A Simple Bijection Between 231-Avoiding and 312-Avoiding Placements

Jonathan Bloom
Dartmouth College

Dan Saracino
Colgate University

Abstract

Stankova and West proved in 2002 that the patterns 231 and 312 are shape-Wilf-equivalent. Their proof was nonbijective. We give a new characterization of 231 and 312 avoiding full rook placements and use this to give a simple bijection that demonstrates the shape-Wilf-equivalence.

1. Introduction

For any pattern $\tau \in S_k$, let $S_n(\tau)$ denote the set of permutations in S_n that avoid τ, in the sense that they have no subsequence order-isomorphic to τ. For any Ferrers board F, let $S_F(\tau)$ denote the set of all full rook placements on F that avoid τ. We say that two patterns τ and σ are *Wilf-equivalent*, and write $\tau \sim \sigma$, if $|S_n(\tau)| = |S_n(\sigma)|$ for all $n > 0$. We say that τ and σ are *shape-Wilf-equivalent*, and write $\tau \sim_s \sigma$, if $|S_F(\tau)| = |S_F(\sigma)|$ for all F. So shape-Wilf-equivalence implies Wilf-equivalence, as we see by considering square Ferrers boards. (The relevant definitions will be reviewed in Section 2.)

The concept of shape-Wilf-equivalence was introduced in [1], as a means for obtaining results about Wilf-equivalence. Since shape-Wilf-equivalence is stronger than Wilf-equivlence, positive results about it are rare. The only "general result" was obtained in [1], where it was shown that the patterns $k \cdots 321$ and $123 \cdots k$ are shape-Wilf-equivalent for every positive k. Later, in [4], Stankova and West proved that the patterns 231 and 312 are shape-Wilf-equivalent, and the motivation for our paper comes from their result. Their proof that $|S_F(231)| = |S_F(312)|$ was nonbijective, and somewhat complicated. Our purpose here it to give a simple bijection between $S_F(231)$ and $S_F(312)$.

We will do so by associating a sequence of nonnegative integers to each full rook placement on F, and characterizing those sequences that arise from 231-avoiding or 312-avoiding placements. We will give a simple way to transform a sequence arising from a 231-avoiding placement into a sequence arising from a 312-avoiding placement, and vice-versa.

In Section 2, we will review the needed definitions, define our bijection, and state the Theorems needed to verify that it is indeed a bijection. In Sections 3 and 4 we will prove these theorems.

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Vit Jelínek has pointed out to us that a bijective proof of the shape-Wilf-equivalence of the patterns 231 and 312 can also be obtained from his work on pattern–avoidance in matchings. See [3], where a bijection is obtained by establishing an isomorphism between generating trees. Examples on small Ferrers boards show that Jelínek's bijection differs from ours.

2. The bijection

Definitions. Let $\mathcal{A}$ be an $n \times n$ array of unit squares and coordinatize it by placing the bottom left corner of $\mathcal{A}$ at the origin in the xy-plane. We refer to the corners of all the squares in $\mathcal{A}$ as *vertices* and reference them by their cardinal position. For example, the upper right corner will be called the NE corner. For any vertex $V = (a, b)$ we define $R(V)$ to be the rectangular array of squares bounded by the lines $x = 0, x = a, y = 0$, and $y = b$.

A *Ferrers board* is any subset F of $\mathcal{A}$ with the property that $R(V) \subseteq F$ for each vertex in F. We define the *right/up border* of F to be the border of F excluding the vertical left hand side and horizontal bottom.

Next we need to define the generalization of a permutation for the context of Ferrers boards.

Definitions. A *rook placement* on a Ferrers board F is a subset of F that contains at most one square from each column of F and at most one square from each row of F. We indicate these squares by putting markers in them. Likewise a *full rook placement* is a rook placement such that each row and each column has exactly one marker in it. We say a rook placement P on a Ferrers board F *avoids* τ if and only if for every vertex V on the right/up border the permutation that is order-isomorphic to the restriction of P to $R(V)$ avoids τ in the usual sense.

Definition. For any rook placement P on F and any vertex V of F, we denote by $S(P, V)$ the maximal length of an increasing sequence of P in $R(V)$.

To define our bijection from $S_F(231)$ to $S_F(312)$, we first associate to each full rook placement P on F a sequence $S(P, F)$.

Notation. For any full rook placement on a Ferrers board F, $S(P, F)$ denotes the sequence of nonnegative integers obtained by taking $S(P, V)$ for all V on the right/up border of F, starting with the vertex at the top left corner of F.

Theorem 1. If P is in $S_F(231)$ or $S_F(312)$, then $S(P, F)$ and F determine P.

We will prove Theorem 1 in Section 3 by giving a "reverse algorithm" for the map $P \to S(P, F)$ consequently establishing injectivity.

Readers familiar with Fomin's growth diagram algorithm will note that the values of $S(P, F)$ are the first entries of the partitions in the oscillating tableaux produced by the algorithm. Theorem 1 may be restated by saying that for P

in $S_F(231)$ or P in $S_F(312)$ the first entries in the partitions determine the oscillating tableaux.

To define our bijection, we will need to characterize those sequences that arise from $P \in S_F(231)$ or $P \in S_F(312)$.

Definition. If F is a Ferrers board, then an *F-sequence* is a sequence of non-negative integers assigned to the vertices on the right/up border of F, starting with the vertex at the top left corner.

Definition (the 231-conditions). If F is a Ferrers board and S is an F-sequence, then the *231-conditions for the pair* (F, S) are the following three conditions:

(i) (monotonicity conditions) If V_1 and V_2 are vertices on the right/up border and V_1 is either directly to the left of V_2 or directly below V_2 then $S(V_1) \leq S(V_2) \leq S(V_1) + 1$.

(ii) (0-conditions) The first and last values of S are 0, and there do not exist consecutive vertices V_1 and V_2 such that $S(V_1) = 0 = S(V_2)$.

(iii) (diagonal condition) If V_1 and V_2 are vertices on the right/up border that are at the left and right ends of a diagonal with slope -1 that lies entirely within F, then $S(V_1) \leq S(V_2)$.

Definition (the 312-conditions). With S as in the preceding definition, the *312-conditions for the pair* (F, S) are the same as the 231-conditions, except that we reverse the inequalities in the diagonal condition.

The following definition is often useful when dealing with the diagonal condition.

Definition. We refer to a pair of vertices V_1, V_2 as *diagonal vertices* or *F-diagonal vertices* if they are on the right/up border of F and are at the left and right ends of a diagonal with slope -1 that lies entirely within F.

Theorem 2. If F is a Ferrers board whose longest row and longest column have the same length, and S is an F-sequence, then there exists $P \in S_F(231)$ (respectively, $P \in S_F(312)$) such that $S(P, F) = S$ if and only if (F, S) satisfies the 231-conditions (respectively, the 312-conditions).

Theorem 2 will be proved in Section 4.

To obtain our bijection, we need a way to take $P \in S_F(231)$ (respectively, $P \in S_F(312)$ and transform $S(P, F)$ into a sequence satisfying the 312-conditions (respectively, the 231-conditions). To do this we need our first lemma.

Lemma 1. For any Ferrers board F and vertex V on its right/up border, there exists an integer $N(F, V)$ such that for every full rook placement P on F, there are exactly $N(F, V)$ markers of P in $R(V)$.

Proof. Take any full rook placement P on F. We proceed inductively, starting with the vertex V at the top left corner. Clearly, P has no markers in $R(V)$. If V_1, V_2 are vertices on the right/up border such that V_1 is either directly to the left of V_2 or directly below it, then the number of markers of P in $R(V_2)$ is one greater than the number in $R(V_1)$. □

Definition. If P is a full rook placement on a Ferrers board F, and $S = S(P, F)$, then we define another F-sequence S^+ by letting $S^+(V) = 0$ if $S(V) = 0$, and $S^+(V) = N(F, V) + 1 - S(V)$ otherwise.

It is clear that S^+ is an F-sequence, because $S(V) \leq N(F, V)$.

Lemma 2. Let P be a full rook placement on F and let $S = S(P, F)$. Then if (F, S) satisfies the 231-conditions (respectively, the 312-conditions), (F, S^+) satisfies the 312-conditions (respectively, the 231-conditions).

Proof. We give the proof when (F, S) satisfies the 231-conditions. The proof of the other case is nearly identical.

To verify the monotonicity conditions for (F, S^+), first let V_1, V_2 be vertices on the right/up border of F such that V_1 is directly to the left of V_2. In the case that $S(V_1) = 0$ then $S(V_2) = 1$ by the 231-conditions. Observe that $N(F, V_2) = 1$ as well, which implies that $S^+(V_1) = 0$ and $S^+(V_2) = 1$. In the case that $S(V_1) \neq 0$ then we know that $S(V_1) \leq S(V_2) \leq S(V_1) + 1$. Since $N(F, V_1) + 1 = N(F, V_2)$ then we get

$$N(F, V_1) + 2 - S(V_1) \geq N(F, V_2) + 1 - S(V_2) \geq N(F, V_1) + 1 - S(V_1)$$

and hence $1 + S^+(V_1) \geq S^+(V_2) \geq S^+(V_1)$. The proof is the same if V_1 is directly below V_2 and therefore monotonicity holds.

The 0-conditions hold for (F, S^+) because $S^+(V) = 0$ if and only if $S(V) = 0$.

To verify the 312-diagonal condition for (F, S^+), let V_1, V_2 be F-diagonal vertices. We note that $N(F, V_1) = N(F, V_2)$, because $N(F, V)$ increases by one each time we move to the right on the right/up border, and decreases by one each time we move downward, and the number of rightward steps between V_1 and V_2 equals the number of downward steps. By the 231-diagonal condition for (F, S), we have $S(V_1) \leq S(V_2)$. If $S(V_1) \neq 0$, then since $N(F, V_1) = N(F, V_2)$, we have $S^+(V_1) \geq S^+(V_2)$. If $S(V_1) = 0$ then $N(F, V_1) = 0$ so $N(F, V_2) = 0$ and thus $S(V_2) = 0$. □

Definitions. Let $P \in S_F(231)$ and let $S = S(P, F)$. By Theorems 1 and 2, let $\alpha(P)$ denote the unique element of $S_F(312)$ such that $S(\alpha(P), F) = S^+$. For $P \in S_F(312)$, define $\beta(P) \in S_F(231)$ analogously.

Theorem 3. The maps $\alpha : S_F(231) \to S_F(312)$ and $\beta : S_F(312) \to S_F(231)$ are inverses, and therefore both are bijections.

Proof. This follows from the fact that if $S = S(P, F)$ for P in either $S_F(231)$ or $S_F(312)$, then $S^{++} = S$. □

Remark 1. Although our proofs depend on the fact that we are working with full rook placements it follows from Theorem 3 that for any Ferrers board F the number of 231-avoiding rook placements on F is equal to the number of 312-avoiding rook placements on F. The idea is as follows. For any rook placement P on F we have the the set $\mathcal{C}$ of column numbers corresponding to columns that contain a marker. Similarly we get the set $\mathcal{R}$ of row numbers. Now we may consider the set of squares

$$F_P = \{(c,r) | c \in \mathcal{C}, r \in \mathcal{R}\}.$$

Observe that we may view F_P as a Ferrers board by sliding all the squares down and then left. Likewise, P may be viewed as a full rook placement on F_P. We may now define an equivalence relation $\sim$ on rook placements by saying two placements P and Q are related if and only if $F_P = F_Q$. Now let A (respectively, B) be the partition under $\sim$ of the set of 231-avoiding (respectively, 312-avoiding) rook placements on F. Clearly $|A| = |B|$ and if $\overline{P} \in A$ then Theorem 3 implies that $|\overline{P}| = |\overline{\alpha(P)}|$ proving our claim.

3. The reverse algorithm

We will prove Theorem 1 by developing an "reverse algorithm" for the map $P \to S(P, F)$. To do this, we must first establish some properties of $S(P, V)$.

Lemma 3. Let P be a rook placement on Ferrers board F, and let V_1 and V_2 be vertices of F. Then if V_1 is directly to the left of V_2, or directly below V_2, we have

$$S(P, V_1) \leq S(P, V_2) \leq S(P, V_1) + 1.$$

Proof. This follows immediately from the definition of $S(P, V)$. □

Lemma 4. Suppose P is a rook placement on a Ferrers board F, and A, B, C are the vertices at the NW, NE, and SE corners, respectively, of a square $\mathcal{B}$ in F. Let a, b, c be the values of $S(P, V)$ at $V = A, B, C$, respectively. Then if P has no marker in $\mathcal{B}$, we have $b = \max(a, c)$. And P has a marker in $\mathcal{B}$ if and only if $b = a + 1 = c + 1$.

Proof. First suppose P has no marker in $\mathcal{B}$. Consider an increasing sequence I of length b in $R(B)$. If I is contained in $R(C)$, then $b \leq c$. If I is not contained in $R(C)$, then I must include a marker in the top row of $R(B)$, so I terminates at this marker, which is to the left of $\mathcal{B}$, and therefore I is contained in $R(A)$, yielding $b \leq a$. In either case, $b \leq \max(a, c)$. Since the reverse inequality follows from Lemma 3, we have $b = \max(a, c)$.

It follows that if P has no marker in $\mathcal{B}$ then we cannot have $b = a+1 = c+1$. It is clear that if P has a marker in $\mathcal{B}$ then $b = a + 1 = c + 1$. □

Lemma 5. Suppose $P \in S_F(231)$ and V_1, V_2 are vertices of F such that V_1 is directly below V_2. Suppose P has a marker X in the top row of $R(V_2)$, and another marker Y in $R(V_2)$ that is to the right of X. Then $S(P, V_1) = S(P, V_2)$.

Proof. Since $P \in S_F(231)$, P has no 231-patterns in $R(V_2)$. If R is the set of markers of P in $R(V_2)$ that are to the right of X, and L is the set of markers of P in $R(V_2)$ that are to the left of X, it follows that all elements of R are in higher rows than all elements of L. Since $R \neq \emptyset$ because of the presence of Y, both $S(P, V_1)$ and $S(P, V_2)$ are the sum of the maximal length of an increasing sequence in L and the maximal length of an increasing sequence in R. This proves the lemma. □

Proof of Theorem 1. It will suffice to prove the result for $P \in S_F(231)$, for then by considering the inverse placement P' on the conjugate board F', we obtain the result for $P \in S_F(312)$.

So let $P \in S_F(231)$. Suppose the bottom row of F contains exactly n squares, and the right-hand column of F contains exactly r squares. Let the values of $S(P, V)$ on the line $x = n$ be $b_r, \dots, b_0$, from top to bottom, and let the values on the line $x = n - 1$ be $a_r, \dots, a_0$, again from top to bottom. The values $a_r, b_r, b_{r-1}, \dots, b_0$ are included in $S(P, F)$, and we will show that from these values we can determine the location of the marker of P in the right-hand column, and the values of $a_{r-1}, \dots, a_0$.

Choose j as large as possible such that $b_j > b_{j-1}$. Then the markers $X_r, \dots, X_{j+1}$ of P in rows $r, \dots, j+1$ are not in the right-hand column, and since there is a marker Y in the right-hand column and there are no 231-patterns in $R((n, r))$, the markers $X_r, \dots, X_{j+1}$ must form a decreasing sequence. Applying Lemma 5 repeatedly, with the X of that lemma being $X_r, \dots, X_{j+1}$ in turn, we conclude that $a_r = \cdots = a_{j+1}$. The marker X_j in row j must be Y, for else, using X_j, Y and Lemma 5, we would have $b_j = b_{j-1}$. It follows that $a_j = b_{j-1}$ and $a_i = b_i$ for $i \leq j - 1$.

We have determined the placement of the marker Y in the right-hand column and the values of $a_{r-1}, \dots, a_0$. If we delete the right-hand column and the row containing the marker Y we obtain a smaller board F^* and a placement $P^* \in S_{F^*}(231)$ such that the sequence of values $S(P^*, F^*)$ is $S(P, F)$ with the terminal $r+1$ values $b_r, \dots, b_0$ replaced by the $r - 1$ values $a_{r-1}, \dots, a_j, a_{j-2}, \dots, a_0$. By iterating the above argument we can proceed to determine the positions of all the markers in P, from right to left. □

4. The proof of Theorem 2

To prove Theorem 2, it will suffice to prove the assertion about $P \in S_F(231)$, for the assertion about $P \in S_F(312)$ then follows by considering the inverse placement P' on the conjugate board F', with $P' \in S_{F'}(231)$.

Notation. Let F be a Ferrers board whose longest row and longest column each contain exactly n squares. Let $\mathcal{B}$ be the square at the top of the right-hand column of F, and suppose $\mathcal{B}$ is in row r. Let A, B, C be the vertices at the NW, NE, and SE corners of $\mathcal{B}$.

We will first prove the necessity of the 231-conditions, then the sufficiency.

Proof of Necessity.

The monotonicity conditions are clear by Lemma 3, and it is also clear that $S(P, F)$ starts and ends with the value 0. If the values of $S(P, F)$ at two successive vertices were both 0, then if one of these vertices were below (respectively, to the left of) the other, F would have a row (respectively, a column) with no marker in it, contradicting the fact that P is a full rook placement.

We now prove the diagonal condition by induction on the number of squares in F. For a board with one square, it is obvious that the 231-diagonal condition holds for the only possible placement. Assume now that $P \in S_F(231)$ and the result holds for all boards with fewer squares than F.

Case 1: $\mathcal{B}$ contains a marker.

Let $V_0, \ldots V_{2n}$ be the sequence of vertices on the right/up border of F starting at the top left corner of F, and let $B = V_k$. Since $S(P, V_{k-1}) = S(P, V_{k+1})$ by Lemma 4, it will suffice to check the diagonal condition for all diagonal vertices not containing V_{k+1}. To this end denote by F^* and P^* the board and placement obtained by deleting the row and column of $\mathcal{B}$ from F. Now let V_i^* be the vertex directly under V_i for $0 \le i \le k-1$ and the vertex directly to the left of V_i for $k+2 \le i \le 2n$. Observe that the sequence

$$V_0^*, \ldots, V_{k-1}^*, V_{k+2}^*, \ldots, V_{2n}^*$$

is precisely the sequence of vertices on the right/up border of F^*. Fix $i, j \notin \{k, k+1\}$. It is clear that

$$S(P, V_i) = S(P^*, V_i^*) \tag{1}$$

and that

$$V_i, V_j \text{ are } F\text{-diagonal vertices iff } V_i^*, V_j^* \text{ are } F^*\text{-diagonal vertices.} \tag{2}$$

By induction $S(P^*, F^*)$ satisfies the diagonal condition. Therefore (1) and (2) directly imply that $S(P, F)$ also satisfies the diagonal condition.

Case 2: $\mathcal{B}$ does not contain a marker.

Note that in this case we must have $r \ge 2$, and consider the smaller board $F^* = F \setminus \mathcal{B}$. By the induction hypothesis the pair $(F^*, S(P, F^*)))$ satisfies the diagonal condition. So we only need to show that $S(P, A) \le S(P, C)$. Since $\mathcal{B}$ contains no marker, Lemma 5 implies that $S(P, C) = S(P, B)$. By monotonicity we must have $S(P, A) \le S(P, B)$ which concludes this case. □

Proof of Sufficiency.

We prove the sufficiency of the 231-conditions by again using induction on the number of squares in F. Let S be an F-sequence such that (F, S) satisfies the 231-conditions, and let a, b, c denote $S(A), S(B), S(C)$, respectively.

Case 1: $b \neq a$ and $b \neq c$.

First note that in this case we must have $a+1=b=c+1$ by monotonicity of S. Define $V_0, \ldots, V_{2n}$ as in the proof of necessity, with $B = V_k$. Also define F^*, P^*, and vertices V_i^* as in that proof, so that the sequence

$$V_0^*, \ldots, V_{k-1}^*, V_{k+2}^*, \ldots, V_{2n}^*$$

is precisely the sequence of vertices on the right/up border of F^*.

Now define $S^*(V_i^*) = S(V_i)$ for $i \notin \{k, k+1\}$. We claim that (F^*, S^*) satisfies the 231-conditions. Since $a = c$ it is clear that (F^*, S^*) satisfies the monotonicity conditions. To see that (F^*, S^*) satisfies the 0-conditions it suffices to show that $a \neq 0$ when $r \geq 2$. So suppose $r \geq 2$ and $a = 0$. Draw the diagonal ℓ extending NW from A and let V_1 be the first vertex on the right/up border where ℓ passes outside of F. Since A is above the diagonal from upper left to lower right, there must be a vertex V_2 on the right/up border directly to the left of V_1. Since $a = 0$, the diagonal and monotonicity conditions for (F, S) yield $S(V_1) = 0$ and $S(V_2) = 0$, contradicting the 0-conditions for (F, S).

To verify the diagonal condition for (F^*, S^*), note that: V_i, V_j are F-diagonal vertices if and only if V_i^*, V_j^* are F^*-diagonal vertices. Therefore since (F, S) satisfies the diagonal condition so must (F^*, S^*).

Since (F^*, S^*) satisfies the 231-conditions, there exists, by the induction hypothesis, $P^* \in S_{F^*}(231)$ such that $S(P^*, F^*) = S^*$. Now restore the row and column we removed from F and place a marker X in square $\mathcal{B}$ to obtain a placement P on F. It is clear that $P \in S_F(231)$, because of the position of X. Lastly we show that $S(P, F) = S$. Note that for $V_i \neq B$ or C

$$S(P, V_i) = S(P^*, V_i^*) = S^*(V_i^*) = S(V_i). \tag{3}$$

Since P has a marker in $\mathcal{B}$, we have $S(P, A) + 1 = S(P, C) + 1 = S(P, B)$. By (3) we know that $S(P, A) = a$. Putting these together we have that $S(P, B) = a + 1 = b$ and $S(P, C) = a = c$.

Case 2: $b = a$ or $b = c$.

Note that in this case we cannot have $r = 1$, because if $r = 1$ then $c = 0$, so by the diagonal condition for (F, S), $a = 0$ and thus $b = 0$, violating the 0-conditions for (F, S). So we can let D be the vertex directly below C. Let E be the vertex at the SW corner of $\mathcal{B}$, and let d denote $S(D)$. Denote by F^* the Ferrers board $F \setminus \mathcal{B}$.

Now consider the function S^* defined by

$$S^*(V) = \begin{cases} S(V) & \text{if } V \neq E \\ \min(a, d) & \text{if } V = E \end{cases}$$

where V is a vertex on the right/up border of F^*.

In order to apply the induction hypothesis to the smaller pair (F^*, S^*) we need to know that (F^*, S^*) satisfies the 231-conditions. Since $r \geq 2$, we have $a \neq 0$ as in Case 1, and $a \leq c$. Since (F, S) satisfies both the monotonicity and 0-conditions it easily follows that (F^*, S^*) satisfies these two conditions as well.

So it only remains to show that (F^*, S^*) satisfies the diagonal condition. Now for the diagonal extending SE from E we have $S^*(E) = \min(a, d) \le d = S^*(D)$. Next consider the diagonal extending NW from E and let its right-most intersection point with the right/up border be E_0. (Note that E_0 exists since $r \ge 2$.) Call the vertex to E_0's immediate right A_0 and note that A_0 must be on the right/up border. Our choice of E_0 implies that A and A_0 are diagonal vertices. Now if $\min(a, d) = a$ then by our definitions we have

$$S^*(E_0) \le S^*(A_0) = S(A_0) \le S(A) = S^*(E).$$

If on the other hand $\min(a, d) = d$ then clearly $S^*(E_0) \le S^*(E)$ since E_0 and D are diagonal vertices in F. Therefore (F^*, S^*) satisfies the diagonal condition.

Since the pair (F^*, S^*) satisfies the 231-conditions then by the induction hypothesis there exists a 231-avoiding full rook placement P on F^* such that $S^* = S(P, F^*)$. We claim that P is also a 231-avoiding full rook placement on F such that $S = S(P, F)$.

To see that $S = S(P, F)$ let V be any vertex on the right/up border of F. If $V \ne B$ then we have $S(P, V) = S^*(V) = S(V)$. If $V = B$ then since $\mathcal{B}$ does not contain a marker we have $S(P, B) = \max(a, c) = b$ where the last equality holds because $\max(a, c) \le b$ by the monotonicity of S and $b \le \max(a, c)$ since $b = a$ or $b = c$ in this case.

Lastly we need to show that P is a 231-avoiding placement on F. Assume it is not and let XYZ be a 231-pattern in F. Let marker Y be in square $\mathcal{B}_1$ and Z be in square $\mathcal{B}_2$. Note that square $\mathcal{B}_1$ must be in row r, along with square $\mathcal{B}$. Likewise, note that $\mathcal{B}_2$ must be in the right-hand column. Since P in F^* has no 231-patterns then all the markers in the columns strictly between $\mathcal{B}_1$ and $\mathcal{B}$ must be above row r. For if not then some marker W is either in a row below X's row in which case XYW is a 231-pattern in F^*, or W is in a row between X's row and Y's row resulting in the 231-pattern XWZ. So if $\bar{A}$ and $\bar{E}$ denote the vertices in the NE and SE corners of $\mathcal{B}_1$ respectively then it follows that, letting $e = \min(a, d)$,

$$S(P, \bar{A}) = S(P, A) = a \quad \text{and} \quad S(P, \bar{E}) = S(P, E) = e.$$

If we could show that $a = e$, it would follow from Lemma 4 that $\mathcal{B}_1$ could not contain a marker. But this would be a contradiction as $\mathcal{B}_1$ contains the marker Y, and we would be done. To show that $a = e$, note that Z cannot be in row $r - 1$, because XYZ is a 231-pattern. Since row $r - 1$ must contain a marker, Lemma 5 implies that $c = d$ and therefore $e = \min(a, d) = a$ since $a \le c$ by the diagonal condition. □

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