\}$, $\{0,1+2j,v/2+1+j\}$, $j\in[0,v/2-2]\backslash\{v/4-1\}$,
$\{0,2+2j,v/2+2+j\}$, $j\in[0,v/2-3]$. □

Construction 2.5 *A* $(v,1,3,12)_0$-DF *over* Z_v *exists for* $v\equiv 6 \pmod{12}$ *and* $v>6$.

Proof The base blocks are $\{0,1,2\}$, $2\{0,1,v-1\}$, $\{0,2,v/2+1\}$,
$2\{0,1+2j,v/2+1+j\}$, $j\in[0,v/2-2]$,
$2\{0,2+2j,v/2+2+j\}$, $j\in[0,v/2-3]$. □

Construction 2.6 *For* $g\equiv 2,4 \pmod 6$ *and* $g>2$, *there exists a* $(3g,g,3,6)_0$-DF *over* Z_{3g}.

Proof When $g\equiv 2 \pmod 6$ and $g>2$, let $g=6t+2$ where $t\geq 1$. The base blocks are

$$\begin{array}{lll} 2\{0,1,6t-1\}, & 3\{0,2,15t+4\}, & \{0,6t-2,18t+2\} \\ 2\{0,6t+1,18t+2\} & 3\{0,1,9t+2\}, & \{0,6t-1,12t+1\} \\ \{0,6t+1,18t+5\}, & 2\{0,6t+2,18t+4\} & \{0,2,12t+4\}, \end{array}$$

and when $t\geq 2$,

$$\begin{array}{ll} 3\{0,6t-7-6r,12t-2-3r\}, & 3\{0,6t-8-6r,18t-1-3r\}, \\ 3\{0,6t-5-6r,12t-1-3r\}, & 3\{0,6t-4-6r,18t+1-3r\}, \end{array}$$

for $r\in[0,t-2]$.

When $g\equiv 4 \pmod 6$, let $g=6t+4$ where $t\geq 0$. The base blocks are

$$\begin{array}{lll} \{0,1,6t+2\}, & \{0,2,12t+7\}, & 2\{0,6t+2,18t+10\} \\ \{0,6t+4,18t+11\}, & \{0,1,12t+8\}, & 2\{0,6t+1,12t+5\}, \end{array}$$

and when $t\geq 1$,

$$\begin{array}{ll} 3\{0,6t-5-6r,12t+2-3r\}, & 3\{0,6t-4-6r,18t+7-3r\}, \\ 3\{0,6t-1-6r,12t+4-3r\}, & 3\{0,6t-2-6r,18t+8-3r\}, \end{array}$$

for $r\in[0,t-1]$. □

Construction 2.7 *For* $g\equiv 1,5 \pmod 6$ *and* $g>1$, *there exists a* $(6g,g,3,12)_0$-DF *over* Z_{6g}.

Proof When $g\equiv 1 \pmod 6$ and $g>1$, let $g=6t+1$ where $t\geq 1$. The base blocks are

$$\begin{array}{lll} 2\{0,1,12t+2\}, & \{0,1,36t+5\}, & 4\{0,12t+1,36t+4\}, \\ \{0,12t+2,36t+4\}, & 2\{0,1,24t+3\}, & \end{array}$$

and

$$\begin{array}{ll} 3\{0,12t-11-12r,24t-4-6r\}, & 3\{0,12t-11-12r,24t-3-6r\}, \\ 3\{0,12t-8-12r,24t-3-6r\}, & 3\{0,12t-8-12r,24t-1-6r\}, \\ 3\{0,12t-5-12r,24t-1-6r\}, & 3\{0,12t-5-12r,24t-2-6r\}, \\ 3\{0,12t-4-12r,24t-2-6r\}, & 3\{0,12t-4-12r,24t+1-6r\}, \\ 3\{0,12t-2-12r,24t+1-6r\}, & 3\{0,12t-2-12r,24t+2-6r\}, \\ 6\{0,12t-10-12r,36t-1-6r\}, & 6\{0,12t-3-12r,36t+2-6r\}, \\ 6\{0,12t-9-12r,36t-2-6r\}, & 6\{0,12t-1-12r,36t+3-6r\}, \\ 6\{0,12t-7-12r,36t+1-6r\}, & \end{array}$$

for $r \in [0, t-1]$.

When $g \equiv 5 \pmod 6$, let $g = 6t+5$ where $t \geq 0$. The base blocks are

$$\begin{array}{lll} 2\{0,1,2\}, & 2\{0,12t+8,24t+21\}, & 3\{0,1,18t+17\} \\ 3\{0,1,30t+26\}, & 2\{0,1,12t+9\}, & 2\{0,12t+9,24t+20\}, \\ 3\{0,2,18t+16\} & 3\{0,2,30t+25\}, & 2\{0,2,12t+9\}, \\ 4\{0,12t+7,24t+20\}, & 3\{0,3,18t+17\} & 3\{0,3,30t+26\}, \\ 2\{0,2,12t+10\}, & 4\{0,12t+10,24t+21\}, & 6\{0,4,18t+19\}, \\ 6\{0,5,30t+27\} & & \end{array}$$

and when $t \geq 1$,

$$\begin{array}{ll} 6\{0,12t-4-12r,24t+15-6r\}, & 6\{0,12t-5-12r,36t+22-6r\}, \\ 6\{0,12t-3-12r,24t+14-6r\}, & 6\{0,12t-2-12r,36t+23-6r\}, \\ 6\{0,12t+1-12r,24t+17-6r\}, & 6\{0,12t-1-12r,36t+25-6r\}, \\ 6\{0,12t+2-12r,24t+16-6r\}, & 6\{0,12t+3-12r,36t+26-6r\}, \\ 6\{0,12t+4-12r,24t+19-6r\}, & 6\{0,12t+5-12r,36t+27-6r\}, \end{array}$$

for $r \in [0, t-1]$. □

3 Constructions of some DFs with 3 short orbits

In this section, we construct some DFs with 3 short orbits. As stated previously, we only display the full base blocks of a $(gv, g, 3, \lambda)_\alpha$-DF over Z_{gv}.

The designs in this section are used to construct $(gv, g, 3, 3)_3$-DFs in [5]. A $(2v, 2, 3, 3)_3$-DF over Z_{2v} from Construction 3.1 is used to get a $(gv, g, 3, 3)_3$-DF over Z_{gv} for $g \equiv 2, 10 \pmod{12}$ and $v \equiv 9 \pmod{12}$, which was listed in Lemma 4.5(3) of [5]. A $(8v, 8, 3, 3)_3$-DF over Z_{8v} from Construction 3.2 together with Construction 3.4 are used to produce a

$(gv, g, 3, 3)_3$-DF over Z_{gv} for $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $v > 3$, which appeared in Lemma 4.5(2) of [5]. A $(3g, g, 3, 3)_3$-DF over Z_{3g} from Construction 3.3 is applied with the known $(gv, 3g, 3, 3)_0$-DF over Z_{gv} to obtain a $(gv, g, 3, 3)_3$-DF over Z_{gv} for $g \equiv 5 \pmod 6$ and $v \equiv 3 \pmod 6$, which appeared in Lemma 4.5(1) of [5].

Construction 3.1 *A* $(2v, 2, 3, 3)_3$-DF *over* Z_{2v} *exists for* $v \equiv 9 \pmod{12}$.

Proof The base blocks are

$\{0, 1, v/3+1\}$, $\{0, (v+3)/6, (v-1)/2\}$, $\{0, v/3, (5v+3)/6\}$, $\{0, (v+3)/4, v+1\}$,

$\{0, 1+2j, (v+1)/2+j\}$, $j \in [0, (v-3)/2] \backslash \{(v-3)/6\}$,

$\{0, 2+2j, (v+3)/2+j\}$, $j \in [0, (v-5)/2] \backslash \{(v-9)/6, (v-5)/4, v/3-1\}$. □

Construction 3.2 *A* $(8v, 8, 3, 3)_3$-DF *over* Z_{8v} *exists for* $v \equiv 3 \pmod 6$ *and* $v > 3$.

Proof The base blocks are

$\{0, 1, 4v/3\}$, $\{0, 4, 2v+3\}$, $\{0, 4v/3-1, 10v/3\}$,
$\{0, 2v-2, 4v-1\}$, $\{0, 1, 3v+2\}$, $\{0, (v+1)/2, 2v+1\}$,
$\{0, 4v/3, 8v/3+1\}$, $\{0, 2v-1, (9v-1)/2\}$, $\{0, 2, v+1\}$,
$\{0, 2v/3, 2v+2\}$,

$\{0, 6+2j, 2v+3+j\}$, $j \in [0, 2v-4] \backslash \{2v/3-3, 2v/3-2, v-3, 4v/3-3\}$,

$\{0, 3+2j, 2v+2+j\}$, $j \in [0, 2v-3] \backslash \{(v-3)/2, 2v/3-2, v-2, v-1, v, (3v-3)/2\}$.
□

Construction 3.3 *For* $g \equiv 5 \pmod 6$, *there exists a* $(3g, g, 3, 3)_3$-DF *over* Z_{3g}.

Proof Let $g = 6t+5$ where $t \geq 0$. The base blocks are

$$\{0, 3t+1, 6t+2\}, \{0, 3t+2, 6t+4\}, \{0, 3t+1, 9t+8\}, \{0, 6t+4, 12t+8\},$$

and when $t \geq 1$,

$2\{0, 3t-2-6r, 6t+2-3r\}$, $2\{0, 3t-1-6r, 9t+7-3r\}$,
$\{0, 3t-1-6r, 6t+1-3r\}$, $\{0, 3t-2-6r, 9t+5-3r\}$,

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-1)/2]$ if t is odd,

$2\{0, 3t-4-6r, 6t+1-3r\}$, $2\{0, 3t-5-6r, 9t+5-3r\}$,
$\{0, 3t-5-6r, 6t-1-3r\}$, $\{0, 3t-4-6r, 9t+4-3r\}$,

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd ($r \in \emptyset$ when $t = 1$). $\square$

Construction 3.4 *For $g \equiv 8 \pmod{12}$, $v \equiv 0 \pmod 3$ and $6 \le v \le 21$, there exists a $(gv, g, 3, 3)_3$-DF over Z_{gv}.*

Proof Let $g = 12t + 8$ where $t \geq 0$.

- $v = 6$: The base blocks are

$2\{0,1,4\}$, $2\{0,2,30t+19\}$, $\{0,12t+7,30t+21\}$,
$2\{0,12t+8,30t+21\}$, $\{0,1,5\}$, $\{0,5,30t+22\}$,
$\{0,12t+7,30t+22\}$, $2\{0,12t+9,30t+20\}$, $\{0,2,5\}$,
$\{0,12t+7,30t+20\}$, $\{0,12t+8,30t+22\}$, $2\{0,18t+10,36t+25\}$,
$\{0,12t+9,30t+19\}$, $\{0,18t+11,36t+25\}$,

and when $t \geq 1$,

$3\{0,12t-5-12r,24t+10-6r\}$, $3\{0,12t-4-12r,36t+17-6r\}$,
$3\{0,12t-2-12r,24t+11-6r\}$, $3\{0,12t-3-12r,36t+19-6r\}$,
$3\{0,12t-1-12r,24t+13-6r\}$, $3\{0,12t+1-12r,36t+20-6r\}$,
$3\{0,12t+3-12r,24t+14-6r\}$, $3\{0,12t+2-12r,36t+22-6r\}$,
$3\{0,12t+5-12r,24t+15-6r\}$, $3\{0,12t+4-12r,36t+21-6r\}$,

for $r \in [0, t-1]$.

- $v = 9$: The base blocks are

$\{0,4,8\}$, $\{0,2,45t+32\}$, $2\{0,3,45t+32\}$,
$3\{0,18t+10,36t+22\}$, $3\{0,1,27t+17\}$, $\{0,3,45t+28\}$,
$2\{0,8,45t+33\}$, $3\{0,18t+11,54t+34\}$, $3\{0,6,27t+19\}$,
$\{0,4,45t+33\}$, $3\{0,5,45t+31\}$, $3\{0,27t+15,54t+35\}$,
$3\{0,7,27t+21\}$, $2\{0,2,45t+30\}$,

and when $t \geq 1$,

$3\{0,18t-8-18r,36t+13-9r\}$, $3\{0,18t-7-18r,54t+26-9r\}$,
$3\{0,18t-5-18r,36t+15-9r\}$, $3\{0,18t-6-18r,54t+25-9r\}$,
$3\{0,18t-3-18r,36t+14-9r\}$, $3\{0,18t-4-18r,54t+28-9r\}$,
$3\{0,18t-2-18r,36t+17-9r\}$, $3\{0,18t-1-18r,54t+29-9r\}$,
$3\{0,18t+1-18r,36t+16-9r\}$, $3\{0,18t+2-18r,54t+31-9r\}$,
$3\{0,18t+3-18r,36t+19-9r\}$, $3\{0,18t+4-18r,54t+30-9r\}$,
$3\{0,18t+6-18r,36t+20-9r\}$, $3\{0,18t+5-18r,54t+33-9r\}$,
$3\{0,18t+8-18r,36t+21-9r\}$, $3\{0,18t+7-18r,54t+32-9r\}$,

for $r \in [0, t-1]$.

- $v = 12$: The base blocks are

$\{0,1,10\}$, $3\{0,24t+16,60t+38\}$, $\{0,7,60t+44\}$, $2\{0,5,60t+40\}$, $2\{0,1,9\}$, $3\{0,36t+20,72t+47\}$, $3\{0,24t+15,60t+45\}$, $3\{0,3,36t+31\}$, $\{0,2,8\}$, $3\{0,24t+17,60t+43\}$, $3\{0,24t+14,48t+33\}$, $2\{0,10,60t+44\}$, $2\{0,4,11\}$, $3\{0,36t+21,72t+46\}$, $\{0,6,60t+40\}$, $2\{0,2,60t+39\}$, $\{0,5,11\}$, $3\{0,24t+18,60t+41\}$, $3\{0,24t+13,60t+42\}$, $\{0,4,60t+39\}$,

and when $t \geq 1$,

$3\{0,24t-10-24r,48t+20-12r\}$, $3\{0,24t-9-24r,48t+22-12r\}$, $3\{0,24t-8-24r,48t+21-12r\}$, $3\{0,24t-4-24r,48t+23-12r\}$, $3\{0,24t-2-24r,48t+26-12r\}$, $3\{0,24t-1-24r,48t+25-12r\}$, $3\{0,24t+2-24r,48t+27-12r\}$, $3\{0,24t+5-24r,48t+28-12r\}$, $3\{0,24t+8-24r,48t+29-12r\}$, $3\{0,24t+9-24r,48t+31-12r\}$, $3\{0,24t+10-24r,48t+30-12r\}$, $3\{0,24t-11-24r,72t+34-12r\}$, $3\{0,24t-7-24r,72t+37-12r\}$, $3\{0,24t-6-24r,72t+35-12r\}$, $3\{0,24t-5-24r,72t+38-12r\}$, $3\{0,24t-3-24r,72t+39-12r\}$, $3\{0,24t+1-24r,72t+41-12r\}$, $3\{0,24t+3-24r,72t+40-12r\}$, $3\{0,24t+4-24r,72t+43-12r\}$, $3\{0,24t+6-24r,72t+44-12r\}$, $3\{0,24t+7-24r,72t+42-12r\}$, $3\{0,24t+11-24r,72t+45-12r\}$,

for $r \in [0, t-1]$.

- $v = 15$: The base blocks are

$\{0,6,12\}$, $3\{0,30t+18,60t+37\}$, $2\{0,8,75t+50\}$, $\{0,3,75t+50\}$, $3\{0,9,45t+34\}$, $3\{0,45t+26,90t+57\}$, $3\{0,7,75t+53\}$, $2\{0,2,75t+51\}$, $3\{0,14,45t+36\}$, $3\{0,5,30t+21\}$, $3\{0,30t+17,90t+56\}$, $2\{0,12,75t+55\}$, $\{0,6,75t+55\}$, $3\{0,10,45t+33\}$, $3\{0,45t+27,90t+59\}$, $3\{0,13,75t+54\}$, $2\{0,3,75t+47\}$, $\{0,2,75t+44\}$, $3\{0,1,45t+29\}$, $3\{0,30t+20,90t+58\}$, $3\{0,4,75t+52\}$, $\{0,8,75t+51\}$, $3\{0,11,45t+35\}$,

and when $t \geq 1$,

$3\{0,30t-14-30r,60t+22-15r\}$, $3\{0,30t-10-30r,60t+25-15r\}$, $3\{0,30t-9-30r,60t+23-15r\}$, $3\{0,30t-8-30r,60t+26-15r\}$, $3\{0,30t-7-30r,60t+24-15r\}$, $3\{0,30t-6-30r,60t+27-15r\}$, $3\{0,30t+1-30r,60t+28-15r\}$, $3\{0,30t+2-30r,60t+31-15r\}$, $3\{0,30t+3-30r,60t+29-15r\}$, $3\{0,30t+4-30r,60t+32-15r\}$, $3\{0,30t+9-30r,60t+33-15r\}$, $3\{0,30t-13-30r,90t+42-15r\}$, $3\{0,30t-12-30r,90t+41-15r\}$, $3\{0,30t-11-30r,90t+43-15r\}$, $3\{0,30t-5-30r,90t+47-15r\}$, $3\{0,30t-4-30r,90t+44-15r\}$, $3\{0,30t-3-30r,90t+46-15r\}$, $3\{0,30t-2-30r,90t+48-15r\}$, $3\{0,30t-1-30r,90t+50-15r\}$, $3\{0,30t+5-30r,90t+49-15r\}$, $3\{0,30t+6-30r,90t+52-15r\}$, $3\{0,30t+7-30r,90t+54-15r\}$,

$$3\{0,30t+10-30r,60t+35-15r\},\quad 3\{0,30t+8-30r,90t+51-15r\},$$
$$3\{0,30t+11-30r,60t+34-15r\},\quad 3\{0,30t+12-30r,90t+53-15r\},$$
$$3\{0,30t+14-30r,60t+36-15r\},\quad 3\{0,30t+13-30r,90t+55-15r\},$$

for $r \in [0, t-1]$.

- $v = 18$: The base blocks are

$$\{0,4,12\},\quad 3\{0,10,54t+45\},\quad 3\{0,36t+20,72t+49\},$$
$$3\{0,36t+25,90t+59\},\quad \{0,8,16\},\quad 3\{0,2,90t+58\},$$
$$3\{0,36t+23,72t+50\},\quad 3\{0,36t+26,90t+64\},\quad 2\{0,4,16\},$$
$$3\{0,5,90t+62\},\quad 3\{0,36t+19,90t+65\},\quad 3\{0,36t+28,90t+68\},$$
$$3\{0,1,14\},\quad 3\{0,7,90t+60\},\quad 3\{0,36t+21,90t+63\},$$
$$3\{0,54t+30,108t+71\},\quad 3\{0,6,17\},\quad 3\{0,9,90t+61\},$$
$$3\{0,36t+22,90t+55\},\quad 3\{0,54t+31,108t+70\},\quad 3\{0,3,54t+47\},$$
$$3\{0,15,90t+66\},\quad 3\{0,36t+24,90t+67\},\quad 3\{0,54t+32,108t+69\},$$

and when $t \geq 1$,

$$3\{0,36t-17-36r,72t+30-18r\},\quad 3\{0,36t-16-36r,108t+51-18r\},$$
$$3\{0,36t-15-36r,72t+31-18r\},\quad 3\{0,36t-14-36r,108t+52-18r\},$$
$$3\{0,36t-11-36r,72t+34-18r\},\quad 3\{0,36t-13-36r,108t+55-18r\},$$
$$3\{0,36t-10-36r,72t+32-18r\},\quad 3\{0,36t-12-36r,108t+53-18r\},$$
$$3\{0,36t-9-36r,72t+35-18r\},\quad 3\{0,36t-7-36r,108t+56-18r\},$$
$$3\{0,36t-8-36r,72t+33-18r\},\quad 3\{0,36t-6-36r,108t+58-18r\},$$
$$3\{0,36t-4-36r,72t+39-18r\},\quad 3\{0,36t-5-36r,108t+57-18r\},$$
$$3\{0,36t-3-36r,72t+37-18r\},\quad 3\{0,36t-2-36r,108t+59-18r\},$$
$$3\{0,36t-1-36r,72t+38-18r\},\quad 3\{0,36t+1-36r,108t+61-18r\},$$
$$3\{0,36t+2-36r,72t+40-18r\},\quad 3\{0,36t+3-36r,108t+62-18r\},$$
$$3\{0,36t+6-36r,72t+41-18r\},\quad 3\{0,36t+4-36r,108t+60-18r\},$$
$$3\{0,36t+7-36r,72t+44-18r\},\quad 3\{0,36t+5-36r,108t+63-18r\},$$
$$3\{0,36t+10-36r,72t+43-18r\},\quad 3\{0,36t+8-36r,108t+65-18r\},$$
$$3\{0,36t+11-36r,72t+45-18r\},\quad 3\{0,36t+9-36r,108t+64-18r\},$$
$$3\{0,36t+12-36r,72t+42-18r\},\quad 3\{0,36t+13-36r,108t+66-18r\},$$
$$3\{0,36t+14-36r,72t+46-18r\},\quad 3\{0,36t+15-36r,108t+67-18r\},$$
$$3\{0,36t+16-36r,72t+47-18r\},\quad 3\{0,36t+17-36r,108t+68-18r\},$$

for $r \in [0, t-1]$.

- $v = 21$: The base blocks are

$$\{0,10,20\},\quad 3\{0,19,63t+51\},\quad 2\{0,2,105t+71\},$$
$$3\{0,16,105t+74\},\quad 3\{0,6,42t+30\},\quad \{0,1,105t+70\},$$
$$2\{0,3,105t+67\},\quad 3\{0,42t+25,84t+52\},\quad 3\{0,7,42t+29\},$$
$$\{0,2,105t+64\},\quad 2\{0,4,105t+70\},\quad 3\{0,42t+23,126t+78\},$$
$$3\{0,8,63t+46\},\quad \{0,3,105t+71\},\quad 2\{0,5,105t+73\},$$
$$3\{0,42t+26,126t+79\},\quad 3\{0,9,63t+45\},\quad \{0,4,105t+77\},$$
$$2\{0,20,105t+77\},\quad 3\{0,42t+28,126t+82\},\quad 3\{0,12,63t+47\},$$
$$\{0,5,105t+66\},\quad 3\{0,11,105t+76\},\quad 3\{0,63t+37,126t+81\},$$

$3\{0,14,63t+48\}$, $\{0,10,105t+67\}$, $3\{0,13,105t+72\}$,
$3\{0,63t+39,126t+80\}$, $3\{0,17,63t+50\}$, $2\{0,1,105t+62\}$,
$3\{0,15,105t+75\}$, $3\{0,63t+40,126t+83\}$, $3\{0,18,63t+49\}$,

and when $t \geq 1$,

$3\{0,42t-19-42r,84t+31-21r\}$, $3\{0,42t-20-42r,126t+57-21r\}$,
$3\{0,42t-17-42r,84t+34-21r\}$, $3\{0,42t-18-42r,126t+58-21r\}$,
$3\{0,42t-15-42r,84t+32-21r\}$, $3\{0,42t-16-42r,126t+59-21r\}$,
$3\{0,42t-13-42r,84t+33-21r\}$, $3\{0,42t-14-42r,126t+60-21r\}$,
$3\{0,42t-12-42r,84t+37-21r\}$, $3\{0,42t-11-42r,126t+61-21r\}$,
$3\{0,42t-10-42r,84t+38-21r\}$, $3\{0,42t-8-42r,126t+62-21r\}$,
$3\{0,42t-9-42r,84t+35-21r\}$, $3\{0,42t-7-42r,126t+66-21r\}$,
$3\{0,42t-6-42r,84t+39-21r\}$, $3\{0,42t-4-42r,126t+65-21r\}$,
$3\{0,42t-5-42r,84t+36-21r\}$, $3\{0,42t-3-42r,126t+68-21r\}$,
$3\{0,42t+1-42r,84t+40-21r\}$, $3\{0,42t-2-42r,126t+64-21r\}$,
$3\{0,42t+3-42r,84t+46-21r\}$, $3\{0,42t-1-42r,126t+67-21r\}$,
$3\{0,42t+4-42r,84t+44-21r\}$, $3\{0,42t+2-42r,126t+69-21r\}$,
$3\{0,42t+5-42r,84t+41-21r\}$, $3\{0,42t+6-42r,126t+71-21r\}$,
$3\{0,42t+7-42r,84t+45-21r\}$, $3\{0,42t+9-42r,126t+73-21r\}$,
$3\{0,42t+8-42r,84t+43-21r\}$, $3\{0,42t+10-42r,126t+70-21r\}$,
$3\{0,42t+13-42r,84t+50-21r\}$, $3\{0,42+11-42r,126t+72-21r\}$,
$3\{0,42t+14-42r,84t+47-21r\}$, $3\{0,42t+12-42r,126t+74-21r\}$,
$3\{0,42t+15-42r,84t+49-21r\}$, $3\{0,42t+17-42r,126t+75-21r\}$,
$3\{0,42t+16-42r,84t+48-21r\}$, $3\{0,42t+18-42r,126t+77-21r\}$,
$3\{0,42t+20-42r,84t+51-21r\}$, $3\{0,42t+19-42r,126t+76-21r\}$,

for $r \in [0, t-1]$. □

4 Some other constructions

To complete the proof of Theorem 1.1, some other designs are needed, which are presented in this section. Constructions 4.1, 4.3 and 4.5 are used to produce a $(gv,g,3,4)_1$-DF over Z_{gv} for $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g,v) \neq (1,3)$, which appeared in Lemma 5.2 of [5]. Constructions 4.2, 4.4 and 4.8 are used to get a $(gv,g,3,6)_6$-DF over Z_{gv} for $g \equiv 2 \pmod 6$, $v \equiv 0 \pmod 3$ and $(g,v) \neq (2,3),(2,6)$, which was stated in Lemma 5.5 of [5]. Construction 4.6 is used to get a $(gv,g,3,8)_1$-DF and a $(gv,g,3,8)_7$-DF over Z_{gv} for $g \equiv 2 \pmod 3$ and $v \equiv 0 \pmod 3$, which appeared in Lemmas 5.6 and 5.8 of [5]. Construction 4.7 is used to get a $(gv,g,3,4)_4$-DF over Z_{gv} for $g \equiv 1 \pmod 3$, $v \equiv 0 \pmod 3$ and $(g,v) \neq (1,6)$, which appeared in Lemma 5.3 of [5]. Construction 4.9 is used to produce a $(gv,g,3,12)_{12}$-DF over Z_{gv} for $g \equiv 2 \pmod 3$ and $v \equiv 0 \pmod 3$, which was involved in Lemma 5.12 of [5].

Construction 4.1 *A* $(v,1,3,4)_1$-DF *over* Z_v *exists for* $v \equiv 0 \pmod 3$ *and* $v > 9$.

Proof For $v \equiv 3 \pmod 6$ and $v > 9$, taking together the base blocks of a $(v,1,3,1)_1$-DF over Z_v from Lemma 4.4(1) and a $(v,1,3,3)_0$-DF over Z_v from Lemma 3.5 both in [5], we can get the desired $(v,1,3,4)_1$-DF over Z_v. For $v \equiv 0 \pmod 6$ and $v > 9$, the base blocks are

- $v \equiv 6 \pmod{12}$ and $v \geq 18$: $\{0, v/6, v/3\}$, $2\{0,(v-2)/4, v/2\}$,

$4\{0, 1+2j, (5v+6)/12+j\}$, $j \in [0,(v-18)/12]$,

$2\{0, 2+2j, (v+6)/4+j\}$, $j \in [0,(v-18)/12]$,

$2\{0, 2+2j, (v+2)/4+j\}$, $j \in [0,(v-18)/12]$.

- $v \equiv 0 \pmod{12}$ and $v \geq 12$:

$\{0, v/6, v/3\}$, $2\{0,1,v/3\}$, $2\{0, v/4-1, v/2-1\}$, $2\{0, v/6-1, 7v/12-1\}$,

$2\{0, 1+2j, 5v/12+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v = 12$),

$2\{0, 3+2j, 5v/12+2+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v = 12$),

$2\{0, 2+2j, v/4+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v = 12$),

$2\{0, 2+2j, v/4+1+j\}$, $j \in [0, v/12-2]$ ($j \in \emptyset$ if $v = 12$). □

Construction 4.2 *A* $(2v,2,3,6)_6$-DF *over* Z_{2v} *exists for* $v \equiv 0 \pmod 3$ *and* $v \geq 9$.

Proof For $v = 9, 12$, repeating the base blocks of a $(2v,2,3,3)_3$-DF over Z_{2v} from Lemma 4.5 of [5] twice to get the conclusion. For $v \equiv 0 \pmod 3$ and $v \geq 15$, the base blocks are

- $v \equiv 0 \pmod 6$ and $v \geq 24$:

$\{0,1,2\}$, $5\{0,2,v/2\}$, $3\{0, v/3-2, 2v/3-2\}$,
$2\{0, v/3-2, 7v/6-1\}$, $\{0,1,v/3-2\}$, $\{0,3,2v/3+2\}$,
$\{0, v/3-1, 2v/3-2\}$, $\{0, v/3-1, 7v/6-2\}$, $\{0,1,5v/6+1\}$,
$5\{0,3,5v/6+2\}$, $3\{0, v/3-1, 2v/3-1\}$, $\{0, v/2-2, 7v/6-1\}$,
$2\{0,1,2v/3-1\}$, $5\{0, v/3-3, v-1\}$, $\{0, v/2-1, v-1\}$,
$5\{0, v/2-1, 7v/6\}$,

$6\{0, 5+2j, 5v/6+3+j\}$, $j \in [0, v/6-5]$ ($j \in \emptyset$ if $v = 24$),

$6\{0, 4+2j, v/2+1+j\}$, $j \in [0, v/6-4]$.

- $v \equiv 3 \pmod 6$ and $v \geq 21$:

$\{0,1,v/3-1\}$, $5\{0,2,(v+1)/2\}$, $5\{0, v/3-1, 2v/3-1\}$,
$\{0, v/3, (7v-3)/6\}$, $\{0,1,2v/3+2\}$, $5\{0, v/3-4, v-2\}$,

$\{0,(v-3)/2,v-2\}$, $\{0,(v-1)/2,(7v+3)/6\}$, $4\{0,1,(5v+3)/6\}$, $4\{0,v/3-2,v-1\}$, $2\{0,(v-1)/2,v-1\}$, $\{0,(v+1)/2,(7v-3)/6\}$, $\{0,2,v/3-2\}$,

$6\{0,3+2j,(5v+9)/6+j\}$, $j \in [0,(v-27)/6]$ ($j \in \emptyset$ if $v=21$),

$6\{0,4+2j,(v+3)/2+j\}$, $j \in [0,(v-21)/6]$.

- $v = 18$:

$\{0,2,8\}$, $\{0,2,13\}$, $\{0,4,13\}$, $2\{0,1,6\}$, $2\{0,6,16\}$, $4\{0,3,17\}$, $4\{0,5,13\}$, $\{0,2,10\}$, $\{0,4,10\}$, $2\{0,1,3\}$, $2\{0,1,15\}$, $2\{0,7,17\}$, $4\{0,4,15\}$, $4\{0,7,16\}$, $\{0,2,11\}$.

- $v = 15$:

$\{0,2,7\}$, $\{0,5,13\}$, $2\{0,1,4\}$, $2\{0,3,14\}$, $4\{0,2,9\}$, $4\{0,4,12\}$, $4\{0,5,11\}$, $\{0,2,8\}$, $\{0,6,13\}$, $2\{0,3,12\}$, $4\{0,1,14\}$.

□

Construction 4.3 *For* $g \equiv 4 \pmod 6$, *there exists a* $(3g,g,3,4)_1$-DF *over* Z_{3g}.

Proof Let $g = 6t+4$ where $t \geq 0$. The base blocks are

$\{0,3t+1,6t+2\}$, $\{0,3t+2,6t+4\}$, $2\{0,3t+1,9t+5\}$, $\{0,3t+2,9t+7\}$,

and when $t \geq 1$,

$\{0,3t-1-6r,6t+1-3r\}$, $\{0,3t-2-6r,9t+5-3r\}$, $3\{0,3t-2-6r,6t+2-3r\}$, $3\{0,3t-1-6r,9t+4-3r\}$,

for $r \in [0,t/2-1]$ if t is even, and $r \in [0,(t-1)/2]$ if t is odd,

$\{0,3t-5-6r,6t-1-3r\}$, $\{0,3t-4-6r,9t+4-3r\}$, $3\{0,3t-4-6r,6t+1-3r\}$, $3\{0,3t-5-6r,9t+2-3r\}$,

for $r \in [0,t/2-1]$ if t is even, and $r \in [0,(t-3)/2]$ if t is odd ($r \in \emptyset$ when $t=1$). □

Construction 4.4 *For* $g \equiv 2 \pmod 6$ *and* $g > 2$, *there exists a* $(3g,g,3,6)_6$-DF *over* Z_{3g}.

Proof Let $g = 6t+2$ where $t \geq 1$. The base blocks are

$2\{0,3t-2,6t-1\}$, $2\{0,3t-1,6t+1\}$, $\{0,3t+2,6t+4\}$, $\{0,6t+1,12t+2\}$, $2\{0,3t-1,6t-2\}$, $2\{0,3t+1,6t+5\}$, $4\{0,3t-2,9t+2\}$,

and when $t \geq 2$,

$$2\{0,3t-4-6r,6t+1-3r\},\quad 2\{0,3t-5-6r,9t+2-3r\},$$
$$2\{0,3t-5-6r,6t-4-3r\},\quad 4\{0,3t-4-6r,9t+1-3r\},$$
$$2\{0,3t-5-6r,6t-1-3r\},$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0,(t-3)/2]$ if t is odd,

$$2\{0,3t-8-6r,6t-1-3r\},\quad 2\{0,3t-7-6r,9t+1-3r\},$$
$$2\{0,3t-7-6r,6t-5-3r\},\quad 4\{0,3t-8-6r,9t-1-3r\},$$
$$2\{0,3t-7-6r,6t-2-3r\},$$

for $r \in [0, t/2-2]$ if t is even ($r \in \emptyset$ when $t=2$), and $r \in [0,(t-3)/2]$ if t is odd. □

Construction 4.5 *For* $g \equiv 10 \pmod{12}$, *there exists a* $(6g,g,3,4)_1$-DF *over* Z_{6g}.

Proof Let $g = 12t+10$ where $t \geq 0$. The base blocks are

$$2\{0,1,18t+14\},\quad 2\{0,5,18t+16\},\quad 2\{0,4,30t+27\},$$
$$3\{0,12t+9,36t+28\},\quad 2\{0,2,18t+15\},\quad 2\{0,1,30t+26\},$$
$$2\{0,5,30t+27\},\quad \{0,12t+8,36t+28\},\quad 2\{0,2,18t+16\},$$
$$2\{0,3,30t+25\},\quad 4\{0,12t+7,24t+17\},\quad \{0,12t+9,36t+29\},$$
$$2\{0,4,18t+15\},\quad 2\{0,3,30t+26\},\quad 3\{0,12t+8,36t+29\},$$
$$\{0,24t+19,48t+39\},$$

and when $t \geq 1$,

$$4\{0,12t-5-12r,24t+11-6r\},\quad 4\{0,12t-4-12r,36t+23-6r\},$$
$$4\{0,12t-2-12r,24t+13-6r\},\quad 4\{0,12t-3-12r,36t+22-6r\},$$
$$4\{0,12t+1-12r,24t+14-6r\},\quad 4\{0,12t-1-12r,36t+25-6r\},$$
$$4\{0,12t+2-12r,24t+16-6r\},\quad 4\{0,12t+3-12r,36t+26-6r\},$$
$$4\{0,12t+4-12r,24t+15-6r\},\quad 4\{0,12t+5-12r,36t+27-6r\},$$

for $r \in [0,t-1]$. □

Construction 4.6 *For* $g \equiv 5 \pmod 6$, $\alpha \in \{1,7\}$, *there exists a* $(6g,g,3,8)_\alpha$-DF *over* Z_{6g}.

Proof Let $g = 6t+5$ where $t \geq 0$.

- $\alpha = 1$: The base blocks are

$$\{0,6t+2,12t+9\},\quad \{0,6t+4,12t+11\},\quad 3\{0,6t+3,12t+7\},$$
$$3\{0,6t+2,18t+11\},\quad \{0,6t+3,12t+8\},\quad 2\{0,6t+2,12t+5\},$$
$$\{0,6t+4,18t+11\},\quad 4\{0,6t+1,18t+14\},\quad \{0,6t+4,12t+8\},$$
$$2\{0,6t+2,12t+10\},\quad 2\{0,6t+5,18t+13\},\quad 4\{0,6t+1,18t+15\},$$
$$\{0,6t+4,12t+9\},\quad 2\{0,6t+3,12t+10\},\quad 2\{0,6t+8,18t+13\},$$
$$3\{0,12t+9,24t+19\},$$

and when $t \geq 2$,

$$
\begin{array}{ll}
4\{0, 6t-11-12r, 12t+3-6r\}, & 4\{0, 6t-10-12r, 18t+5-6r\}, \\
4\{0, 6t-8-12r, 12t+5-6r\}, & 4\{0, 6t-9-12r, 18t+7-6r\}, \\
4\{0, 6t-7-12r, 12t+4-6r\}, & 4\{0, 6t-5-12r, 18t+8-6r\}, \\
4\{0, 6t-3-12r, 12t+7-6r\}, & 4\{0, 6t-4-12r, 18t+10-6r\}, \\
4\{0, 6t-1-12r, 12t+8-6r\}, & 4\{0, 6t-2-12r, 18t+9-6r\}, \\
4\{0, 6t-11-12r, 12t-1-6r\}, & 4\{0, 6t-10-12r, 18t+9-6r\}, \\
4\{0, 6t-8-12r, 12t+1-6r\}, & 4\{0, 6t-9-12r, 18t+11-6r\}, \\
4\{0, 6t-5-12r, 12t+2-6r\}, & 4\{0, 6t-7-12r, 18t+10-6r\}, \\
4\{0, 6t-4-12r, 12t+4-6r\}, & 4\{0, 6t-3-12r, 18t+13-6r\}, \\
4\{0, 6t-2-12r, 12t+3-6r\}, & 4\{0, 6t-1-12r, 18t+14-6r\},
\end{array}
$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$$
\begin{array}{llll}
4\{0, 2, 9t+6\}, & 4\{0, 3, 9t+10\}, & 4\{0, 5, 9t+11\}, & 4\{0, 1, 15t+17\}, \\
4\{0, 4, 15t+12\}, & 4\{0, 3, 9t+8\}, & 4\{0, 5, 9t+7\}, & 4\{0, 1, 15t+13\}, \\
4\{0, 2, 15t+13\}, & 4\{0, 4, 15t+14\}. & &
\end{array}
$$

- $\alpha = 7$: The base blocks are

$$
\begin{array}{lll}
\{0, 6t+2, 12t+9\}, & \{0, 6t+4, 12t+11\}, & 3\{0, 6t+3, 12t+7\}, \\
3\{0, 6t+2, 18t+11\}, & \{0, 6t+3, 12t+8\}, & 2\{0, 6t+2, 12t+5\}, \\
\{0, 6t+4, 18t+11\}, & 4\{0, 6t+1, 18t+14\}, & \{0, 6t+4, 12t+8\}, \\
2\{0, 6t+2, 12t+9\}, & 2\{0, 6t+5, 18t+13\}, & 4\{0, 6t+1, 18t+15\}, \\
\{0, 6t+4, 12t+9\}, & 2\{0, 6t+3, 12t+11\}, & 2\{0, 6t+8, 18t+13\}, \\
\{0, 12t+9, 24t+19\}, & &
\end{array}
$$

and when $t \geq 2$,

$$
\begin{array}{ll}
4\{0, 6t-11-12r, 12t+3-6r\}, & 4\{0, 6t-10-12r, 18t+5-6r\}, \\
4\{0, 6t-8-12r, 12t+5-6r\}, & 4\{0, 6t-9-12r, 18t+7-6r\}, \\
4\{0, 6t-7-12r, 12t+4-6r\}, & 4\{0, 6t-5-12r, 18t+8-6r\}, \\
4\{0, 6t-3-12r, 12t+7-6r\}, & 4\{0, 6t-4-12r, 18t+10-6r\}, \\
4\{0, 6t-1-12r, 12t+8-6r\}, & 4\{0, 6t-2-12r, 18t+9-6r\}, \\
4\{0, 6t-11-12r, 12t-1-6r\}, & 4\{0, 6t-10-12r, 18t+9-6r\}, \\
4\{0, 6t-8-12r, 12t+1-6r\}, & 4\{0, 6t-9-12r, 18t+11-6r\}, \\
4\{0, 6t-5-12r, 12t+2-6r\}, & 4\{0, 6t-7-12r, 18t+10-6r\}, \\
4\{0, 6t-4-12r, 12t+4-6r\}, & 4\{0, 6t-3-12r, 18t+13-6r\}, \\
4\{0, 6t-2-12r, 12t+3-6r\}, & 4\{0, 6t-1-12r, 18t+14-6r\},
\end{array}
$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$$
\begin{array}{llll}
4\{0, 2, 9t+6\}, & 4\{0, 3, 9t+10\}, & 4\{0, 5, 9t+11\}, & 4\{0, 1, 15t+17\}, \\
4\{0, 4, 15t+12\}, & 4\{0, 3, 9t+8\}, & 4\{0, 5, 9t+7\}, & 4\{0, 1, 15t+13\}, \\
4\{0, 2, 15t+13\}, & 4\{0, 4, 15t+14\}. & &
\end{array}
$$

□

Construction 4.7 *For* $g \equiv 1 \pmod 6$ *and* $g > 1$, *there exists a* $(6g, g, 3, 4)_4$-DF *over* Z_{6g}.

Proof Let $g = 6t + 1$ where $t \geq 1$. The base blocks are

$$\begin{array}{lll}
\{0, 6t-5, 12t-1\}, & 2\{0, 6t-3, 12t-1\}, & \{0, 6t-5, 18t+2\}, \\
2\{0, 6t-4, 18t-1\}, & \{0, 6t-2, 12t-1\}, & 2\{0, 6t-2, 12t+1\}, \\
\{0, 6t-2, 18t+2\}, & 2\{0, 6t-4, 18t+1\}, & \{0, 6t+1, 12t+4\}, \\
2\{0, 6t-1, 12t-2\}, & 2\{0, 6t-5, 18t+3\}, & 2\{0, 6t-3, 18t-2\}, \\
\{0, 6t+3, 12t+7\}, & 2\{0, 6t+1, 12t+3\}, &
\end{array}$$

and when $t \geq 3$,

$$\begin{array}{ll}
2\{0, 6t-16-12r, 12t-7-6r\}, & 2\{0, 6t-17-12r, 18t-3-6r\}, \\
2\{0, 6t-15-12r, 12t-5-6r\}, & 2\{0, 6t-14-12r, 18t-1-6r\}, \\
2\{0, 6t-11-12r, 12t-4-6r\}, & 2\{0, 6t-13-12r, 18t-2-6r\}, \\
2\{0, 6t-10-12r, 12t-2-6r\}, & 2\{0, 6t-9-12r, 18t+1-6r\}, \\
2\{0, 6t-8-12r, 12t-3-6r\}, & 2\{0, 6t-7-12r, 18t+2-6r\}, \\
2\{0, 6t-16-12r, 12t-8-6r\}, & 2\{0, 6t-17-12r, 18t-8-6r\}, \\
2\{0, 6t-14-12r, 12t-7-6r\}, & 2\{0, 6t-15-12r, 18t-7-6r\}, \\
2\{0, 6t-13-12r, 12t-4-6r\}, & 2\{0, 6t-11-12r, 18t-4-6r\}, \\
2\{0, 6t-10-12r, 12t-5-6r\}, & 2\{0, 6t-9-12r, 18t-5-6r\}, \\
2\{0, 6t-7-12r, 12t-3-6r\}, & 2\{0, 6t-8-12r, 18t-3-6r\},
\end{array}$$

for $r \in [0, t/2-2]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is even,

$$\begin{array}{llll}
2\{0, 1, 9t+2\}, & 2\{0, 3, 9t+2\}, & 2\{0, 5, 9t+4\}, & 2\{0, 2, 15t+1\}, \\
2\{0, 4, 15t+2\}, & 2\{0, 2, 9t+3\}, & 2\{0, 5, 9t+3\}, & 2\{0, 1, 15t+4\}, \\
2\{0, 3, 15t+8\}, & 2\{0, 4, 15t+7\}. & &
\end{array}$$

□

Construction 4.8 *For* $g \equiv 2 \pmod{12}$ *and* $g > 2$, *there exists a* $(6g, g, 3, 6)_6$-DF *over* Z_{6g}.

Proof Let $g = 12t + 2$ where $t \geq 1$. The base blocks are

$$\begin{array}{llll}
\{0, 1, 10\}, & 6\{0, 4, 18t+5\}, & \{0, 11, 30t+10\}, & 6\{0, 3, 30t+7\}, \\
\{0, 1, 11\}, & 6\{0, 5, 18t+4\}, & 4\{0, 10, 30t+9\}, & 6\{0, 12t+1, 24t+3\}, \\
4\{0, 1, 18t+3\}, & \{0, 7, 30t+9\}, & 5\{0, 7, 30t+8\}, & 2\{0, 18t-3, 36t+5\}, \\
4\{0, 9, 18t+7\}, & \{0, 8, 30t+9\}, & 5\{0, 8, 30t+10\}, & 2\{0, 18t-2, 36t+5\}, \\
4\{0, 11, 18t+8\}, & \{0, 9, 30t+8\}, & 6\{0, 2, 30t+5\}, & 2\{0, 18t+2, 36t+5\},
\end{array}$$

and when $t \geq 2$,

$$
\begin{array}{ll}
6\{0,12t-11-12r,24t-3-6r\}, & 6\{0,12t-10-12r,36t-1-6r\},\\
6\{0,12t-8-12r,24t-1-6r\}, & 6\{0,12t-9-12r,36t+1-6r\},\\
6\{0,12t-7-12r,24t-2-6r\}, & 6\{0,12t-5-12r,36t+2-6r\},\\
6\{0,12t-3-12r,24t+1-6r\}, & 6\{0,12t-4-12r,36t+4-6r\},\\
6\{0,12t-1-12r,24t+2-6r\}, & 6\{0,12t-2-12r,36t+3-6r\},
\end{array}
$$

for $r \in [0, t-2]$. □

Construction 4.9 *For $g \equiv 5 \pmod 6$, there exists a $(6g, g, 3, 12)_{12}$-DF over Z_{6g}.*

Proof Let $g = 6t + 5$ where $t \geq 0$. The base blocks are

$$
\begin{array}{lll}
\{0,6t+1,12t+8\}, & 3\{0,6t+4,12t+11\}, & 6\{0,6t+4,12t+13\},\\
2\{0,12t+7,24t+16\}, & 2\{0,6t+1,12t+9\}, & 4\{0,6t+2,12t+7\},\\
6\{0,6t+1,18t+14\}, & \{0,12t+8,24t+16\}, & 2\{0,6t+2,12t+9\},\\
4\{0,6t+3,12t+11\}, & 3\{0,6t+1,18t+15\}, & \{0,12t+8,24t+19\},\\
2\{0,6t+3,12t+8\}, & 6\{0,6t+2,12t+5\}, & 3\{0,6t+4,18t+15\},
\end{array}
$$

and when $t \geq 2$,

$$
\begin{array}{ll}
6\{0,6t-10-12r,12t-1-6r\}, & 6\{0,6t-11-12r,18t+9-6r\},\\
6\{0,6t-8-12r,12t+2-6r\}, & 6\{0,6t-9-12r,18t+10-6r\},\\
6\{0,6t-7-12r,12t+1-6r\}, & 6\{0,6t-5-12r,18t+11-6r\},\\
6\{0,6t-4-12r,12t+3-6r\}, & 6\{0,6t-3-12r,18t+14-6r\},\\
6\{0,6t-1-12r,12t+4-6r\}, & 6\{0,6t-2-12r,18t+13-6r\},\\
6\{0,6t-10-12r,12t+5-6r\}, & 6\{0,6t-11-12r,18t+8-6r\},\\
6\{0,6t-9-12r,12t+4-6r\}, & 6\{0,6t-8-12r,18t+9-6r\},\\
6\{0,6t-7-12r,12t+7-6r\}, & 6\{0,6t-5-12r,18t+11-6r\},\\
6\{0,6t-3-12r,12t+8-6r\}, & 6\{0,6t-4-12r,18t+10-6r\},\\
6\{0,6t-1-12r,12t+9-6r\}, & 6\{0,6t-2-12r,18t+13-6r\},
\end{array}
$$

for $r \in [0, t/2-1]$ if t is even, and $r \in [0, (t-3)/2]$ if t is odd,

and when t is odd,

$$
\begin{array}{llll}
6\{0,2,9t+6\}, & 6\{0,3,9t+8\}, & 6\{0,5,9t+7\}, & 6\{0,1,15t+14\},\\
6\{0,4,15t+16\}, & 6\{0,2,9t+12\}, & 6\{0,4,9t+11\}, & 6\{0,1,15t+13\},\\
6\{0,3,15t+17\}, & 6\{0,5,15t+16\}. & &
\end{array}
$$

□

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