Domination Number of the Complete Directed Grid Graphs

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Abstract

We use dynamic programming to compute the domination number of the cartesian product of two directed paths, $\vec{P}_m$ and $\vec{P}_n$, for $m \leq 25$ and all n. This suggests that the domination number for $\min(m,n) \geq 4$ is $\lfloor (m+1)(n+1)/3 \rfloor - 1$, which we then confirm by showing that this is both an upper and a lower bound on the domination number.

keywords: directed grid graph, domination number

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1 Introduction

A vertex v dominates vertex w in a digraph G if (v, w) is an arc of G. A dominating set S for a digraph G is a subset of the vertices of G with the property that every vertex v is in S or is dominated by some vertex in S. The domination number of G, $\gamma(G)$, is the minimum size of a dominating set.

Let $\vec{P}_n$ denote the directed path on n vertices; the complete directed grid graph $\vec{G}_{m,n}$ is the product $\vec{P}_m \times \vec{P}_n$. Fisher [3] used a dynamic programming algorithm to compute $\gamma(G_{m,n})$ for $m \leq 21$ and all n, where $G_{m,n}$ is the undirected grid graph. In particular, when $16 \leq m \leq 21$, he found that

$$\gamma(G_{m,n}) = \gamma_U \doteq \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor - 4.$$

Chang [1] showed that γ_U is an upper bound for $\gamma(G_{m,n})$ when $\min(m,n) \geq 8$, and conjectured that it gives $\gamma(G_{m,n})$ exactly when m and n are large enough. Fisher conjectured that in fact γ_U is the correct value for $\gamma(G_{m,n})$ when $\min(m,n) \geq 16$. Guichard [4] showed that $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 9$ is a lower bound for the domination number, and Gonçalves et al. [2] have recently confirmed the conjecture by improving the lower bound to $\lfloor \frac{(m+2)(n+2)}{5} \rfloor - 4$. We use similar techniques to compute the domination numbers of $\vec{G}_{m,n}$ for small m and all n, and then to confirm that the domination number for $\min(m,n) \geq 4$ is $\lfloor (m+1)(n+1)/3 \rfloor - 1$.

2 Exact values

Our approach is very similar to Fisher [3]. We picture $\vec{G}_{m,n}$ as consisting of n columns of m vertices each, with all arcs directed down or to the right. Given a subset S of the vertices, we code the state of a column as a vector $s_1, s_2, \ldots, s_m$, where $s_i = 0$ if the ith vertex (from the top) in the column is in S; $s_i = 1$ if the ith vertex is not in S but is dominated by an element of S, and $s_i = 2$ otherwise. If s and t are states, we say $s \geq t$ if $s_i \leq t_i$ for all i; a column in state s is "more dominated" than one in state t. An s-domination of $\vec{G}_{m,n}$ is a set of vertices that dominates $\vec{G}_{m,n}$ and leaves the last column in state s. An s-overdomination of $\vec{G}_{m,n}$ is a set of vertices that dominates $\vec{G}_{m,n}$ and leaves the last column in state u, $u \geq s$. Let $\gamma_n(s)$ be the minimum size of an s-domination of $\vec{G}_{m,n}$, and $\gamma_n^*(s)$ the minimum size of an s-overdomination. Then $\gamma_n^*(s) = \min_{u \geq s} \gamma_n(u)$, and $\gamma(\vec{G}_{m,n}) = \gamma_n^*(\mathbf{1})$, where $\mathbf{1}$ is the state consisting of all ones.

We use Fisher's algorithm, modified slightly for directed grid graphs. For the current section, we need only consider state vectors consisting of zeros and ones. Denote the number of zeros in state s by $|s|$, and define $\mathcal{P}(s)$, the set of previous states of s, to be the possible states for column $n-1$ in an s-domination of $\vec{G}_{m,n}$. It is easy to see that $t \in \mathcal{P}(s)$ if and only if

$$t_i = \begin{cases} 0 & \text{if } s_i = 1 \text{ and } s_{i-1} = 1 \\ 0 \text{ or } 1 & \text{otherwise,} \end{cases}$$

where, for convenience, we let $u_0 = 1$ for all states u. Then $\psi(s)$, the minimum previous state of s, is

$$\psi(s)_i = \begin{cases} 0 & \text{if } s_i > 0 \text{ and } s_{i-1} > 0 \\ 1 & \text{otherwise.} \end{cases}$$

That is, $t \in \mathcal{P}(s)$ if and only if $t \geq \psi(s)$. Let $u \succ s$ if for some i, $u_i = s_i - 1$ and $u_j = s_j$ for $j \neq i$.

The algorithm is:

Initialization. Let $\gamma_0^*(s) = 0$ if $s \geq \mathbf{1}$, and $\gamma_0^*(s) = \infty$ otherwise.

Iteration. Compute $\gamma_i^*(s) = \min(|s| + \gamma_{i-1}^*(\psi(s)), \min_{s' \succ s} \gamma_i^*(s')),\ i = 1, \ldots, n.$

Domination number. $\gamma(\overrightarrow{G}_{m,n}) = \gamma_n^*(\mathbf{1})$.

The correctness of the algorithm depends on a theorem of Fisher, modified for the directed case, which we include for completeness. In fact, the theorem is simpler for us, since we need consider only states consisting of zeros and ones.

Theorem 1. $\gamma_n^*(s) = \min(|s| + \gamma_{n-1}^*(\psi(s)), \min_{s' \succ s} \gamma_n^*(s'))$.

Proof. We have first that

$$\begin{aligned}\gamma_n^*(s) \leq \gamma_n(s) &= |s| + \min_{t \in \mathcal{P}(s)} \gamma_{n-1}(t)\\ &= |s| + \min_{t \geq \psi(s)} \gamma_{n-1}(t)\\ &= |s| + \gamma_{n-1}^*(\psi(s)).\end{aligned}$$

Second, $\gamma_n^*(s) \leq \gamma_n^*(s')$ for all $s' \succ s$, by definition of γ^*. Thus, $\gamma_n^* \leq \min(|s| + \gamma_{n-1}^*(\psi(s)), \min_{s' \succ s} \gamma_n^*(s'))$.

For the reverse inequality, there are two cases. If $\gamma_n^*(s) = \gamma_n(u)$ for some $u > s$, then there is an s' such that $u \geq s' \succ s$, so $\gamma_n^*(s') = \gamma_n(u) = \gamma_n^*(s)$. Otherwise, $\gamma_n^*(s) = \gamma_n(s)$, and $\gamma_n(s) = |s| + \gamma_{n-1}^*(\psi(s))$. This completes the proof. □

With this algorithm, we can compute $\gamma(\overrightarrow{G}_{m,n})$ for any m and n, provided with enough computational time. Suppose that for some values q, p, and n_0 we find that $\gamma_{n_0}^*(s) = q + \gamma_{n_0-p}^*(s)$ for all s. Then $\gamma(\overrightarrow{G}_{m,n}) = q + \gamma(\overrightarrow{G}_{m,n-p})$ for $n \geq n_0$, that is, we have found $\gamma(\overrightarrow{G}_{m,n})$ for a single m and all n. Of course, if n_0 is very large, this will not help, but in practice it appears that n_0 is small when m is small. (Conceivably, there is no such n_0 for some values of m. In the undirected case, Fisher showed that such an n_0 does always exist, and this is likely true here as well, but we don't need such a theorem.)

We implemented the algorithm in C, and on a modestly fast computer found $\gamma(\overrightarrow{G}_{m,n})$ for $1 \leq m \leq 25$ (memory is more limiting than speed). We found that

$$\gamma(\vec{G}_{m,n}) = \begin{cases} \lceil n/2 \rceil & m = 1, n \geq 1 \\ n & m = 2, n \geq 2 \\ \lceil 5n/4 \rceil & m = 3, n \geq 3 \\ \lfloor (m+1)(n+1)/3 \rfloor - 1 & 4 \leq m \leq 25, n \geq m \end{cases}$$

Since $G_{m,n} \cong G_{n,m}$, these results are valid with m and n interchanged as well. For $m \leq 5$, the domination numbers have been independently discovered by Shaheen [5].

3 An upper bound

We label the vertices of $\vec{G}_{m,n}$ with pairs of integers, assigning (r, c) to the point in row r and column c, with $(1,1)$ at the upper left corner. We present dominating sets of size $\lfloor (m+1)(n+1)/3 \rfloor - 1$ for $\vec{G}_{m,n}$. There are three cases, depending on the residues of m and n modulo 3. We assume $n \geq m$; since $G_{m,n} \cong G_{n,m}$, the result applies without this restriction as well.

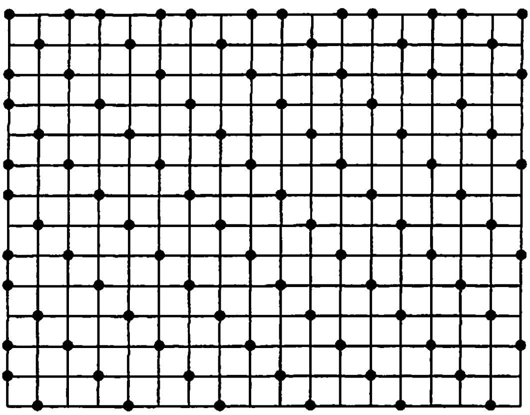

Figure 1: Dominating set when $m \equiv 2$ or $n \equiv 2$.

Case 1. When $m \equiv 2 \pmod 3$ or $n \equiv 2 \pmod 3$, the set consists of:

1. Points $(1, i)$ and $(i, 1)$, where $i \equiv 0 \pmod 3$
2. Points $(1, i)$ and $(i, 1)$, where $i \equiv 1 \pmod 3$.
3. Points on diagonals beginning at each of the points in (2), that is, $(1+j, i+j)$ and $(i+j, 1+j)$, where $i \equiv 1 \pmod 3$ and $j = 0, 1, 2, \ldots$

See Figure 1 for a 14×18 example with a dominating set of size 94. Note that the arrows on the arcs have been suppressed; all arcs are directed down or to the right. In general, the size of this set is

$$\left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + m\left(\frac{n - m - (n-m) \bmod 3}{3} + 1\right) + \sum_{i=1}^{\lfloor (m-1)/3 \rfloor} (m - 3i) + \sum_{i=1}^{\lfloor (m+(n-m) \bmod 3 - 1)/3 \rfloor} (m - 3i + (n-m) \bmod 3).$$

The third term counts the full diagonals; the fourth term counts the short diagonals in the southwest; the last term counts the short diagonals in the northeast; and the first two terms count the extra vertices in the first row and column, that is, the points in item (1) above. Some rather tedious algebraic manipulation (a computer algebra system is helpful) shows that this expression is in fact equal to $\lfloor (m+1)(n+1)/3 \rfloor - 1$.

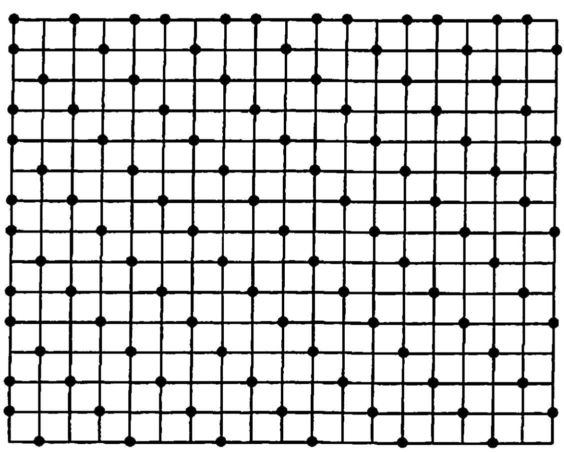

Figure 2: Dominating set when $n \equiv 1$ and $m \not\equiv 2$.

Case 2. When $n \equiv 1 \pmod 3$ and $m \not\equiv 2 \pmod 3$, the set consists of:

1. Points $(1, i)$ where $i \equiv 2 \pmod 3$ and $i \geq 5$.
2. Points $(1, i)$ where $i \equiv 0 \pmod 3$.
3. Points on diagonals beginning at each of the points in (2).
4. Points $(i, 1)$ where $i \equiv 1 \pmod 3$.
5. Points $(i, 1)$ where $i \equiv 2 \pmod 3$.
6. Points on diagonals beginning at each of the points in (5).

See Figure 2 for a 15 × 19 example with a dominating set of size 105. In general, the size of this set is

$$\left\lceil \frac{m}{3} \right\rceil - 1 + \left\lfloor \frac{n}{3} \right\rfloor + m\left(\frac{n-m-1+m \bmod 3}{3}\right) + \sum_{i=1}^{\lfloor m/3 \rfloor}(m-1-3(i-1)) + \sum_{i=1}^{\lfloor m/3 \rfloor}(2+3(i-1)).$$

The fourth term counts the full diagonals; the fifth term counts the short diagonals in the southwest; the last term counts the short diagonals in the northeast; and the first three terms count the extra vertices in the first row and column. Again, some algebraic manipulation shows that this expression is equal to $\lfloor (m+1)(n+1)/3 \rfloor - 1$.

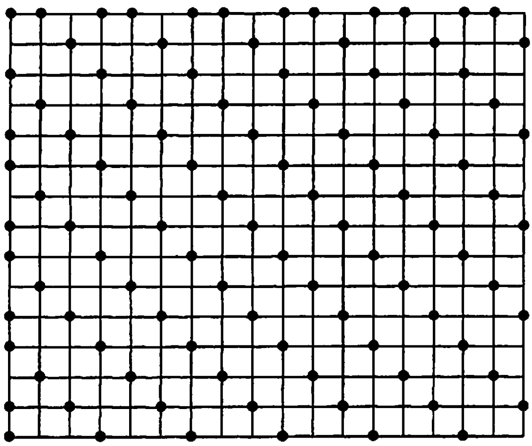

Figure 3: Dominating set when $n \equiv 0$ and $m \not\equiv 2$.

Case 3. When $n \equiv 0 \pmod 3$ and $m \not\equiv 2 \pmod 3$, the set consists of:

1. Points $(1, i)$ where $i \equiv 1 \pmod 3$.
2. Points $(1, i)$ where $i \equiv 2 \pmod 3$.
3. Points on diagonals beginning at each of the points in (2).
4. Points $(i, 1)$ where $i \equiv 2 \pmod 3$ and $i \geq 5$.
5. Points $(i, 1)$ where $i \equiv 0 \pmod 3$.
6. Points on diagonals beginning at each of the points in (5).

See Figure 3 for a 15×18 example with a dominating set of size 100. In general, the size of this set is

$$\left\lfloor \frac{m}{3} \right\rfloor - 1 + \left\lfloor \frac{n}{3} \right\rfloor + m\left(\frac{n - m + m \bmod 3}{3}\right) + \sum_{i=1}^{\lfloor m/3 \rfloor} (m - 2 - 3(i-1)) + \sum_{i=1}^{\lfloor m/3 \rfloor} (2 + 3(i-1)).$$

The fourth term counts the full diagonals; the fifth term counts the short diagonals in the southwest; the last term counts the short diagonals in the northeast; and the first three terms count the extra vertices in the first row and column. Again, some algebraic manipulation shows that this expression is equal to $\lfloor (m+1)(n+1)/3 \rfloor - 1$.

4 The lower bound

A vertex in the graph $\vec{G}_{m,n}$ dominates at most three vertices, including itself, so certainly $\gamma(\vec{G}_{m,n}) \geq nm/3$. If we could keep the sets dominated by individual vertices from overlapping, and use only vertices with outdegree 2, we could get a dominating set with $nm/3$ vertices, and indeed we can arrange this for 'most' of the graph, as shown in the figures of the previous section. At the edges we are forced to overlap some of the sets dominated by individual vertices, and also to use some vertices with outdegree less than 2.

Suppose S is a subset of the vertices of $\vec{G}_{m,n}$. Let $N[S]$ be the set of vertices v that are either in S or dominated by some w in S. Define the *wasted domination* of S as $w(S) = 3|S| - |N[S]|$, that is, the number of vertices we could dominate with $|S|$ vertices in the best case, less the number actually dominated. When S is a dominating set, $|N[S]| = mn$, and if $w(S) \geq L$ then $|S| \geq (L + mn)/3$. Our goal is to find a lower bound L for $w(S)$ that leads to a lower bound on $|S|$ equal to the upper bound.

Suppose $\vec{G}_{m,n}$ is partitioned into five parts as shown in Figure 4, where the widths of G_1 through G_4 are the same, say i, and the lengths are $m - i$ for G_2 and G_4, and $n - i$ for G_1 and G_3. Let S be a dominating set for $\vec{G}_{m,n}$, and let $S_k = S \cap V(G_k)$. Then

$$w(S) \geq \sum_{k=1}^{5} w(S_k) \geq \sum_{k=1}^{4} w(S_k).$$

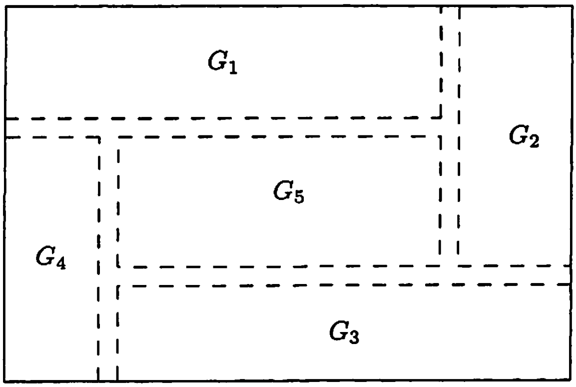

Figure 4: Partitioned digraph $\vec{G}_{m,n}$.

Note that in computing $w(S_k)$ we consider S_k to be a subset of $V(G)$, not of $V(G_k)$ (this affects the computation of $N[S_k]$). Since we expect that $w(S_5) = 0$, we hope that we are not giving up anything in the second inequality, but the first might well be strict, since the neighborhoods of two of the S_k may overlap.

Note that each G_k itself is a directed grid graph. The sets S_k are not necessarily dominating sets for the corresponding G_k, except when $k = 1$, but they are in some sense "almost" dominating sets: they might not dominate some vertices on the boundary of G_k. Using dynamic programming algorithms we can compute $\min_{A_k} w(A_k)$, taking the minimum over sets A_k that almost dominate G_k in the appropriate way, and of course computing $w(A_k)$ in the context of the whole graph $\vec{G}_{m,n}$. We can in fact do this for fixed width i but arbitrary length using algorithms very similar to the one in section 2; since $i = 4$ provides us with the bound we seek, we assume from this point that the width of the G_k is 4.

The sets $A_k^{\min}$ that realize the minimums $\min_{A_k} w(A_k)$ will not necessarily have the form $S \cap G_k$ for a dominating set S of $\vec{G}_{m,n}$, but of course $w(S_k) \geq w(A_k^{\min})$. Thus, $\sum_{k=1}^{4} A_k^{\min}$ will give us a lower bound for $\sum_{k=1}^{4} w(S_k)$, but not a best possible lower bound, and as noted above, $\sum_{k=1}^{4} w(S_k)$ may be strictly less than $w(S)$.

To improve the lower bound, we want to limit the sets A_k that we consider, ignoring those that cannot arise from a dominating set S for the entire graph, and also eliminate the underestimate induced by $w(S) > \sum_{k=1}^{4} w(S_k)$, due to overlapping neighborhoods. Gonçalves et al. [2] did this in the undirected case by paying careful attention to the boundaries between the graphs G_1 through G_4, and we do something similar here. There are two issues at the boundary where G_i meets G_j, if we have in hand two almost dominating sets A_i and A_j: some vertices which can only be dominated by vertices in G_i and G_j may be left undominated, and the neighborhoods of the two almost dominating sets may overlap, so that the wasted domination is underestimated. Denote the union of G_1, G_2, G_3, and G_4 by G_B, the boundary of $\vec{G}_{m,n}$. Then

$$w(S) \geq w(S \cap G_B) + w(S_5) \geq w(S \cap G_B),$$

and our goal is to find a good lower bound for $w(S \cap G_B)$.

Given a directed grid graph and a subset S of its vertices, we attach a label to each vertex of the graph: 0 if the vertex is in S; 1 if it is not in S but is dominated by an element of S, and 2 otherwise. Let us focus our attention first on G_1. As above, we picture $G_1 = \vec{G}_{4,j}$ as consisting of j columns of 4 vertices each, with all arcs directed down or to the right. Given a subset S of the vertices, we code the state of a column as a vector (s_1, s_2, s_3, s_4), using the labels 0, 1, 2 as described.

Let $r = (r_1, r_2, r_3, r_4)$ be a state vector, and imagine that it describes the status of the four vertices at the very left of row 5 in $\vec{G}_{m,n}$, immediately below G_1, that is, the four vertices in the top row of G_4. Let R_0 be the vertices labeled 0 by r. We interpret this state vector in isolation, so that the vertices are labeled relative to the subset corresponding to the 0 elements in r only.

An (r, s)-domination of $\vec{G}_{4,j}$ is a set S of vertices that dominates $\vec{G}_{4,j}$, leaves the last column in state s, and is *compatible* with r; that is, the elements of S together with the vertices labeled 0 by r dominate the first four vertices of row 5. Specifically, any of these 4 vertices labeled 2 must have a vertex of S immediately above in row 4. We want to compute the minimum possible value of $w(S)$ over (r, s)-dominations S, computing $w(S)$ in the context of R_0, namely, we set $w(S) = 3|S| - |N_r[S]|$, where $N_r[S]$ is $N[S] - N[R_0]$. Denote this minimum by $w_{1,j}(r, s)$. Note that this calculation addresses both issues at the boundary between G_1 and G_4.

Denote the number of zeros in state s by $|s|$, and define $\mathcal{P}_j(r, s)$, the set of previous states of s, to be the possible states for column $j-1$ in an (r, s)-domination of $\vec{G}_{4,j}$. The set $\mathcal{P}_j(r, s)$ depends only on s, not on r or j, when $j > 5$. In outline, an algorithm to compute $w_{1,j}(r, s)$ for any j is this:

1. **Initialization.** Set $w_{1,0}(r, s) = 0$ if $s = \mathbf{1}$ (the vector of all 1s), and

∞ otherwise.

2. **Iteration.** For $k > 0$, for each s, compute $w_{1,k}(r,s) = \min_{t \in \mathcal{P}_j(r,s)} (3|s| - \text{nd}(t,s) + w_{1,k-1}(r,t))$.

In essence, to compute $w_{1,k}(r,s)$, we are forming dominating sets S_s by adding elements of column k to a dominating set S_t of the first $k-1$ columns, computing $w(S_s)$, and minimizing over all t.

In the iteration step, the terms $3|s| - \text{nd}(t,s)$ represent the increment to the wasted domination when new vertices in column k are added to S_t, namely those labeled 0 by s. The term $3|s|$ reflects the potential number of new vertices that are dominated, and $\text{nd}(t,s)$ is the number of vertices that are newly dominated when a column with state s is appended. This is of course computed in the context of r as explained above: "newly dominated" means dominated by a vertex in column k labeled 0, but not already dominated by a vertex of $S_t \cup R_0$. The value of $\text{nd}(t,s)$ depends on s and t but not k once $k > 5$, that is, once the influence of r disappears.

With this algorithm we may compute $w_{1,j}(r,s)$ for all r and s and any j, given enough time. But suppose that for some values q, p, and $j_0 > 5+p$ we find that $w_{1,j_0}(r,s) = q + w_{1,j_0-p}(r,s)$ for all r and s. This allows us to compute $w_{1,j}(r,s)$ for all r, s, and j after a finite amount of time. Of course, if j_0 is very large (or does not exist), this will not help, but in practice it appears that j_0 is small when i is small, and in particular when $i = 4$. When we run the algorithm, we find that $w_{1,19}(r,s) = 2 + w_{1,16}(r,s)$, which implies that $w_{1,j}(r,s) = 2 + w_{1,j-3}(r,s)$ for $j \geq 19$, so the values of $w_{1,16}$, $w_{1,17}$, $w_{1,18}$ suffice to compute all $w_{1,j}$, $j \geq 19$.

We can do the same thing for the other three graphs, G_2, G_3, G_4. The algorithm in each case is slightly different, though it has the same form. We describe briefly the differences. Instead of working with G_2 as it appears, with 4 columns, we work with its transpose G_2^T, so that it is very much like G_1. The input word r we now imagine to describe the four vertices immediately above the beginning of the first row in G_2^T, namely, the rightmost column of G_1. We are now interested in sets A_2 that almost dominate G_2^T: vertices in the first row are allowed to remain undominated, as they can be dominated by vertices of G_5 in the full graph $\overrightarrow{G}_{m,n}$. The first four vertices in the first row, however, must be dominated either by A_2 or by the vertices in R_0, that is, A_2 must be compatible with r. As before, we compute $w_{2,j}(r,s)$ by taking r into account. When we run this algorithm we find that $w_{2,11}(r,s) = 1 + w_{2,8}(r,s)$ so that $w_{2,j}(r,s) = 1 + w_{2,j-3}(r,s)$ when $j \geq 11$.

For G_3, the algorithm changes somewhat more; the input word is now above the "wrong" end of the grid, so we work essentially with the mirror image of G_3, which we denote $\overleftarrow{G}_3$. Since this means the horizontal arcs are directed right-to-left, some adjustment is required to compute $\mathcal{P}_j$ and $\mathrm{nd}(t,s)$, but both are straightforward. The vertices in the top row may be undominated by A_3, and also the last column, with state s, may have undominated vertices, as they might be dominated by vertices of G_4. As before, we consider only those sets compatible with r: the first four vertices in the top row of $\overleftarrow{G}_3$ must be dominated either by A_3 or by the vertices described by r. When we run this algorithm, we find that $w_{3,13}(r,s) = 1 + w_{3,10}(r,s)$. Since G_3 always has the same size as G_1, we will make use of this fact starting at 19, that is, we note that $w_{3,j}(r,s) = 1 + w_{3,j-3}(r,s)$ when $j \geq 19$.

Finally, for G_4, we work with the mirror image of the transpose. Now all vertices of G_4 must be dominated by A_4 except those in the last column, as they may be dominated by vertices of G_1. We find that $w_{4,16}(r,s) = 2 + w_{4,13}(r,s)$ so that $w_{4,j}(r,s) = 1 + w_{4,j-3}(r,s)$ when $j \geq 16$. Since G_2 and G_4 are the same size, we note that this applies to $w_{2,j}$ as well.

Once we discovered the point at which the periodicity of each $w_{i,j}$ begins, we used slightly modified programs to save all values of $w_{1,16}$, $w_{1,17}$, $w_{1,18}$, $w_{2,13}$, $w_{2,14}$, $w_{2,15}$, $w_{3,16}$, $w_{3,17}$, $w_{3,18}$, $w_{4,13}$, $w_{4,14}$, and $w_{4,15}$ to disk files. Then we wrote an additional program to read all of these values in and compute the nine values

$$m_{a,b} = \min_{r,s,t,u} \left(w_{1,a}(r,s) + w_{2,b}(s,t) + w_{3,a}(t,u) + w_{4,b}(u,r) \right)$$

for $a = 16, 17, 18$ and $b = 13, 14, 15$. If S is a dominating set for $\overrightarrow{G}_{b+4,a+4}$, then $w(S) \geq w(S \cap G_B) \geq m_{a,b}$, so

$$\gamma(\overrightarrow{G}_{b+4,a+4}) \geq \left\lceil \frac{mn + m_{a,b}}{3} \right\rceil,$$

for $a = 16, 17, 18$ and $b = 13, 14, 15$. When we do the computation, we find that these lower bounds are $\lfloor (m+1)(n+1)/3 \rfloor - 1$, for $m = 17, 18, 19$ and $n = 20, 21, 22$.

We know that $w_{1,a}(r,s) + w_{3,a}(t,u) = 3 + w_{1,a-3}(r,s) + w_{3,a-3}(t,u)$ and $w_{2,b}(r,s) + w_{4,b}(t,u) = 3 + w_{2,b-3}(r,s) + w_{4,b-3}(t,u)$, so by a straightforward induction we see that

$$\gamma(\overrightarrow{G}_{m,n}) \geq \left\lceil \frac{mn + m_{n-4,m-4}}{3} \right\rceil = \left\lfloor \frac{(m+1)(n+1)}{3} \right\rfloor - 1$$

for all $m \geq 17$ and $n \geq 20$. This finishes the proof.

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