Decomposing λK_v into the graphs with seven vertices, seven edges and one 5-cycle for any index λ*

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Abstract. Let λK_v be the complete multigraph of order v and index λ, where any two distinct vertices x and y are joined exactly by λ edges $\{x, y\}$. Let G be a finite simple graph. A G-design of λK_v, denoted by (v, G, λ)-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called blocks, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. There are four graphs with seven vertices, seven edges and one 5-cycle, denoted by G_i, $i = 1, 2, 3, 4$. In [9], we have solved the existence problems of $(v, G_i, 1)$-GD. In this paper, we obtain the existence spectrum of (v, G_i, λ)-GD for any index λ.

Keywords: G-design, G-holey design, G-incomplete design.

1 Introduction

A *complete multigraph* of order v and index λ, denoted by λK_v, is a graph with v vertices, where any two distinct vertices x and y are joined by exactly λ edges $\{x, y\}$. A *t-partite graph* is one whose vertex set can be

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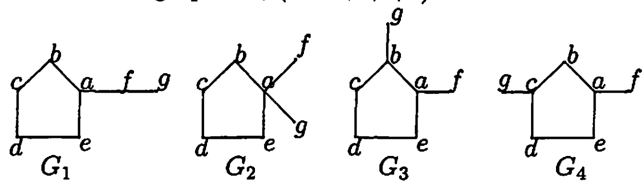
partitioned into t subsets $X_1, X_2, \ldots, X_t$, such that two ends of each edge lie in distinct subsets respectively. Such a partition $(X_1, X_2, \ldots, X_t)$ is called a *t-partition* of the graph. A *complete t-partite graph* with replication λ is a t-partite graph with t-partition $(X_1, X_2, \ldots, X_t)$, in which each vertex of X_i is joined to each vertex of X_j by λ times (where $i \neq j$). Such a graph is denoted by $\lambda K_{n_1,n_2,\ldots,n_t}$ if $|X_i| = n_i$ $(1 \leq i \leq t)$.

Let $(X_1, X_2, \ldots, X_t)$ be the t-partition of $K_{n_1,n_2,\ldots,n_t}$, and $|X_i| = n_i$. Denote $v = \sum_{i=1}^{t} n_i$ and $\mathcal{G}=\{X_1, X_2, \ldots, X_t\}$. For any given graph G, if the edge set of $\lambda K_{n_1,n_2,\ldots,n_t}$ can be decomposed into a collection of subgraphs $\mathcal{A}$, each of which is isomorphic to G and is called *block*, then the system $(X, \mathcal{G}, \mathcal{A})$ is called a *holey G-design* with index λ, denoted by $G\text{-}HD_\lambda(T)$, where $T = n_1^1 n_2^1 \ldots n_t^1$ is the *type* of the holey G-design. Usually, the type is denoted by exponential form, for example, the type $1^i 2^r 3^k \ldots$ denotes i occurrences of 1, r occurrences of 2, etc. A $G\text{-}HD_\lambda(1^{v-w} w^1)$ is called an *incomplete G-design*, denoted by $G\text{-}ID_\lambda(v; w) = (V, W, \mathcal{A})$, where $|V| = v$, $|W| = w$ and $W \subset V$.

Let G be a finite simple graph. A *G-design* of λK_v, denoted by (v, G, λ)-GD, is a pair $(X, \mathcal{B})$, where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly λ blocks of $\mathcal{B}$. Obviously, a (v, G, λ)-GD is a $G\text{-}HD_\lambda(1^v)$ or a $G\text{-}ID_\lambda(v; w)$ with $w = 0$ or 1. It is well known that if there exists a (v, G, λ)-GD, then

$$\lambda v(v-1) \equiv 0 \pmod{2e(G)} \quad \text{and} \quad \lambda(v-1) \equiv 0 \pmod{d},$$

where $e(G)$ denotes the number of edges in G and d is the greatest common divisor of the degrees of the vertices of G. For the path P_k and the star $K_{1,k}$, the existence problems of their graph designs have been solved (see [1] and [2]). For the graphs which have fewer vertices and fewer edges, the problem of their graph designs has already been researched (see [3]-[8]). There are four graphs with seven vertices, seven edges and one 5-cycle, denoted by $G_i, i = 1, 2, 3, 4$. In [9], we have solved the existence problems of $(v, G_i, 1)$-GD. In this paper, we obtain the existence spectrum of (v, G_i, λ)-GD for any index λ. The four graphs G_i $(i = 1, 2, 3, 4)$ are drawn as follows.



For convenience, the graphs $G_1 - G_4$ above are denoted by (a, b, c, d, e, f, g).

2 General structures

Theorem 2.1 *Let G be a simple graph. For positive integers h, m, λ and nonnegative integers w, if there exist a G-$HD_\lambda(h^m)$, a G-$ID_\lambda(h+w; w)$ and a (w, G, λ)-GD (or an $(h+w, G, \lambda)$-GD), then there exists a $(mh+w, G, \lambda)$-GD.*

Proof. Let $X = (Z_h \times Z_m) \bigcup W$, where W is a w-set. Suppose there exist

G-$HD_\lambda(h^m) = (Z_h \times Z_m, \mathcal{A})$,

G-$ID_\lambda(h + w; w) = ((Z_h \times \{i\}) \bigcup W, \mathcal{B}_i)$, $i \in Z_m$ or $i \in Z_m \setminus \{0\}$, and

(w, G, λ)-$GD = (W, \mathcal{C})$ or $(h + w, G, \lambda)$-$GD = ((Z_h \times \{0\}) \bigcup W, \mathcal{D})$,

then (X, Ω) is a $(mh + w, G, \lambda)$-GD, where

$$\Omega = \mathcal{A} \bigcup (\bigcup_{i=0}^{m-1} \mathcal{B}_i) \bigcup \mathcal{C} \quad \text{or} \quad \mathcal{A} \bigcup (\bigcup_{i=1}^{m-1} \mathcal{B}_i) \bigcup \mathcal{D}.$$

Note that

$$|\Omega| = \frac{\lambda\binom{mh+w}{2}}{e(G)} = \begin{cases} \frac{\lambda\binom{m}{2}h^2}{e(G)} + \frac{m(\lambda\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w}{2}}{e(G)} \\ \frac{\lambda\binom{m}{2}h^2}{e(G)} + \frac{\lambda(m-1)(\binom{h}{2}+wh)}{e(G)} + \frac{\lambda\binom{w+h}{2}}{e(G)} \end{cases}$$

$$= \begin{cases} |\mathcal{A}| + \sum_{i=0}^{m-1} |\mathcal{B}_i| + |\mathcal{C}| \\ |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{D}| \end{cases} . \square$$

The necessary conditions for the existence of (v, G_i, λ)-GD are $\lambda v(v - 1) \equiv 0 \pmod{14}$ and $v \geq 7$, i.e.,

$v \equiv 0, 1 \pmod 7$ for any λ, $v \equiv 2, 3, 4, 5, 6 \pmod 7$ for $\lambda \equiv 0 \pmod 7$.

In [9], we have obtained the following three lemmas:

Lemma 2.2 *There exists a $(v, G_i, 1)$-GD if and only if $v \equiv 0, 1 \pmod 7$ and $v \geq 7$, where $i = 1, 3, 4$.*

Lemma 2.3 *There exists a $(v, G_2, 1)$-GD if and only if $v \equiv 0, 1 \pmod 7$ and $v \neq 7$.*

Lemma 2.4 *There exists a G_i-$HD(7^{2t+1})$ for $i = 1, 2, 3, 4$.*

Theorem 2.5 *For $v \equiv 0, 1 \pmod 7$ and any $\lambda > 1$, there exists a (v, G_i, λ)-GD, where $i = 1, 3, 4$.*

Proof. By Lemma 2.2, let each block in $(v, G_i, 1)$-GD repeat λ times. □

Lemma 2.6 *There exists a $(7, G_2, \lambda)$-GD for any $\lambda > 1$.*

Proof. $(7, G_2, 2)$-GD: $X = Z_6 \bigcup \{\infty\}$ $(5, \infty, 0, 1, 3, 2, 4)$ mod 6.

$(7, G_2, 3)$-GD: $X = (Z_3 \times Z_2) \bigcup \{\infty\}$

$(0_1, 0_0, \infty, 1_1, 2_0, 1_0, 2_1)$ $(0_0, 1_1, 1_0, 2_1, \infty, 2_0, 0_1)$

$(0_0, \infty, 1_1, 0_1, 2_1, 1_0, 2_0)$ mod $(3, -)$.

Obviously, there are nonnegative integers m and n such that $\lambda = 2m + 3n$ for any $\lambda > 1$. So there exists a $(7, G_2, \lambda)$-GD for any $\lambda > 1$. □

Theorem 2.7 *For $v \equiv 0, 1 \pmod 7$ and any $\lambda > 1$, there exists a (v, G_2, λ)-GD.*

Proof. By Lemma 2.3 and Lemma 2.6. □

For $v \equiv 2, 3, 4, 5, 6 \pmod 7$ and $\lambda \equiv 0 \pmod 7$, in order to solve (v, G_i, λ)-GD, we only need to construct a $(v, G_i, 7)$-GD. By Theorem 2.1 and the following table, considering the existence of the needed HD (see Lemma 2.4), we only need to give the constructions of ID and GD for the pointed orders in the Table 2.1.

v(mod 14)	HD	ID	GD ($\lambda = 7$)
2	7^{2t+1}	(16; 9)	9
3	7^{2t+1}	(17; 10)	10
4	7^{2t+1}	(18; 11)	11
5	7^{2t+1}	(19; 12)	12
6	7^{2t+1}	(20; 13)	13
9	7^{2t+1}	(9 ; 2)	9
10	7^{2t+1}	(10; 3)	10
11	7^{2t+1}	(11; 4)	11
12	7^{2t+1}	(12; 5)	12
13	7^{2t+1}	(13; 6)	13

(Table 2.1) For G_1, G_2, G_3 and G_4

3 Incomplete G_i-designs

Lemma 3.1 *There exists a G_1-$ID(7+w;w)$ for $2 \le w \le 6$ and $9 \le w \le 13$.*

Proof. Let $X = Z_7 \bigcup \{a_1, a_2, \ldots, a_w\}$ and a G_1-$ID(7+w;w)$=$(X, \mathcal{B})$. The family $\mathcal{B}$ consists of the following blocks, where $|\mathcal{B}| = w+3$.

$$\begin{array}{llll}
w=2: & (0,4,1,3,6,5,2) & (1,a_1,4,5,6,0,a_2) & (2,0,a_1,5,a_2,4,3) \\
 & (3,a_2,4,6,2,5,1) & (a_1,2,1,a_2,6,3,0) &
\end{array}$$

$$\begin{array}{llll}
w=3: & (0,2,a_1,6,1,a_3,3) & (1,a_2,0,6,5,4,3) & (2,1,3,a_1,5,4,a_3) \\
 & (3,6,2,a_2,5,0,4) & (a_1,0,5,a_3,1,4,6) & (a_2,3,2,a_3,6,4,5)
\end{array}$$

$$\begin{array}{llll}
 & (0,a_1,2,1,3,a_2,6) & (1,a_2,3,a_4,4,6,0) & (2,5,a_4,6,3,0,1) \\
w=4: & (3,a_1,6,5,a_3,4,2) & (a_1,1,a_4,0,4,5,3) & (a_2,5,1,a_3,4,2,6) \\
 & (a_3,0,5,4,6,2,a_4) & &
\end{array}$$

$$\begin{array}{llll}
 & (0,a_1,2,4,a_2,5,a_5) & (1,a_1,4,5,a_2,6,a_5) & (2,0,6,3,a_3,1,a_5) \\
w=5: & (3,a_1,5,6,a_2,0,a_3) & (4,3,2,a_4,1,6,a_1) & (5,a_3,1,3,a_4,2,a_2) \\
 & (6,a_3,4,0,a_4,2,a_5) & (a_5,0,1,5,3,4,a_4) &
\end{array}$$

$$\begin{array}{llll}
 & (0,5,3,1,a_5,a_6,4) & (1,a_1,4,0,a_3,2,3) & (2,5,6,4,a_5,0,a_4) \\
w=6: & (3,a_2,0,6,a_5,4,a_4) & (4,1,a_6,2,a_2,5,a_5) & (a_1,0,3,a_3,2,6,a_6) \\
 & (a_2,5,a_6,3,6,1,0) & (a_3,5,a_4,2,4,6,1) & (a_4,3,a_1,5,1,6,2)
\end{array}$$

$$\begin{array}{llll}
 & (0,a_9,6,a_8,1,3,a_3) & (1,a_7,0,6,a_5,4,a_3) & (2,a_9,4,a_6,5,3,a_8) \\
w=9: & (3,a_1,2,0,a_2,6,a_4) & (4,a_1,1,3,5,0,a_4) & (5,a_8,4,3,a_7,0,a_5) \\
 & (6,a_7,2,a_2,1,5,a_5) & (a_1,5,1,a_3,6,0,a_6) & (a_2,5,a_4,2,6,4,a_7) \\
 & (a_3,2,1,a_9,5,0,a_8) & (a_4,4,2,a_6,1,3,a_9) & (a_5,4,6,a_6,3,2,a_8)
\end{array}$$

$$\begin{array}{llll}
 & (0,a_8,4,a_3,6,a_1,1) & (1,a_7,6,a_{10},3,5,2) & (2,1,a_5,6,a_1,a_3,0) \\
 & (a_1,5,6,1,4,3,a_7) & (a_2,3,5,a_4,0,2,a_7) & (a_3,1,a_2,4,5,3,a_8) \\
w=10: & (a_4,1,0,a_5,3,2,a_8) & (a_5,5,0,3,2,4,a_9) & (a_6,4,0,2,6,1,a_9) \\
 & (a_7,4,3,a_6,5,0,a_{10}) & (a_8,5,a_9,3,6,1,a_{10}) & (a_9,6,a_4,4,2,0,a_6) \\
 & (a_{10},4,6,a_2,5,2,a_6) & &
\end{array}$$

$$\begin{array}{llll}
 & (0,5,a_4,4,a_5,a_1,3) & (1,3,a_8,5,a_5,a_1,4) & (a_{11},3,5,4,0,1,6) \\
 & (2,4,a_9,3,a_2,a_{11},5) & (a_1,5,1,a_2,6,2,a_6) & (a_2,4,3,2,5,0,a_{10}) \\
w=11: & (a_3,3,0,a_4,6,1,a_8) & (a_4,3,a_6,4,1,2,a_9) & (a_5,2,1,0,6,3,a_7) \\
 & (a_6,5,a_7,4,6,0,a_8) & (a_8,2,0,a_3,4,6,a_{10}) & (a_7,2,a_3,5,6,0,a_9) \\
 & (a_9,5,a_{10},2,6,1,a_7) & (a_{10},3,6,a_{11},4,1,a_6) &
\end{array}$$

$$\begin{array}{llll}
 & (4,6,a_1,5,a_{11},a_{10},0) & (1,a_2,4,0,2,a_{12},3) & (2,a_2,3,1,a_7,a_{12},4) \\
 & (3,a_{11},0,a_{12},5,a_4,1) & (0,a_9,1,4,a_1,6,2) & (a_1,2,5,a_3,3,1,a_{11}) \\
w=12: & (a_2,5,4,a_3,0,6,a_{12}) & (a_3,2,a_4,0,1,6,a_7) & (a_4,5,1,a_6,6,4,a_8) \\
 & (a_5,4,3,a_6,2,1,a_{10}) & (a_6,0,a_5,6,5,4,a_9) & (a_7,4,2,3,0,5,a_8) \\
 & (a_9,2,a_{11},6,3,5,a_{10}) & (a_8,3,a_5,5,0,6,a_9) & (a_{10},2,a_8,1,6,3,a_7)
\end{array}$$

$$
w = 13: \begin{array}{lll}
(0, a_{12}, 3, 4, a_5, a_9, 5) & (1, a_7, 5, 0, 3, a_{10}, 6) & (2, 6, a_{13}, 3, a_{10}, 0, a_7) \\
(a_{12}, 1, a_8, 3, 5, 6, a_5) & (3, a_9, 4, a_8, 6, 2, a_5) & (a_1, 2, a_{13}, 1, 0, 4, a_{10}) \\
(a_1, 1, 2, a_7, 3, 6, a_4) & (a_2, 0, a_8, 5, 4, 1, a_3) & (a_{11}, 2, a_{12}, 4, 1, 0, a_6) \\
(a_2, 6, a_7, 4, 2, 5, a_1) & (a_5, 1, a_9, 2, 5, 3, a_2) & (a_{13}, 4, a_6, 6, 5, 0, a_3) \\
(a_3, 3, a_4, 0, 6, 2, a_8) & (a_4, 4, a_3, 5, 1, 2, a_6) & (4, a_{11}, 5, a_{10}, 0, 6, a_9) \\
(a_6, 3, a_{11}, 6, 1, 5, a_4) & &
\end{array}
$$

□

Lemma 3.2 *There exists a G_2-$ID(7+w; w)$ for $2 \le w \le 6$ and $9 \le w \le 13$.*

Proof. Let $X = Z_7 \bigcup \{a_1, a_2, \ldots, a_w\}$ and a G_2-$ID(7+w; w)$=$(X, \mathcal{B})$. The family $\mathcal{B}$ consists of the following blocks, where $|\mathcal{B}| = w + 3$.

$$
w = 2: \begin{array}{lll}
(4, 0, 3, 2, 1, a_2, 5) & (5, 1, 3, 4, 2, a_1, 6) & (6, a_2, 2, a_1, 0, 3, 4) \\
(a_1, 3, 5, 0, 1, 4, 6) & (a_2, 0, 2, 6, 1, 3, 5) &
\end{array}
$$

$$
w = 3: \begin{array}{lll}
(0, 6, 2, 5, 4, a_3, a_2) & (1, 4, 6, 3, 5, a_3, 0) & (2, a_1, 0, 5, a_2, a_3, 4) \\
(3, a_3, 5, 6, a_2, 1, 2) & (4, a_3, 6, 1, a_2, 3, a_1) & (a_1, 1, 2, 0, 3, 5, 6)
\end{array}
$$

$$
w = 4: \begin{array}{lll}
(a_1, 5, a_4, 1, 4, 2, 3) & (a_2, 3, 1, a_1, 6, 0, 2) & (a_3, 4, 2, 5, 3, 1, 6) \\
(a_4, 4, a_2, 5, 6, 0, 3) & (0, 5, 4, 6, 3, a_1, 1) & (1, 2, 0, a_3, 5, a_2, 6) \\
(2, 6, 0, 4, 3, a_3, a_4) & &
\end{array}
$$

$$
w = 5: \begin{array}{lll}
(0, 1, 6, a_3, 4, a_1, a_2) & (1, a_4, 0, 6, a_5, a_1, a_3) & (2, 0, a_5, 3, a_3, a_1, 1) \\
(3, 0, a_3, 5, 1, a_2, a_4) & (4, a_4, 5, 2, 6, a_2, a_5) & (5, a_5, 2, 3, 6, 4, 0) \\
(a_1, 4, 2, a_4, 6, 3, 5) & (a_2, 1, 4, 3, 5, 2, 6) &
\end{array}
$$

$$
w = 6: \begin{array}{lll}
(0, a_4, 2, 6, a_5, a_3, 4) & (1, 0, 5, 3, 4, a_4, a_2) & (2, 1, 3, a_6, 0, a_5, a_1) \\
(3, a_4, 5, 4, a_3, 0, a_2) & (4, a_5, 1, a_3, 2, 6, a_6) & (5, 6, a_4, 4, a_1, 2, 1) \\
(6, a_3, 5, a_5, 3, 1, a_2) & (a_1, 3, 2, a_6, 1, 0, 6) & (a_2, 0, 6, a_6, 5, 2, 4)
\end{array}
$$

$$
w = 9: \begin{array}{lll}
(a_1, 2, a_9, 4, 3, 6, 0) & (1, a_9, 0, a_6, 6, a_1, a_4) & (a_3, 1, 5, a_8, 0, 4, 2) \\
(a_4, 5, 2, a_6, 4, 3, 6) & (4, 0, 6, a_3, 5, a_7, a_8) & (0, 2, a_8, 1, a_7, a_4, a_2) \\
(a_2, 3, a_6, 1, 4, 6, 5) & (2, a_2, 1, 0, 3, a_4, a_5) & (3, a_7, 6, 5, a_9, a_3, a_5) \\
(a_5, 0, 5, 3, 6, 4, 1) & (5, a_7, 2, 4, a_1, a_5, a_6) & (6, a_8, 3, 1, 2, 4, a_9)
\end{array}
$$

$$
w = 10: \begin{array}{lll}
(a_5, 4, 2, a_3, 5, 6, 1) & (3, a_2, 5, a_1, 4, a_3, a_5) & (a_7, 2, 0, , a_8, 4, 6, 3) \\
(a_8, 1, a_9, 5, 3, 6, 2) & (6, 3, 1, a_7, 0, a_9, a_{10}) & (a_{10}, 1, a_3, 6, 2, 0, 3) \\
(0, a_1, 6, 1, a_4, 4, 5) & (1, a_2, 0, a_5, 2, 4, a_1) & (2, a_6, 0, 3, a_1, 5, a_2) \\
(a_6, 3, 2, a_4, 5, 6, 1) & (4, a_{10}, 5, 6, a_2, a_4, a_6) & (5, 4, a_3, 0, 1, a_7, a_8) \\
(a_9, 3, a_4, 6, 4, 0, 2) & &
\end{array}
$$

$$
w = 11: \begin{array}{lll}
(a_1, 5, 1, a_7, 6, 0, 3) & (1, a_3, 6, 2, a_5, a_1, a_4) & (a_6, 2, a_{10}, 0, 3, 4, 5) \\
(a_4, 3, a_8, 0, 5, 4, 6) & (5, a_8, 1, 0, 2, a_9, a_{10}) & (0, a_2, 3, 2, a_4, 6, a_3) \\
(a_2, 4, 2, a_8, 6, 1, 5) & (3, a_9, 1, a_{10}, 6, a_5, a_{11}) & (a_{11}, 0, a_9, 2, 1, 4, 5) \\
(a_3, 4, 0, a_7, 3, 2, 5) & (2, a_1, 4, 5, a_7, a_2, a_{11}) & (4, a_9, 6, 5, 3, a_7, a_8) \\
(a_5, 0, a_6, 1, 4, 6, 5) & (6, 1, 3, a_{10}, 4, a_{11}, a_6) &
\end{array}
$$

$$
w = 12: \quad \begin{array}{lll}
(a_1, 5, a_{10}, 2, 6, 0, 1) & (6, a_9, 3, 0, a_{11}, a_5, 5) & (a_3, 4, 0, a_{10}, 6, 1, 2) \\
(a_4, 1, 4, a_{12}, 6, 3, 5) & (1, a_{12}, 3, 2, a_5, a_2, a_{10}) & (a_6, 1, 3, a_{11}, 5, 4, 6) \\
(a_7, 1, a_9, 5, 2, 4, 6) & (2, a_{11}, 1, 0, a_{12}, a_1, a_9) & (0, a_8, 1, 2, a_4, a_3, 6) \\
(a_5, 0, a_6, 2, 4, 3, 5) & (4, 3, a_7, 0, 5, a_4, a_{11}) & (3, a_1, 4, 6, a_2, a_6, a_8) \\
(a_8, 2, 0, a_9, 4, 6, 5) & (5, a_3, 3, 6, 1, a_7, a_{12}) & (a_2, 4, a_{10}, 3, 5, 0, 2)
\end{array}
$$

$$
w = 13: \quad \begin{array}{lll}
(a_1, 3, 2, a_{11}, 4, 0, 1) & (3, a_{13}, 1, a_8, 0, a_2, a_9) & (4, a_{13}, 0, a_6, 5, a_3, a_{10}) \\
(a_4, 0, a_5, 1, 4, 2, 3) & (a_6, 2, a_{12}, 1, 6, 3, 4) & (6, a_2, 0, 5, a_{12}, a_5, a_{13}) \\
(a_3, 3, a_{12}, 4, 6, 1, 2) & (0, a_3, 5, 3, a_7, 6, a_{12}) & (a_9, 1, a_{11}, 0, 4, 5, 6) \\
(a_8, 4, 2, a_{10}, 3, 5, 6) & (1, a_4, 6, a_{11}, 3, a_7, a_6) & (2, a_9, 0, a_{10}, 5, a_1, a_8) \\
(a_2, 4, 3, 6, 5, 1, 2) & (a_7, 2, 1, a_{10}, 6, 4, 5) & (5, a_1, 6, 2, a_{13}, a_4, a_{11}) \\
(a_5, 2, 0, 1, 5, 3, 4) & &
\end{array}
$$

□

Lemma 3.3 *There exists a G_3-$ID(7+w;w)$ for $2 \le w \le 6$ and $9 \le w \le 13$.*

Proof. Let $X = Z_7 \bigcup \{a_1, a_2, \ldots, a_w\}$ and a G_3-$ID(7+w;w)$=$(X, \mathcal{B})$. The family $\mathcal{B}$ consists of the following blocks, where $|\mathcal{B}| = w + 3$.

$$
w = 2: \quad \begin{array}{lll}
(a_1, 0, 5, a_2, 4, 2, 6) & (a_1, 1, 0, 4, 6, 3, 5) & (a_2, 2, 3, 6, 1, 0, 4) \\
(4, 3, 0, 2, 5, 1, a_2) & (6, 5, 3, 1, 2, a_2, a_1) &
\end{array}
$$

$$
w = 3: \quad \begin{array}{lll}
(0, 4, 5, a_3, 6, 2, 3) & (0, 5, a_1, 4, a_3, 3, 6) & (1, 6, a_2, 3, a_3, 5, 4) \\
(1, 3, 6, 2, 4, 0, 5) & (2, a_1, 0, a_2, 5, 1, 6) & (2, a_2, 1, a_1, 3, a_3, 4)
\end{array}
$$

$$
w = 4: \quad \begin{array}{lll}
(0, a_1, 6, 2, 1, a_4, 5) & (1, a_1, 4, 0, 3, a_4, 2) & (2, a_2, 1, 5, 3, a_3, 4) \\
(3, a_2, 5, 6, 4, a_1, 0) & (4, a_3, 0, 2, 5, a_4, 6) & (5, a_3, 3, 6, 0, a_4, 1) \\
(6, a_4, 2, 4, 1, a_2, 3) & &
\end{array}
$$

$$
w = 5: \quad \begin{array}{lll}
(a_1, 0, a_4, 2, 1, 6, 3) & (a_2, 0, 2, 3, 6, 5, 4) & (a_2, 1, 6, 5, 2, 4, 0) \\
(a_1, 3, 4, 6, 2, 5, a_2) & (a_3, 4, a_5, 5, 3, 0, a_1) & (a_3, 5, 4, a_4, 1, 6, 0) \\
(a_4, 6, a_5, 1, 5, 3, 0) & (a_5, 2, 4, 1, 3, 0, a_3) &
\end{array}
$$

$$
w = 6: \quad \begin{array}{lll}
(1, 5, a_4, 6, a_2, a_6, 4) & (0, 5, a_2, 2, a_6, a_5, a_1) & (0, 4, a_1, 2, 1, 3, a_3) \\
(1, 6, 5, a_6, 3, a_4, a_3) & (2, 6, a_1, 1, a_3, 3, a_5) & (2, 0, a_1, 3, 5, 4, 6) \\
(3, a_2, 4, a_6, 6, a_5, 0) & (3, a_3, 5, a_5, 4, a_4, 0) & (4, a_4, 2, a_5, 1, 6, 0)
\end{array}
$$

$$
w = 9: \quad \begin{array}{lll}
(a_5, 0, a_2, 1, 4, 6, a_7) & (a_5, 1, 0, 3, 5, 2, a_3) & (a_6, 2, 6, a_2, 3, 1, a_8) \\
(a_6, 0, a_1, 1, 5, 4, 2) & (a_7, 1, 2, a_1, 3, 4, a_4) & (a_7, 2, 4, a_1, 5, 6, a_2) \\
(a_8, 3, 2, a_3, 0, 5, 6) & (a_8, 4, a_4, 3, 1, 6, a_9) & (a_9, 5, a_3, 6, 1, 2, 0) \\
(a_9, 3, a_3, 4, 0, 6, a_5) & (6, 4, a_2, 5, a_4, a_1, 3) & (6, 5, 2, a_4, 0, a_6, 4)
\end{array}
$$

$$
w = 10: \quad \begin{array}{lll}
(2, a_{10}, 0, 4, a_8, a_5, 3) & (0, a_9, 6, 1, a_8, a_2, 5) & (1, a_9, 4, 3, a_1, a_3, 2) \\
(1, a_{10}, 6, 2, 4, a_4, 5) & (0, 6, 5, 3, 1, a_1, 4) & (2, a_1, 6, a_5, 0, a_2, 4) \\
(3, a_2, 6, a_3, 0, a_4, 1) & (3, a_3, 5, 1, a_7, 6, 2) & (3, a_8, 5, 2, a_6, a_9, 6) \\
(4, a_5, 3, 2, a_7, a_3, 1) & (5, a_6, 1, 2, a_4, a_5, 0) & (5, a_7, 6, a_6, 4, a_1, 0) \\
(4, a_4, 0, 5, a_2, a_{10}, 6) & &
\end{array}
$$

$w = 11:$

$(0,a_4,4,3,2,a_9,5)$ $(0,a_5,6,a_6,4,a_{11},3)$ $(1,a_6,5,a_5,4,a_2,0)$
$(6,a_{11},1,a_5,2,0,5)$ $(2,a_2,5,0,a_8,a_{10},4)$ $(2,a_3,0,3,a_6,a_{11},6)$
$(3,a_1,2,5,1,a_7,0)$ $(3,a_2,6,4,a_{11},a_{10},0)$ $(2,a_7,5,a_{10},4,a_9,0)$
$(5,a_8,3,a_9,6,4,1)$ $(4,a_3,1,6,a_8,a_1,3)$ $(5,a_9,1,a_4,3,a_3,4)$
$(6,a_7,1,0,a_{10},3,4)$ $(1,a_1,6,a_4,2,a_{10},5)$

$w = 12:$

$(a_2,0,3,a_8,5,1,a_{11})$ $(a_4,0,1,a_{11},6,5,a_1)$ $(a_8,2,0,a_{10},4,6,a_{12})$
$(a_5,1,3,a_6,0,5,2)$ $(a_6,1,4,a_7,6,2,a_{12})$ $(3,4,a_9,5,a_1,a_2,a_4)$
$(a_3,0,5,2,6,4,a_{12})$ $(a_9,2,a_2,4,0,3,a_{11})$ $(3,5,a_6,4,a_{11},a_4,a_{12})$
$(a_7,1,a_3,2,3,0,a_9)$ $(a_{10},2,a_4,1,6,5,a_1)$ $(3,6,a_1,1,a_{10},a_3,a_9)$
$(5,6,0,a_8,1,a_3,a_{12})$ $(4,6,a_5,3,a_{12},2,a_2)$ $(4,5,a_7,2,a_5,a_1,a_{11})$

$w = 13:$

$(a_1,6,a_3,1,5,2,a_{12})$ $(a_2,0,a_{13},1,3,5,a_5)$ $(a_5,1,2,a_{13},6,5,a_{11})$
$(a_4,1,6,2,3,0,a_2)$ $(a_3,0,6,a_7,3,2,a_{11})$ $(5,4,a_8,3,a_9,a_{13},a_{12})$
$(a_7,2,a_6,0,1,5,a_4)$ $(a_8,2,a_5,4,1,0,a_{11})$ $(6,5,a_3,4,a_2,a_4,a_{12})$
$(a_{10},3,a_1,0,2,5,a_5)$ $(a_{11},3,0,a_7,4,6,a_{13})$ $(4,3,a_{12},0,a_9,a_{13},a_6)$
$(a_6,1,a_{12},2,5,4,a_1)$ $(a_9,2,4,a_{10},1,6,a_2)$ $(6,4,a_4,5,a_8,a_6,a_1)$
$(0,5,3,6,a_{10},4,a_{11})$ □

Lemma 3.4 *There exists a G_4-$ID(7+w;w)$ for $2 \le w \le 6$ and $9 \le w \le 13$.*

Proof. Let $X = Z_7 \bigcup \{a_1, a_2, \ldots, a_w\}$ and a G_4-$ID(7+w;w)$=$(X, \mathcal{B})$. The family $\mathcal{B}$ consists of the following blocks, where $|\mathcal{B}| = w + 3$.

$w = 2:$

$(a_1,3,0,6,1,5,a_2)$ $(a_1,0,1,5,4,2,3)$ $(6,4,2,3,a_2,a_1,0)$
$(a_2,5,3,6,2,1,4)$ $(5,0,4,1,2,6,a_2)$

$w = 3:$

$(0,5,3,a_3,1,4,6)$ $(0,3,4,a_2,2,a_3,1)$ $(1,a_1,5,4,6,3,a_3)$
$(1,5,6,0,a_2,2,a_3)$ $(2,3,a_1,4,a_3,6,0)$ $(2,5,a_2,6,a_1,4,3)$

$w = 4:$

$(0,2,4,6,3,a_4,a_2)$ $(1,3,5,4,0,a_4,a_1)$ $(2,1,6,0,5,a_4,a_3)$
$(3,2,a_1,1,4,a_4,0)$ $(a_2,3,a_1,6,5,2,4)$ $(a_3,0,a_2,6,2,5,1)$
$(a_4,4,a_3,1,5,6,3)$

$w = 5:$

$(0,1,a_1,3,a_5,6,4)$ $(0,4,a_2,5,2,3,1)$ $(1,5,a_3,2,a_4,4,3)$
$(2,3,a_4,6,a_5,a_1,0)$ $(3,5,a_1,6,1,a_2,0)$ $(4,2,a_2,6,3,5,0)$
$(5,0,a_3,4,a_4,6,1)$ $(6,4,a_5,1,2,a_3,5)$

$w = 6:$

$(0,a_3,1,a_2,4,5,a_4)$ $(0,a_5,5,6,1,a_2,4)$ $(1,a_6,5,a_3,3,2,a_2)$
$(1,a_5,6,3,4,5,2)$ $(2,a_2,6,0,a_1,a_5,4)$ $(2,5,a_1,6,a_6,0,1)$
$(3,5,a_4,0,a_6,a_2,4)$ $(3,2,a_3,4,a_5,0,6)$ $(4,2,a_4,3,a_1,a_6,6)$

$w = 9:$

$(a_5,2,0,a_3,4,1,a_7)$ $(a_5,0,1,a_1,3,5,a_6)$ $(a_9,4,2,a_1,0,5,a_4)$
$(a_6,3,0,a_2,6,5,4)$ $(6,3,1,a_2,2,0,a_9)$ $(a_7,1,2,a_3,3,4,5)$
$(a_8,2,3,a_4,0,5,4)$ $(a_8,1,4,a_2,3,6,5)$ $(a_6,0,5,6,4,2,a_2)$
$(a_9,3,5,a_7,2,6,a_3)$ $(6,a_1,4,a_4,1,a_5,a_8)$ $(6,a_4,5,1,a_3,a_7,a_1)$

$$w=10:\ \begin{array}{lll}
(0,a_8,6,a_1,2,a_5,1) & (0,1,a_9,2,a_7,a_1,6) & (1,3,a_9,4,a_4,5,0)\\
(1,2,a_{10},5,a_7,4,0) & (2,3,a_{10},1,a_6,a_3,6) & (2,4,a_1,1,a_2,5,3)\\
(3,4,a_2,5,6,a_3,0) & (3,0,a_3,6,a_4,a_2,1) & (4,0,a_4,5,a_3,a_5,2)\\
(4,6,a_5,1,a_8,a_{10},2) & (5,4,a_6,3,a_5,a_1,0) & (5,3,a_7,6,a_6,a_9,4)\\
(6,2,a_8,5,0,a_2,3) & &
\end{array}$$

$$w=11:\ \begin{array}{lll}
(0,1,a_1,2,a_6,4,6) & (1,5,a_1,0,a_7,a_9,4) & (0,5,a_2,3,a_9,a_{11},2)\\
(3,0,a_3,6,a_4,5,2) & (2,1,a_2,0,a_4,a_{10},6) & (4,2,a_7,6,a_8,a_9,3)\\
(3,4,a_4,5,6,a_1,1) & (6,0,a_{10},3,a_5,a_{11},1) & (5,4,a_6,1,a_8,a_3,6)\\
(2,3,a_3,1,a_5,0,4) & (5,2,a_8,3,a_6,a_{10},0) & (4,6,a_9,5,a_7,a_2,2)\\
(1,4,a_5,5,a_{11},3,0) & (6,2,a_{11},4,a_{10},1,3) &
\end{array}$$

$$w=12:\ \begin{array}{lll}
(a_2,0,a_8,1,5,4,2) & (2,a_6,0,4,a_4,5,a_{11}) & (2,a_{10},0,5,a_5,a_2,a_1)\\
(a_9,4,a_5,1,3,6,0) & (5,a_8,4,6,a_1,a_{11},1) & (4,a_1,1,a_6,5,a_{11},a_{12})\\
(a_4,6,a_7,2,3,0,4) & (1,a_2,3,0,a_9,a_4,6) & (5,a_{12},6,a_8,3,a_4,a_3)\\
(a_3,3,a_6,4,2,0,6) & (5,a_9,2,0,a_7,a_3,a_1) & (6,a_5,3,a_7,1,a_{11},a_{12})\\
(a_{10},1,0,6,5,4,a_{12}) & (1,a_3,4,a_{12},2,a_{11},3) & (6,a_{10},3,a_{11},2,a_2,a_1)
\end{array}$$

$$w=13:\ \begin{array}{lll}
(a_4,2,1,a_7,6,0,a_{13}) & (5,a_{13},6,2,a_7,1,a_{12}) & (a_3,6,0,a_2,2,3,a_5)\\
(a_5,5,0,a_{11},2,1,a_6) & (5,a_{10},1,a_{12},4,a_3,a_{11}) & (a_1,1,6,a_8,4,0,a_5)\\
(a_{10},3,1,a_9,2,0,a_3) & (4,a_{13},3,6,a_{10},a_3,a_{11}) & (a_8,3,2,a_{13},0,1,5)\\
(a_6,4,2,a_8,5,1,a_{12}) & (a_{11},5,3,a_6,6,4,a_{12}) & (a_7,0,2,a_1,3,4,a_6)\\
(a_9,0,3,a_5,4,5,a_4) & (4,a_2,6,a_9,3,a_4,a_1) & (4,6,5,a_4,1,0,a_1)\\
(a_2,1,0,a_{12},5,3,a_3) & &
\end{array}$$

□

4 Graph designs

Lemma 4.1 *There exists a* $(v, G_1, 7)$ *for* $v = 9, 10, 11, 12, 13$.

Proof. $v = 9$: $X = Z_9$

$(4,0,1,2,3,5,8)$ $(8,0,2,4,6,5,3)$ $(5,0,3,6,2,8,7)$

$(7,0,4,8,3,5,2)$ mod 9

$v = 10$: $X = Z_9 \bigcup \{\infty\}$

$(4,0,1,2,3,5,8)$ $(\infty,0,2,4,6,5,8)$ $(5,0,3,6,2,\infty,7)$

$(7,0,4,8,3,\infty,2)$ $(7,0,1,3,6,5,8)$ mod 9

$v = 11$: $X = Z_{11}$

$(4,0,1,2,3,5,6)$ $(8,0,2,4,6,10,1)$ $(1,0,3,6,9,4,7)$

$(5,0,4,8,1,9,2)$ $(9,0,5,10,4,3,8)$ mod 11

$v = 12$: $X = Z_{11} \bigcup \{\infty\}$

$(\infty,0,1,2,3,4,9)$ $(8,0,2,4,6,3,1)$ $(0,\infty,9,6,3,4,1)$

$(4,0,\infty,1,8,10,7)$ $(4,0,1,2,3,7,9)$ $(9,0,5,10,4,2,6)$ mod 11

$v = 13$: $X = Z_{13}$

$(4,0,1,2,3,10,6)$ $(8,0,2,4,6,1,5)$ $(12,0,3,6,9,7,4)$ $(3,0,4,8,12,6,7)$

$(7,0,5,10,2,12,1)$ $(11,0,6,12,5,10,8)$ mod 13 □

Theorem 4.2 *For* $v \equiv 2,3,4,5,6 \pmod 7$ *and* $\lambda \equiv 0 \pmod 7$, *there exists a* (v, G_1, λ)*-GD.*

Proof. By Theorem 2.1, Lemma 2.4, Lemma 3.1 and Lemma 4.1. □

Lemma 4.3 *There exists a* $(v, G_2, 7)$ *for* $v = 9, 10, 11, 12, 13$.

Proof. $v = 9$: $X = Z_9$

$(4,0,1,2,3,7,6)$ $(8,0,2,4,6,7,5)$ $(5,0,3,6,2,7,8)$

$(7,0,4,8,3,1,6)$ mod 9

$v = 10$: $X = Z_9 \bigcup \{\infty\}$

$(\infty,0,1,2,3,5,6)$ $(6,\infty,7,2,4,1,3)$ $(0,1,3,6,8,\infty,4)$

$(7,0,1,2,4,3,5)$ $(5,0,3,6,2,1,8)$ mod 9

$v = 11$: $X = Z_{11}$

$(4,0,1,2,3,9,8)$ $(8,0,2,4,6,3,1)$ $(1,0,3,6,9,2,10)$

$(5,0,4,8,1,6,3)$ $(9,0,5,10,4,6,1)$ mod 11

$v = 12$: $X = Z_{11} \bigcup \{\infty\}$

$(\infty,0,1,2,3,4,5)$ $(6,\infty,0,2,4,3,8)$ $(0,3,6,9,1,\infty,4)$

$(5,0,4,8,1,9,2)$ $(9,0,5,10,4,3,6)$ $(0,1,2,3,5,4,9)$ mod 11

$v = 13$: $X = Z_{13}$

$(4,0,1,2,3,10,9)$ $(8,0,2,4,6,3,7)$ $(12,0,3,6,9,8,11)$

$(3,0,4,8,12,9,1)$ $(7,0,5,10,2,3,4)$ $(11,0,6,12,5,8,9)$ mod 13 □

Theorem 4.4 *For* $v \equiv 2,3,4,5,6 \pmod 7$ *and* $\lambda \equiv 0 \pmod 7$, *there exists a* (v, G_2, λ)*-GD.*

Proof. By Theorem 2.1, Lemma 2.4, Lemma 3.2 and Lemma 4.3. □

Lemma 4.5 *There exists a* $(v, G_3, 7)$ *for* $v = 9, 10, 11, 12, 13$.

Proof. $v = 9$: $X = Z_9$

$(4,0,1,2,3,7,5)$ $(8,0,2,4,6,1,3)$ $(4,0,3,6,1,2,8)$

$(5,0,4,8,2,3,1)$ mod 9

$v = 10$: $X = Z_9 \bigcup \{\infty\}$

$(\infty,0,1,2,3,5,6)$ $(\infty,0,2,4,6,1,3)$ $(0,3,6,1,4,2,\infty)$

$(0,4,8,3,7,2,6)$ $(0,1,2,3,4,6,7)$ mod 9

$v = 11$: $X = Z_{11}$

$(0,1,2,3,4,9,10)$ $(8,0,2,4,6,9,1)$ $(1,0,3,6,9,4,8)$

$(5,0,4,8,1,10,6)$ $(9,0,5,10,4,2,7)$ mod 11

$v = 12$: $X = Z_{11} \bigcup \{\infty\}$

$(\infty, 0, 1, 2, 3, 4, 8)$ $(\infty, 0, 2, 4, 6, 1, 3)$ $(0, 3, 6, 9, 1, 2, \infty)$

$(0, 4, 8, 1, 5, 3, 6)$ $(0, 5, 10, 4, 9, 2, 6)$ $(0, 4, 9, 3, 7, 1, 5)$ mod 11

$v = 13$: $X = Z_{13}$

$(0, 1, 2, 3, 4, 6, 7)$ $(0, 2, 4, 6, 8, 3, 5)$ $(0, 3, 6, 9, 12, 4, 7)$

$(0, 4, 8, 12, 3, 5, 9)$ $(0, 5, 10, 2, 7, 1, 6)$ $(0, 6, 12, 5, 11, 2, 8)$ mod 13 □

Theorem 4.6 *For* $v \equiv 2, 3, 4, 5, 6 \pmod 7$ *and* $\lambda \equiv 0 \pmod 7$, *there exists a* (v, G_3, λ)*-GD.*

Proof. By Theorem 2.1, Lemma 2.4, Lemma 3.3 and Lemma 4.5. □

Lemma 4.7 *There exists a* $(v, G_4, 7)$ *for* $v = 9, 10, 11, 12, 13$.

Proof. $v = 9$: $X = Z_9$

$(4, 0, 1, 2, 3, 7, 5)$ $(8, 0, 2, 4, 6, 1, 3)$ $(4, 0, 3, 6, 1, 2, 5)$

$(5, 0, 4, 8, 2, 6, 7)$ mod 9

$v = 10$: $X = Z_9 \bigcup \{\infty\}$

$(\infty, 0, 1, 2, 3, 5, 4)$ $(\infty, 0, 2, 4, 6, 1, 5)$ $(0, 3, 6, 1, 4, 2, 8)$

$(0, 4, 8, 3, 7, 2, 5)$ $(0, 1, 2, 3, 4, 6, \infty)$ mod 9

$v = 11$: $X = Z_{11}$

$(0, 1, 2, 3, 4, 9, 6)$ $(8, 0, 2, 4, 6, 9, 7)$ $(1, 0, 3, 6, 9, 4, 5)$

$(5, 0, 4, 8, 1, 10, 3)$ $(9, 0, 5, 10, 4, 2, 8)$ mod 11

$v = 12$: $X = Z_{11} \bigcup \{\infty\}$

$(\infty, 0, 1, 2, 3, 4, 9)$ $(\infty, 0, 2, 4, 6, 1, 3)$ $(0, 3, 6, 9, 1, 2, 7)$

$(0, 4, 8, 1, 5, 3, \infty)$ $(0, 5, 10, 4, 9, 2, 8)$ $(0, 4, 9, 3, 7, 1, 6)$ mod 11

$v = 13$: $X = Z_{13}$

$(0, 1, 2, 3, 4, 6, 11)$ $(0, 2, 4, 6, 8, 3, 9)$ $(0, 3, 6, 9, 12, 4, 5)$

$(0, 4, 8, 12, 3, 5, 6)$ $(0, 5, 10, 2, 7, 1, 4)$ $(0, 6, 12, 5, 11, 2, 9)$ mod 13 □

Theorem 4.8 *For* $v \equiv 2, 3, 4, 5, 6 \pmod 7$ *and* $\lambda \equiv 0 \pmod 7$, *there exists a* (v, G_4, λ)*-GD.*

Proof. By Theorem 2.1, Lemma 2.4, Lemma 3.4 and Lemma 4.7. □

Theorem 4.9 *For* $i = 1, 3, 4$, *there exists a* (v, G_i, λ)*-GD if and only if* $\lambda v(v-1) \equiv 0 \pmod{14}$ *and* $v \geq 7$.

Proof. By Lemma 2.2, Theorem 2.5, Theorem 4.2, Theorem 4.6 and Theorem 4.8. □

Theorem 4.10 *There exists a (v, G_2, λ)-GD if and only if* $\lambda v(v-1) \equiv 0 \pmod{14}$, $v \geq 7$ *and* $(v, \lambda) \neq (7, 1)$.

Proof. By Lemma 2.3, Theorem 2.7 and Theorem 4.4. □

References

[1] K. Heinrich, *Path-decompositions*, Le Matematiche (Catania), XLVII (1992), 241-258.

[2] J. Bosak, *Decompositions of graphs*, Kluwer Academic Publishers, Boston, 1990.

[3] J. C. Bermond and J. Schönheim, *G-decomposition of K_n, where G has four vertices or less*, Discrete Math., 19(1977), 113-120.

[4] J. C. Bermond, C.Huang, A. Rosa and D. Sotteau, *Decomposition of complete graphs into isomorphic subgraphs with five vertices*, Ars Combin., 10(1980), 211-254.

[5] Qingde Kang,Yanke Du and Zihong Tian, *Docomposition of λK_v into some graph with six vertices and seven edges*, J. Statist. Plann. Inference, 136(2006), 1394-1409.

[6] Qingde Kang, Lan dang Yuan, Shuxia Liu, *Graph Designs for all Graphs with Six Vertices and Eight Edges* , Acta Math. Appl. Sin. Engl. Ser., 21(2005), 469-484.

[7] Qingde Kang, Chunping Ma and Hongtao Zhao, *G-decomposition of λK_v, where G has six vertices and nine edges*, Ars Combin., 94(2010), 485-510.

[8] Yinzhi Gao, Huijuan Zuo and Qingde Kang, *Decomposition of λK_v into the graphs with 7 points, 7 edges and an even-circle*, Acta Math. Appl. Sin., 27(2004), 646-658.

[9] Yanfang Zhang, *Decompositions of K_v into the graphs with 7 points, 7 edges and a 5-circle*, Ars Combin., 103(2012), 193-203.

[10] Qingde Kang and Zhihe Liang, *Optimal packings and coverings of λDK_v with k-circuits*, J. Combin. Math. Combin. Comput., 39(2001), 203-253.